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On finite products of groups and supersolubility ☆

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ABSTRACT

Two subgroups X and Y of a group G are said to be *conditionally permutable* in G if X permutes with Y^g for some element $g \in G$, i.e., XY^g is a subgroup of G. Using this permutability property new criteria for the product of finite supersoluble groups to be supersoluble are obtained and previous results are recovered. Also the behaviour of the supersoluble residual in products of finite groups is studied.

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1. Introduction and preliminaries

All groups considered in the paper are finite.

Two subgroups A and B of a group G are called *totally permutable* if every subgroup X of A is permutable with every subgroup Y of B, i.e., XY is a subgroup of G. In this case, if $G = AB$ we say that G is the *totally permutable product* of the subgroups A and B.

This condition was first considered by M. Asaad and A. Shaalan in [4] to provide criteria for the product of supersoluble groups to be supersoluble.

It is a well-known fact in the theory of groups that the product of two normal supersoluble subgroups is not necessarily supersoluble and there is a general interest in finding conditions for positive results. A classical R. Baer's result [5] states that if a group G is the product of two normal supersoluble subgroups and its derived subgroup G' is nilpotent, then G is supersoluble. Later on a vast

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research has been leaded by this aim. Asaad and Shaalan prove in particular that a totally permutable product of supersoluble subgroups is supersoluble.

In fact their paper has originated a fruitful research on products of groups whose factors are linked by certain permutability conditions on different families of subgroups in the factor groups (see [7,10] for a first approach to the topic). In particular the structure of totally permutable products of subgroups has been widely investigated and nowadays much information is known.

But initially R. Maier in [24] proved that Asaad and Shaalan result is a particular case of a more general one when considering the class $\mathcal{U}$ of supersoluble groups as a saturated formation (containing $\mathcal{U}$). Later on it was proved that Maier's result extends to non-saturated formations which contain all supersoluble groups [6] and totally permutable products of groups have been deeply studied both in the frameworks of formation theory (see [7,10]) as well as in the theory of Fitting classes [17–19].

W. Guo, K.P. Shum and A.N. Skiba in [16] have extended previous results by considering a weaker condition of subgroups permutability, namely *conditional permutability*. More precisely they consider the following concepts:

Definition. Let G be a group. Two subgroups X and Y of G are called *conditionally permutable* (*c-permutable*, for brevity) in G if X permutes with Y^g for some element $g \in G$.

The subgroups X and Y are called *completely c-permutable* in G if X permutes with Y^g for some element $g \in \langle X, Y\rangle$, the subgroup generated by X and Y.

Such type of permutability conditions has been considered by other authors in extending classical results about the influence of permutability properties of certain families of subgroups on the structure of groups; to be mentioned for instance [25,21,3,15].

In relation to products of groups W. Guo, K.P. Shum and A.N. Skiba in [16] extend Asaad and Shaalan result on products of supersoluble groups by weakening permutability to complete c-permutability. More exactly, they prove the following result (see Section 2, Corollary 2):

Let the group $G = AB$ be the product of supersoluble subgroups A and B. If every subgroup of A is completely c-permutable in G with every subgroup of B, then G is supersoluble.

This result has been recently improved by X. Liu, W. Guo, K.P. Shum in [22] by assuming that subnormal subgroups of each factor are completely c-permutable with the subgroups of the other factor (see Section 2, Corollary 3). Other related results can be also found in the same reference.

Also a related extension of Asaad and Shaalan result has been obtained by X. Liu, B. Li, X. Yi in [23] (see Section 2, Corollary 4).

Certainly permutable subgroups are completely c-permutable and these ones are in turn c-permutable. But easy examples show that these concepts are all different as we see below in the paper. In particular, it is remarkable that the property of persistence in intermediate subgroups turns to be a main difference between c-permutability and complete c-permutability. In fact complete c-permutability appears when requiring c-permutability to satisfy this persistence property and becomes a much stronger hypothesis, as we show next. (See Section 4, Example 4 and Final Remark.)

On the other hand significant structural properties of totally permutable products of subgroups are missed when considering c-permutability, even complete c-permutability, instead of permutability. (See Section 4, Examples 2, 3.)

This paper is devoted to the study of c-permutability in relation to products of groups and supersolubility. In Section 2 we prove that the aforesaid Asaad and Shaalan result and its extension by Guo, Shum and Skiba remain true in the following more general form for c-permutability (Corollary 1):

Let the group $G = AB$ be the product of supersoluble subgroups A and B. If every subgroup of A is c-permutable in G with every subgroup of B, then G is supersoluble.

In fact this result is obtained as a consequence of an even stronger result (Theorem 1) by considering a weaker permutability hypothesis, called NS-permutability (see Definition 1 and Lemma 2). This way also the above-mentioned recent results by Liu, Guo and Shum and by Liu, Li and Yi (Corollaries 3 and 4, respectively) follow as consequences of Theorem 1.

In Section 3 we study the behaviour of the supersoluble residual in products of groups within this framework. We recall that the supersoluble residual $G^{\mathcal{U}}$ of a group G is the smallest normal subgroup of G such that the factor group $G/G^{\mathcal{U}}$ is supersoluble. We prove in Theorem 2 that if the group $G = AB$ is the totally c-permutable product of the subgroups A and B (i.e., every subgroup of A is c-permutable in G with every subgroup of B), then $G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}$. Example 1 shows that this result is not further true for NS-permutable products of subgroups. We also apply Theorem 2 to obtain a local version for p-supersolubility of Corollary 1 which extends again previous results (Corollary 5).

We shall adhere to the notation used in [12], in particular $\sigma(G)$ denotes the set of all primes dividing the order of the group G. For subgroups X, A of a group G, we denote $\langle X^A\rangle = \langle x^a\colon\ x \in X, a \in A\rangle$; in particular, $\langle X^G\rangle$ is the normal closure of X in G.

