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Bounds on the number of maximal sum-free sets

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ABSTRACT

We show that the number of maximal sum-free subsets of $\{1, 2, \dots, n\}$ is at most $2^{3n/8+o(n)}$. We also show that $2^{0.406n+o(n)}$ is an upper bound on the number of maximal product-free subsets of any group of order n.

1. Introduction

A subset S of $[n] = \{1, 2, \dots, n\}$ is sum-free if for every $y, z \in S$, we have $y + z \notin S$. We say that S is maximal sum-free if it is sum-free and is properly contained in no other sum-free subset of $[n]$. Define $f(n)$ to be the number of sum-free subsets of $[n]$ and $f_{\max}(n)$ to be the number of maximal sum-free subsets of $[n]$. It is known that $f(n) = \Theta(2^{n/2})$ [3]. Cameron and Erdös [2] showed that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$ and asked whether or not $f_{\max}(n) = f(n)/2^{\epsilon n}$ for some constant $\epsilon > 0$. Luczak and Schoen [5] answered that question affirmatively, proving that $f_{\max}(n) \leq 2^{n/2-2^{-28}n}$, provided that n is sufficiently large. In this paper we prove the following, improved upper bound on $f_{\max}(n)$.

Theorem 1.1. $f_{\max}(n) \leq 2^{3n/8+o(n)}$.

If $(G, \cdot)$ is a group and $S \subseteq G$, we say that S is product-free if for every $y, z \in S$, we have $y \cdot z \notin S$. S is maximal product-free if it is not strictly contained in any other product-free subset of G. Denote by $f(G)$ and $f_{\max}(G)$ the numbers of product-free subsets and maximal product-free subsets of G, respectively. From Alon [1], it is known that for every group G of order n, $f(G) \leq 2^{n/2+o(n)}$ and that there exists a group G of order n for which $f(G) \geq 2^{n/2+\Omega(\ln n)}$. From [5,1] it follows that for a sufficiently large n, if G is a group of order n then $f_{\max}(G) \leq 2^{n/2-2^{-28}n+o(n)}$. Our second result improves the best known upper bound on the number of maximal product-free subsets in a group G of order n.

Theorem 1.2. *For any group G of order n, $f_{\max}(G) \leq 2^{0.406n+o(n)}$.*

1.1. Overview

The following fact is the starting point of the proof of Theorem 1.2.

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Fact 1.3. *Let Ω be an arbitrary set, let $\Omega' \subseteq \Omega$ and let $\mathcal{D}$ be a distribution over Ω. Then $\Pr_{\mathcal{D}}(x) \leq 1/|\Omega'|$ for some $x \in \Omega'$.*

In light of Fact 1.3, in order to give an upper bound on the size of a set $\Omega' \subseteq \Omega$, it is enough to define a distribution $\mathcal{D}$ over the set Ω and then lower bound the probability $\Pr_{\mathcal{D}}(x)$ for every $x \in \Omega'$. If we are able to show that for all $x \in \Omega'$, $\Pr_{\mathcal{D}}(x) \geq p$, it will then follow by Fact 1.3 that $|\Omega'| \leq 1/p$. To use this idea in the proof of Theorem 1.2 we define for every fixed subset A of G, a distribution over the set of product-free subsets of A. Our distribution is defined by means of a randomized, greedy process: We start with an empty product-free set R and with a uniformly random permutation of A. Then, according to the order implied by the permutation, we take each element in A and add it to R with probability ϵ, unless its addition creates a set R which is not product-free. Trivially, for every $\epsilon \in (0, 1)$, such a process induces a distribution over the set of all product-free subsets of A. We then, essentially, give a lower bound on the probability that the above process produces a maximal product-free subset of G (that is contained in A). Doing this by Fact 1.3 gives an upper bound on the number of maximal product-free subsets of G that are contained in A. This upper bound on the number of maximal product-free subsets of G that are contained in a fixed subset A of G is combined together with a covering lemma for independent sets in regular graphs and with an appropriate, small family, of Cayley graphs defined on G, to obtain the proof of Theorem 1.2. The covering lemma for independent sets in regular graphs follows from