2. Products of supersoluble groups

We introduce first NS-permutable products of subgroups and gather conditional permutability concepts. Then we apply them to study products of supersoluble groups.

Definition 1. Two subgroups A and B of a group G are said to be NS-*permutable* if they satisfy the following conditions:

- Whenever X is a normal subgroup of A and $p \in \sigma(B)$, there exists a Sylow p-subgroup B_p of B such that X permutes with B_p.
- Whenever Y is a normal subgroup of B and $p \in \sigma(A)$, there exists a Sylow p-subgroup A_p of A such that Y permutes with A_p.

(In particular it follows that A permutes with B.)

Moreover, if $G = AB$, we say that G is an NS-*permutable product* of the subgroups A and B.

Definition 2. (See [15,16].) Let G be a group.

Two subgroups X and Y of G are called *conditionally permutable* (*c-permutable*, for brevity) in G if X permutes with Y^g for some element $g \in G$.

If the group $G = AB$ is the product of subgroups A and B such that every subgroup of A is c-permutable in G with every subgroup of B, we will say that $G = AB$ is the *totally c-permutable product* of the subgroups A and B.

The subgroups X and Y are called *completely c-permutable* in G if X permutes with Y^g for some element $g \in \langle X, Y\rangle$, the subgroup generated by X and Y.

Lemma 1. *(See [11].) Let the group $G = HK$ be the product of subgroups H and K. If $L \trianglelefteq H$ and $L \leqslant K$, then $L \leqslant \mathrm{Core}_G(K)$.*

Proof. It follows easily since $L \leqslant \langle L^G\rangle = \langle L^{HK}\rangle = \langle L^K\rangle \leqslant \mathrm{Core}_G(K)$. □

Theorem 1. *Let the group $G = AB$ be the* NS*-permutable product of the subgroups A and B. If A and B are supersoluble, then G is supersoluble.*

Proof. Assume that the result is false and let $G = AB$ be a counterexample of minimal order. In particular, $A \neq 1$ and $B \neq 1$. We split the proof into two steps:

Step 1. G is a soluble primitive group, $\mathrm{Soc}(G)$ is a p-group for the largest prime p dividing $|G|$ and $G/\mathrm{Soc}(G)$ is supersoluble.

Let $1 \neq N$ be a normal subgroup of G. It is easily checked that $G/N = (AN/N)(BN/N)$ is the NS-permutable product of the supersoluble subgroups AN/N and BN/N. The choice of G implies that G/N is supersoluble. If N_1 and N_2 are distinct minimal normal subgroups of G, it follows that $G \cong G/(N_1 \cap N_2)$ is supersoluble, a contradiction. Then G has a unique minimal normal subgroup, which is also not contained in $\Phi(G)$, the Frattini subgroup of G. Consequently G is a primitive group of type 1.

We consider the largest prime p dividing $|G|$. Without loss of generality we may assume that p divides $|A|$. Let A_p be the Sylow p-subgroup of A (since A is a supersoluble, we notice that A_p is normal in A). Let X be a minimal normal subgroup of A such that $X \leqslant A_p$. Then X is a cyclic group of order p. By hypothesis we deduce that X permutes with B since X permutes with some Sylow subgroup of B for each prime dividing $|B|$. Consequently XB is a subgroup of G.

We distinguish next the following cases:

Case 1. $A_p \cap B \neq 1$.

Since A and B are supersoluble, we have that $A_p \cap B$ is a subnormal subgroup of both A and B. Therefore $A_p \cap B$ is a subnormal p-subgroup of $G = AB$ (see [1, Theorem 7.5.7]). It follows in this case that $\mathrm{Soc}(G)$ is a p-group and G is a soluble primitive group.

Case 2. $A_p \cap B = 1$.

We claim that XB satisfies the hypothesis of the theorem. Since X is a normal subgroup of A we have that X permutes with some Sylow subgroup of B for each prime divisor of $|B|$. Now we consider a normal subgroup T of B. By hypothesis T permutes with the Sylow p-subgroup A_p of A. We notice that $TA_p \cap XB = XT(A_p \cap B) = XT$ is a subgroup of G, which implies that T permutes with X. Since X and B are supersoluble, the claim is proved.

We assume first that XB is a proper subgroup of G. The choice of G implies that XB is supersoluble. By Lemma 1, $X \leqslant \mathrm{Core}_G(XB) \in \mathcal{U}$. Hence the Sylow p-subgroup of $\mathrm{Core}_G(XB)$ is a normal p-subgroup of G. We obtain again that $\mathrm{Soc}(G)$ is a p-group and G is a soluble primitive group.

We assume now that $G = XB$. We consider a minimal normal subgroup Y of the supersoluble group B. Then Y is a cyclic group. We notice that X permutes with Y and the group XY is supersoluble because it is the product of cyclic groups (see [20, VI, 10.1]). Since $1 \neq \langle Y^G \rangle = \langle Y^{BX} \rangle = \langle Y^X \rangle$ is a subgroup XY, it follows that $\langle Y^G \rangle = Y(X \cap \langle Y^G \rangle)$ is supersoluble.

If $X \cap \langle Y^G \rangle = 1$, then $\langle Y^G \rangle = Y$ is a cyclic normal subgroup of G. Since G/Y is supersoluble, we deduce that G is supersoluble, a contradiction. Consequently, $X \cap \langle Y^G \rangle \neq 1$ and the Sylow p-subgroup of $\langle Y^G \rangle$ is a normal subgroup of G. Again we deduce finally that $\mathrm{Soc}(G)$ is a p-group and G is a soluble primitive group.

Step 2. The final contradiction.

From Step 1 we have that $G = NM$, where $N = O_p(G) = \mathrm{Soc}(G)$ is abelian, M is a maximal subgroup of G such that $O_p(M) = 1$ and $M \cong G/N$ is supersoluble.

Since p is the largest prime dividing $|G|$ we can deduce that N is a Sylow p-subgroup of G. On the other hand there are Hall p'-subgroup $A_{p'}$ and $B_{p'}$ of A and B, respectively, such that $A_{p'}B_{p'}$ is a Hall p'-subgroup of G (see [1, Lemma 1.3.2]). Then we may consider that $M = A_{p'}B_{p'}$ is a maximal subgroup of G.

As above we assume that p divides $|A|$ and consider X a minimal normal subgroup of A such that $X \leqslant A_p \in \mathrm{Syl}_p(A)$.

We claim that X permutes with $B_{p'}$. This is clear if $B_{p'} = 1$. Assume that $B_{p'} = B_{p_1} \cdots B_{p_r}$ being $p_1 > \cdots > p_r$ the prime divisors of $|B_{p'}|$ and B_{p_i} a Sylow p_i-subgroup of B for $i = 1, \ldots, r$.

We notice that $B = B_p B_{p'}$ being B_p the Sylow p-subgroup of B (eventually $B_p = 1$). Moreover $B_p, X \leqslant N$ and, in particular, B_p centralizes X.

For formal purposes we set $B_{p_0} := 1$, assume inductively that X permutes with $B_{p_0} \cdots B_{p_{i-1}}$, for $i > 0$, and prove that X permutes with $B_{p_0} \cdots B_{p_{i-1}} B_{p_i}$. In particular it will follow that X permutes with $B_{p'}$ and the claim will be proved.

We set $B_j = B_p B_{p_0} \cdots B_{p_j}$, for $j \in \{0, 1, \ldots, r\}$. We notice that B_j is a normal subgroup of B for every j because B is supersoluble.

Since G is an NS-permutable product of the subgroups A and B, we have that X permutes with $B_{p_i}^z$ for some $z \in B$. Moreover, by the Frattini argument, $B = B_i N_B(B_{p_i})$ and consequently, X permutes with $B_{p_i}^{b_i}$ for some $b_i \in B_i = B_p B_{p_0} \cdots B_{p_i}$. We can write $b_i = wtx$ with $x \in B_p$, $t \in B_{p_0} \cdots B_{p_{i-1}}$ and $w \in B_{p_i}$. It follows that $XB_{p_i}^t = B_{p_i}^t X$. Since we are assuming that X permutes with $B_{p_0} \cdots B_{p_{i-1}}$, we can deduce that X permutes with $B_{p_0} \cdots B_{p_i}$ and the claim follows.

Hence we have that X permutes with $B_{p'}$ and obviously it permutes with $A_{p'}$. Consequently, $M = A_{p'}B_{p'} < XA_{p'}B_{p'} \leqslant G$, which implies $XA_{p'}B_{p'} = G$. Then $N = X$ is a cyclic normal subgroup of G. Since G/N is supersoluble it follows that G is supersoluble, the final contradiction. □

Lemma 2. *Let the group $G = AB$ be the product of the subgroups A and B. Then*:

1. *If the subgroup $X \trianglelefteq A$ is c-permutable in G with the subgroup $Y \leqslant B$, there exists $b \in B$ such that X permutes with Y^b. In particular, if $X \trianglelefteq A$ is c-permutable in G with B, then X permutes with B.*
2. *If the group $G = AB$ is the totally c-permutable product of the subgroups A and B, then $G = AB$ is an* NS-*permutable product of the subgroups A and B.*

Proof. 1. Let X be a normal subgroup of A which is c-permutable in G with a subgroup Y of B. Then there exists $g = ba \in G = BA$ with $b \in B$ and $a \in A$ such that X permutes with $Y^g = Y^{ba}$. It follows that $X^{a^{-1}} = X$ permutes with $(Y^g)^{a^{-1}} = Y^b$. The rest of Part 1 is clear.

2. Let X be a normal subgroup of A and Y a Sylow p-subgroup of B for any $p \in \sigma(B)$. By hypothesis X is c-permutable with Y in G, which implies by Part 1 that there exists $b \in B$ such that X permutes with $Y^b \in \mathrm{Syl}_p(B)$. Changing the roles of A and B it is easily deduced that A and B are NS-permutable. □

From Theorem 1 and Lemma 2 the following result follows.

Corollary 1. *Let the group $G = AB$ be the totally c-permutable product of the subgroups A and B. If A and B are supersoluble, then G is supersoluble.*

As mentioned in the introduction from Corollary 1 the following result due to Guo, Shum and Skiba is deduced:

Corollary 2. *(See [16, Theorem A].) Let the group $G = AB$ be the product of supersoluble subgroups A and B. If every subgroup of A is completely c-permutable in G with every subgroup of B, then G is supersoluble.*

Also from Theorem 1 the following recent improvement of the previous corollary by Liu, Guo and Shum, as well as the next related result due to Liu, Li and Yi, are recovered.

Corollary 3. *(See [22, Theorem 3.2].) A group G is supersoluble if and only if $G = AB$ is a product of supersoluble subgroups A and B such that every subnormal subgroup of A is completely c-permutable with every subgroup of B in G and every subnormal subgroup of B is completely c-permutable with every subgroup of A in G.*

Corollary 4. *(See [23, Theorem 3.4].) A group G is supersoluble if and only if $G = AB$ is the product of supersoluble subgroups A and B such that every normal subgroup of A is permutable with every Sylow subgroup of B and every normal subgroup of B is permutable with every Sylow subgroup of A.*

3. $\mathcal{U}$-residual of totally c-permutable products of subgroups

In this section we study the behaviour of the supersoluble residual in totally c-permutable products of subgroups. We gather first some previous results.

Lemma 3. *Let the group $1 \neq G = AB$ be the totally c-permutable product of the subgroups A and B. Then:*

1. *If A and B possess Hall π-subgroups for a set of primes π, then G possesses a Hall π-subgroup; more precisely, there exist Hall π-subgroups A_π of A and B_π of B such that $A_\pi B_\pi$ is a Hall π-subgroup of G.*
2. *If A and B have coprime orders, then G is $\sigma(A)$-separable. In particular either A or B contains a nontrivial normal subgroup of G.*
3. *If B is a group of prime order, then either A or B contains a nontrivial normal subgroup of G.*

Proof. 1. Let A_π and B_π be Hall π-subgroups of A and B, respectively. By hypothesis there exists $g \in G$ such that A_π permutes with B_π^g. But $g = ba$ for some $b \in B$ and $a \in A$. It follows easily that $A_\pi^{a^{-1}}$ permutes with B_π^b, which are also respective Hall π-subgroups of A and B. W.l.o.g. we assume that A_π permutes with B_π. Now

$$|G|_\pi = \frac{|A|_\pi |B|_\pi}{|A \cap B|_\pi} \leqslant \frac{|A_\pi||B_\pi|}{|A_\pi \cap B_\pi|} = |A_\pi B_\pi| \leqslant |G|_\pi,$$

which implies that $A_\pi B_\pi$ is a Hall π-subgroup of G.

2. Set $\pi = \sigma(A)$. Obviously A is a Hall π-subgroup of G and B is a Hall π'-subgroup of G. Moreover by Sylow's theorems and the previous result the group G possesses Hall $\pi \cup \{q\}$-subgroups and Hall $\pi' \cup \{p\}$-subgroups for all $p \in \pi$ and all $q \in \pi'$. This implies by a Z. Du's result [13, Theorem 1] that the group G is π-separable. In this case either $1 \neq O_\pi(G) \leqslant A$ or $1 \neq O_{\pi'}(G) \leqslant B$ and the result follows.

3. Assume that B is a group of order prime p. Using Part 2 we may assume that p divides the order of A. We can also assume that $B \nleqslant A$ and so $A \cap B = 1$. By Lemma 2 there exists a Sylow p-subgroup A_p of A which permutes with B. Then $|BA_p : A_p| = p$ and A_p is normal in BA_p. Consequently $A_p \leqslant \mathrm{Core}_G(A)$, by Lemma 1, and A contains a nontrivial normal subgroup of G. □

Remark. From the well-known Hall's theorem, which characterizes soluble groups by the existence of Hall π-subgroups for all sets π of primes (see [12, I, 3.6]), and Lemma 3 it follows that the totally c-permutable product of soluble subgroups is soluble. Though again this result can be deduced from Lemma 2 and the following result for NS-permutable products of subgroups.

"If the group $G = AB$ is the NS*-permutable product of the soluble subgroups A and B, then G is soluble."*

Proof. Assume that the result is not true and let $G = AB$ be a counterexample of minimal order. In particular, $A \neq 1$ and $B \neq 1$. Let N be a minimal normal subgroup of G. Then $G/N = (AN/N)(BN/N)$ is the NS-permutable product of the soluble subgroups AN/N and BN/N. The choice of G implies that G/N is soluble. If M is a minimal normal subgroup of G, $N \neq M$, then $G \cong G/(N \cap M)$ is soluble, a contradiction. Consequently N is the unique minimal normal subgroup of G and G/N is soluble.

Since A and B are soluble there exist corresponding primes p and q such that $O_q(A) \neq 1$ and $O_p(B) \neq 1$. By hypothesis we have that $AO_p(B)$ and $BO_q(A)$ are subgroups of G. By Lemma 1 we deduce that $O_p(B) \leqslant \mathrm{Core}_G(AO_p(B)) \neq 1$ and also $O_q(A) \leqslant \mathrm{Core}_G(BO_q(A)) \neq 1$; in particular,

$$N \leqslant AO_p(B) \cap BO_q(A) = O_p(B)O_q(A)(A \cap B) = \langle O_p(B), O_q(A)\rangle(A \cap B).$$

By hypothesis we have that $O_p(B)$ permutes with some Sylow q-subgroup A_q of A. Then $\langle O_p(B), O_q(A)\rangle \leqslant O_p(B)A_q$ which is a soluble group by Burnside's $p^a q^b$-theorem. Moreover $A \cap B$ is a soluble group which normalizes $\langle O_p(B), O_q(A)\rangle$. This implies that $\langle O_p(B), O_q(A)\rangle(A \cap B)$ is a soluble group. In particular it follows that N is abelian and G is soluble, a contradiction. □

Theorem 2. *Let the group $G = AB$ be the totally c-permutable product of the subgroups A and B. Then $G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$.*

Proof. We notice first that $A^{\mathcal{U}}$ and $B^{\mathcal{U}}$ permute by Lemma 2 and so $A^{\mathcal{U}}B^{\mathcal{U}}$ is a subgroup of G.

Assume that the result is false and let $G = AB$ be a counterexample with $|G| + |A| + |B|$ minimal. Since supersoluble groups are closed under taking subgroups and from Corollary 1 we have that $1 \neq A^{\mathcal{U}}B^{\mathcal{U}} \leqslant G^{\mathcal{U}}$. Also we notice that A and B are proper subgroups of G. We split the proof into the following steps:

Step 1. $G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}N$ for all minimal normal subgroup N of G and $\mathrm{Core}_G(A^{\mathcal{U}}B^{\mathcal{U}}) = 1$; in particular, $\mathrm{Soc}(G) \leqslant G^{\mathcal{U}}$.

Let N be a minimal normal subgroup of G. It is easy to prove that $G/N = (AN/N)(BN/N)$ is the totally c-permutable product of the subgroups AN/N and BN/N. By the choice of G we have that $(G/N)^{\mathcal{U}} = (AN/N)^{\mathcal{U}}(BN/N)^{\mathcal{U}}$. This implies that $G^{\mathcal{U}}N = A^{\mathcal{U}}B^{\mathcal{U}}N$ and, consequently, $G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}(G^{\mathcal{U}} \cap N)$. Then we can deduce that $N \leqslant G^{\mathcal{U}}$ and $G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}N$ and the desired conclusions follow.

Step 2. B is not a group of prime order.

Assume that B is a group of prime order. Then A is a maximal subgroup of G and $A \cap B = 1$. Assume that $\mathrm{Core}_G(A) \neq 1$. Then from Step 1 we have that $G^{\mathcal{U}} \leqslant A$. Moreover from Lemmas 2 and 1, $A^{\mathcal{U}}B$ is a subgroup of G and $G^{\mathcal{U}} \leqslant A^{\mathcal{U}}B$. Then $G^{\mathcal{U}} = G^{\mathcal{U}} \cap A^{\mathcal{U}}B = A^{\mathcal{U}}(G^{\mathcal{U}} \cap B) \leqslant A^{\mathcal{U}}(A \cap B) = A^{\mathcal{U}}$, a contradiction. Hence $\mathrm{Core}_G(A) = 1$. From Lemma 3 we deduce now that B is a normal subgroup of G, which has order prime. Consequently G is a primitive group of type 1 with abelian minimal normal subgroup B. Then $G/B = G/C_G(B)$ is cyclic and G is supersoluble, a contradiction.

Step 3. $A^{\mathcal{U}}B^{\mathcal{U}}$ is a normal subgroup of $G^{\mathcal{U}}$.

We distinguish the following cases:

B is supersoluble. In this case $A^{\mathcal{U}}B^{\mathcal{U}} = A^{\mathcal{U}}$.

Let B_0 be a minimal normal subgroup of the supersoluble group B; then B_0 is a cyclic group of prime order. By Lemma 2 we have that A permutes with B_0 and, moreover, AB_0 is the totally c-permutable product of the subgroups A and B_0, since B_0 has order prime. The choice of (G, A, B) and Step 2 imply that $(AB_0)^{\mathcal{U}} = A^{\mathcal{U}} \trianglelefteq AB_0$. On the other hand, since $B_0 \triangleleft B$, we deduce from Lemma 1 and Step 1 that $A^{\mathcal{U}} \leqslant G^{\mathcal{U}} \leqslant AB_0$, and then $A^{\mathcal{U}} \trianglelefteq G^{\mathcal{U}}$.

None A or B is supersoluble. In this case $A^{\mathcal{U}} \neq 1$ and $B^{\mathcal{U}} \neq 1$.

By Lemma 2 we have that A permutes with $B^{\mathcal{U}}$ and B permutes with $A^{\mathcal{U}}$. Again from Lemma 1 and Step 1, we deduce that both $AB^{\mathcal{U}}$ and $BA^{\mathcal{U}}$ contain some corresponding minimal normal subgroup of G and consequently

$$G^{\mathcal{U}} \leqslant AB^{\mathcal{U}} \cap BA^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}(A \cap B).$$

Since $A \cap B$ normalizes both $A^{\mathcal{U}}$ and $B^{\mathcal{U}}$, it is clear now that $A^{\mathcal{U}}B^{\mathcal{U}} \trianglelefteq G^{\mathcal{U}}$.

Step 4. G is a soluble group.

Assume that the result is false. In particular, the soluble residual $G^{\mathcal{S}}$ of G is nontrivial and there exists a minimal normal subgroup of N of G such that $N \leqslant G^{\mathcal{S}} \leqslant G^{\mathcal{U}}$. It follows from Step 1 that

$$G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}N = A^{\mathcal{U}}B^{\mathcal{U}}G^{\mathcal{S}}.$$

By Step 3 we have that

$$G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}} \cong G^{\mathcal{S}}/\big(A^{\mathcal{U}}B^{\mathcal{U}} \cap G^{\mathcal{S}}\big).$$

We claim that $G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ is a soluble group. It will follow that $N \leqslant G^{\mathcal{S}} = (G^{\mathcal{S}})^{\mathcal{S}} \leqslant A^{\mathcal{U}}B^{\mathcal{U}}$, a contradiction which will prove Step 4.

We prove next that $G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ has Hall π-subgroups for any set of primes π. By Hall's theorem [12, I, 3.6], $G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ will be soluble and the claim will be proved.

Let π be a set of primes. We consider subgroups $A^{\mathcal{U}} \leqslant X$ and $B^{\mathcal{U}} \leqslant Y$ of A and B, respectively, such that $X/A^{\mathcal{U}}$ and $Y/B^{\mathcal{U}}$ are corresponding Hall π-subgroups of $A/A^{\mathcal{U}}$ and $B/B^{\mathcal{U}}$. By hypothesis there exists $g \in G$ such that X permutes with Y^g. But $g = ba$ with $b \in B$ and $a \in A$. It follows that $X^{a^{-1}}$ permutes with Y^b. We notice that $A^{\mathcal{U}} \leqslant X^{a^{-1}}$ and $B^{\mathcal{U}} \leqslant Y^b$ and $X^{a^{-1}}/A^{\mathcal{U}}$ and $Y^b/B^{\mathcal{U}}$ are Hall π-subgroups of $A/A^{\mathcal{U}}$ and $B/B^{\mathcal{U}}$, respectively. Hence w.l.o.g. we may assume that X permutes with Y. Next we argue that

$$XY \cap G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$$

is a Hall π-subgroup of $G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ and we will be done.

It is clear that $|XY \cap G^{\mathcal{U}} : A^{\mathcal{U}}B^{\mathcal{U}}|$ divides

$$\left|XY : A^{\mathcal{U}}B^{\mathcal{U}}\right| = \frac{|X||Y|}{|A^{\mathcal{U}}||B^{\mathcal{U}}|\frac{|X\cap Y|}{|A^{\mathcal{U}}\cap B^{\mathcal{U}}|}}$$

which is a π-number; hence, $XY \cap G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ is a π-group.

On the other hand,

$$\left|\left(G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}\right) : \left(XY \cap G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}\right)\right| = \left|G^{\mathcal{U}} : XY \cap G^{\mathcal{U}}\right| = \left|G^{\mathcal{U}}XY : XY\right|$$

divides

$$|AB : XY| = \frac{|A||B|}{|X||Y||\frac{A\cap B}{X\cap Y}|}$$

which is a π'-number because

$$|A : X| = \left|\left(A/A^{\mathcal{U}}\right) : \left(X/A^{\mathcal{U}}\right)\right|, \qquad |B : Y| = \left|\left(B/B^{\mathcal{U}}\right) : \left(Y/B^{\mathcal{U}}\right)\right|$$

are π'-numbers.

This proves finally that $XY \cap G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ is a Hall π-subgroup of $G^{\mathcal{U}}/A^{\mathcal{U}}B^{\mathcal{U}}$ and concludes the proof of Step 4.

Step 5. $G^{\mathcal{U}}$ is an abelian p-group for some prime p.

Let N be a minimal normal subgroup of G. We have that $A^{\mathcal{U}}B^{\mathcal{U}} \trianglelefteq G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}N$ and N is an abelian p-group for some prime p, from Steps 1, 3 and 4. Consequently $G^{\mathcal{U}}/(A^{\mathcal{U}}B^{\mathcal{U}}) \cong N/(N \cap A^{\mathcal{U}}B^{\mathcal{U}})$ is an abelian p-group. Therefore $O^p(G^{\mathcal{U}})(G^{\mathcal{U}})' \leqslant A^{\mathcal{U}}B^{\mathcal{U}}$ which implies $O^p(G^{\mathcal{U}})(G^{\mathcal{U}})' = 1$ since $\mathrm{Core}_G(A^{\mathcal{U}}B^{\mathcal{U}}) = 1$ by Step 1, that is, $G^{\mathcal{U}}$ is an abelian p-group.

Step 6. The final contradiction.

From Lemma 2 we have that $A^{\mathcal{U}}B$ and $AB^{\mathcal{U}}$ are subgroups of G. We consider also $A = A^{\mathcal{U}}U_A$ and $B = B^{\mathcal{U}}U_B$ where U_A and U_B are $\mathcal{U}$-projectors of A and B, respectively. Then

$$A^{\mathcal{U}}B \cap G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}U_B \cap G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}\left(U_B \cap G^{\mathcal{U}}\right) \trianglelefteq A^{\mathcal{U}}B$$

and analogously

$$B^{\mathcal{U}}A \cap G^{\mathcal{U}} = B^{\mathcal{U}}A^{\mathcal{U}}U_A \cap G^{\mathcal{U}} = B^{\mathcal{U}}A^{\mathcal{U}}\left(U_A \cap G^{\mathcal{U}}\right) \trianglelefteq B^{\mathcal{U}}A.$$

If $U_B \cap G^{\mathcal{U}} = 1$ and also $U_A \cap G^{\mathcal{U}} = 1$, it follows that $A^{\mathcal{U}} B^{\mathcal{U}} \trianglelefteq G$ and so $A^{\mathcal{U}} B^{\mathcal{U}} = 1$ by Step 1, a contradiction.

W.l.o.g. we assume now that $U_B \cap G^{\mathcal{U}} \neq 1$. Since $U_B \cap G^{\mathcal{U}}$ is a normal subgroup of U_B, we may consider a minimal normal subgroup N_0 of U_B such that $N_0 \leqslant U_B \cap G^{\mathcal{U}}$. We notice that N_0 is a cyclic group of order p, the prime divisor of $|G^{\mathcal{U}}|$, because U_B is supersoluble. Moreover by Step 5 we have also that $G^{\mathcal{U}}$ is abelian, and consequently $G^{\mathcal{U}}$ centralizes N_0 and N_0 is a normal subgroup of $B = B^{\mathcal{U}} U_B$. In particular we can deduce that $N_0 \leqslant Z(B_p)$ for every Sylow p-subgroup B_p of B.

On the other hand by Lemma 2 there exists a Hall p'-subgroup $A_{p'}$ of A which permutes with N_0. Hence

$$N_0 = N_0\big(G^{\mathcal{U}} \cap A_{p'}\big) = G^{\mathcal{U}} \cap N_0 A_{p'} \trianglelefteq N_0 A_{p'},$$

that is, $A_{p'}$ normalizes N_0.

Consequently,

$$\big\langle N_0^G \big\rangle = N_0[N_0, G] = N_0[N_0, A_p] \leqslant G^{\mathcal{U}}$$

because $G = BA = B(A_{p'} A_p)$ for any Sylow p-subgroup A_p of A, and $N_0 \leqslant G^{\mathcal{U}}$. Also by Lemma 2 we notice that

$$N_0 A_p^a = N_0 A_p^{xy} = (N_0 A_p)^y$$

is a subgroup of G, for some $a = xy \in A$ with $x \in A_p$ and $y \in A_{p'}$, that is, N_0 permutes with any Sylow p-subgroup A_p of A.

Taking into account the previous facts we prove next that $[N_0, A_p]$ is a normal subgroup of G.

We know by [1, Lemma 1.3.2] that there exist Sylow p-subgroups A_p and B_p of A and B, respectively, such that $G_p := A_p B_p$ is a Sylow p-subgroup of $G = AB$. Consequently $[N_0, G_p] = [N_0, A_p]$ is normalized by G_p.

We claim that $[N_0, A_p]$ is a normal subgroup of A.

We notice that either $N_0 \leqslant A_p$ or $N_0 \cap A_p = 1$. In the second case $|N_0 A_p : A_p| = p$, and then, in any case, $[N_0, A_p] \leqslant A_p$.

If $N_0 \cap A_p = 1$, then $N_0 \cap A = 1$ and consequently

$$\big\langle N_0^G \big\rangle \cap A = N_0[N_0, A_p] \cap A = [N_0, A_p](N_0 \cap A) = [N_0, A_p] \trianglelefteq A.$$

Assume now that $N_0 \leqslant A_p$. Then

$$\big\langle N_0^G \big\rangle = N_0[N_0, A_p] \trianglelefteq A.$$

If $N_0 \leqslant A^{\mathcal{U}}$, then $\langle N_0^G \rangle \leqslant \mathrm{Core}_G(A^{\mathcal{U}}) \leqslant \mathrm{Core}_G(A^{\mathcal{U}} B^{\mathcal{U}}) = 1$ by Step 1, a contradiction.

Consequently,

$$1 \neq \big\langle N_0^G \big\rangle A^{\mathcal{U}} / A^{\mathcal{U}} \cong_A \big\langle N_0^G \big\rangle / \big\langle N_0^G \big\rangle \cap A^{\mathcal{U}},$$

and we can consider an A-chief factor C/D such that

$$\big\langle N_0^G \big\rangle \cap A^{\mathcal{U}} \leqslant D < C \leqslant \big\langle N_0^G \big\rangle$$

and satisfying that $N_0 \nleqslant D$ but $N_0 \leqslant C$. We notice that in addition C/D is a cyclic group of order p and, in particular, is centralized by A_p. Therefore, this implies that $C = DN_0 = \langle N_0^G \rangle = N_0[N_0, A_p]$ and $D = [N_0, A_p]$, which is a normal subgroup of A, and the claim is proved.

Now it follows by Lemma 2 that there exists a Hall p'-subgroup $B_{p'}$ of B such that $[N_0, A_p]$ permutes with $B_{p'}$. Consequently,

$$[N_0, A_p] = [N_0, A_p]\big(B_{p'} \cap G^{\mathcal{U}}\big) = [N_0, A_p]B_{p'} \cap G^{\mathcal{U}} \trianglelefteqslant [N_0, A_p]B_{p'},$$

that is, $B_{p'}$ normalizes $[N_0, A_p]$, and we can deduce finally that $[N_0, A_p]$ is normal in G.

Since $[N_0, A_p] \leqslant A_p$ we notice that $[N_0, A_p] < \langle N_0^G \rangle = N_0[N_0, A_p]$.

Hence $\langle N_0^G \rangle/[N_0, A_p] \cong N_0 \cong C_p$ is a $\mathcal{U}$-central chief factor of G. By [12, V, 3.2(e)] this chief factor is covered by any $\mathcal{U}$-normalizer U of G. But, since $G^{\mathcal{U}}$ is abelian, U is a $\mathcal{U}$-projector of G and $G^{\mathcal{U}} \cap U = 1$ by [12, V, 4.2; IV, 5.18]. Therefore

$$\big\langle N_0^G \big\rangle \leqslant [N_0, A_p]U \cap G^{\mathcal{U}} = [N_0, A_p]\big(G^{\mathcal{U}} \cap U\big) = [N_0, A_p] < \big\langle N_0^G \big\rangle,$$

the final contradiction. □

As an application of Theorem 2 we deduce next a local version for p-supersoluble groups of Corollary 1.

Corollary 5. *Let p be a prime. If the group $G = AB$ is a totally c-permutable product of p-supersoluble subgroups A and B, then G is p-supersoluble.*

Proof. We recall that for any group X, $O_{p',p}(X)$ is the centralizer of all chief factors of X whose orders are divisible by p (see for instance [12, A, 13.8(a)]). If, in addition, X is p-supersoluble, these chief factors are cyclic of order p. Hence, in this case, $X/O_{p',p}(X)$ is in particular abelian and $X/O_{p'}(X)$ is supersoluble, which implies $X^{\mathcal{U}} \leqslant O_{p'}(X)$.

Since A and B are p-supersoluble, it follows that $A^{\mathcal{U}} \leqslant O_{p'}(A)$, $B^{\mathcal{U}} \leqslant O_{p'}(B)$ and then $G^{\mathcal{U}} = A^{\mathcal{U}}B^{\mathcal{U}}$ is a p'-group by Theorem 2. It follows that G is p-supersoluble. □

A corresponding result for complete c-permutability was considered in [16] by Guo, Shum and Skiba, as an extension of a result by A. Carocca in [9] for totally permutable products of groups. It follows now from Corollary 5.

Corollary 6. *(See [16, Theorem 4.1].) Let p be a prime. Let the group $G = AB$ be the product of p-supersoluble subgroups A and B. If every subgroup of A is completely c-permutable with every subgroup of B, then G is p-supersoluble.*

The next example shows the failure of Theorem 2 for NS-permutable products of subgroups.

Example 1. We construct a group $G = AB$ which is the NS-permutable product of subgroups A and B such that $G^{\mathcal{U}} = \langle A^{\mathcal{U}}, B^{\mathcal{U}} \rangle \neq A^{\mathcal{U}}B^{\mathcal{U}}$; in particular, $A^{\mathcal{U}}$ and $B^{\mathcal{U}}$ are not permutable.

We consider $H = \mathrm{Alt}(4) = VS$ the alternating group on $\{1, 2, 3, 4\}$, being $V = \langle v_1, v_2 \rangle$ with $v_1 = (1\,2)(3\,4)$, $v_2 = (1\,3)(2\,4)$, and $S = \langle x \rangle$ with $x = (1\,2\,3)$. Let M be the natural permutation module for $\mathrm{Alt}(4)$ over $\mathbb{F}_2$ with permutation basis $\{x_1, x_2, x_3, x_4\}$. We set

$$y_1 = x_1x_2, \qquad y_2 = x_1x_3, \qquad z = x_1x_2x_3x_4 \in M,$$

$Y = \langle y_1, y_2 \rangle$, $Z = \langle z \rangle$ and $W = \langle y_1, y_2, z \rangle = Y \times Z$ which is an H-submodule of M. Let $G = [W]H$ be the corresponding semidirect product.

More precisely H acts on W as follows:

$$\begin{aligned} y_1^x &= y_1 y_2, & y_2^x &= y_1, & z^x &= z; \\ y_1^{v_1} &= y_1, & y_2^{v_1} &= y_2 z, & z^{v_1} &= z; \\ y_1^{v_2} &= y_1 z, & y_2^{v_2} &= y_2, & z^{v_2} &= z. \end{aligned}$$

In particular:

- $Z = Z(G)$,
- Y is a nontrivial irreducible S-submodule of W (hence, $YS \cong VS = H$),
- $\langle Y, V\rangle = YZV = WV \neq YV$.

We let $A = WS = ZYS$ and $B = ZVS = ZH$. Then $G = AB$, $A^{\mathcal{U}} = Y$, $B^{\mathcal{U}} = V$ and $G^{\mathcal{U}} = WV = \langle A^{\mathcal{U}}, B^{\mathcal{U}}\rangle \neq A^{\mathcal{U}} B^{\mathcal{U}}$.

(We notice that $A \cong B$ being Y and V in correspondence.)

We prove finally that A and B are NS-permutable.

Let $1 \neq N \trianglelefteq A$. If $3 \notin \sigma(N)$, then $N = Y$, $N = Z$ or $N = W$. If $3 \in \sigma(N)$, then either $N = YS$ or $N = ZYS = A$. In all cases N permutes with $ZV \in \mathrm{Syl}_2(B)$ and with $S \in \mathrm{Syl}_3(B)$.

With the same arguments we deduce also that any normal subgroup of B permutes with a Sylow p-subgroup of A for any $p = 2, 3$.

4. Final examples and remarks

The first two examples next show the relation between permutability and complete c-permutability and how significant structural properties of totally permutable products of subgroups are missed when replacing permutability by (complete) c-permutability.

Example 2. Assume that X and Y are subgroups of a group G.

If X and Y are permutable, then X and Y are obviously completely c-permutable in G, though the converse is not true.

To see this it is enough to consider the symmetric group $G = \mathrm{Sym}(3)$ of degree 3. In fact a Sylow 2-subgroup X of G is completely c-permutable in G with all subgroups of G though it is not permutable with all subgroups of G.

In particular, for the trivial factorization $G = AB$ being $A = G$ and $B = X$, we can see that every subgroup of A is completely c-permutable in G with every subgroup of B, but $X = X \cap G \nleqslant F(G)$, the Fitting subgroup of G, differently to the behaviour of totally permutable products of subgroups.

(We recall that if a group $G = HK$ is the totally permutable product of subgroups H and K, then $H \cap K \leqslant F(G)$; [24, Lemma 2].)

Example 3. We consider $V = \langle a, b\rangle \cong Z_5 \times Z_5$ and $Z_6 \cong C = \langle \alpha, \beta\rangle \leqslant \mathrm{Aut}(V)$ given by

$$a^\alpha = a^{-1}, \qquad b^\alpha = b^{-1}; \qquad a^\beta = b, \qquad b^\beta = a^{-1}b^{-1}.$$

Let $G = [V]C$ be the corresponding semidirect product of V with C. Set $A = \langle \alpha \rangle$ and $B = V\langle \beta\rangle$. Then $G = AB$ and every subgroup of A is completely c-permutable in G with every subgroup of B but A and B are not totally permutable. In fact we notice that $B^{\mathcal{N}} = B^{\mathcal{U}} = V$ does not centralize A, in contrast to properties of totally permutable products of subgroups.

We denote by $Z_{\mathcal{U}}(G)$ the $\mathcal{U}$-hypercentre of the group G, i.e., the largest normal subgroup of G such that every chief factor H/K of G with $K < H \leqslant Z_{\mathcal{U}}(G)$ is cyclic of prime order. We remark that in this example $Z_{\mathcal{U}}(G) = 1$ and obviously G modulo $Z_{\mathcal{U}}(G)$ is not a direct product of the images of A and B.

(It is know that if $G = HK$ is the totally permutable product of subgroups H and K, then $X^{\mathcal{N}}$ centralizes Y for $\{H, K\} = \{X, Y\}$ [8, Theorem 1]; also G modulo $Z_{\mathcal{U}}(G)$ is a direct product of the images of A and B (see [14, p. 859, Remarks (3)]).)

As mentioned in the introduction we see next that c-permutability fails to satisfy the property of persistence in intermediate subgroups; i.e., if X and Y are c-permutable subgroups in a group G, then X and Y are not necessarily c-permutable in any subgroup M of G such that $X, Y \leqslant M \leqslant G$. This makes a relevant difference between c-permutability and complete c-permutability.

Example 4. Let $G = \mathrm{Sym}(4)$ be the symmetric group of degree 4, Y a subgroup of G of order 2 generated by a transposition, V the normal subgroup of G of order 4 and X a subgroup of V of order 2, $X \neq Z(VY)$. Then we observe that X and Y are c-permutable in G but they are not c-permutable in $\langle Y, X\rangle$.

Final Remark. Inspired by the previous research on totally permutable products of subgroups it is natural to wonder whether this study on conditional permutability and supersolubility can be extended in the framework of formation theory. In this sense it is to be mentioned that totally c-permutable products of subgroups is a too weak structure to obtain positive results for general saturated formations, containing $\mathcal{U}$, even in the universe of soluble groups. We mention here the following example. We consider as in Example 4, $G = \mathrm{Sym}(4)$, $A = \mathrm{Alt}(4)$ the alternating subgroup of G and Y a subgroup of G of order 2 generated by a transposition. Then one can check that $G = AY$ is the totally c-permutable product of A and Y; but for $\mathcal{N}^2$ the saturated formation of metanilpotent groups, we have that $\mathcal{U} \subseteq \mathcal{N}^2$, $A, Y \in \mathcal{N}^2$ but $G \notin \mathcal{N}^2$.

In contrast, for saturated formations of soluble groups containing all supersoluble groups, previous developments on totally permutable products of subgroups have been extended by weakening permutability to complete c-permutability in [2].

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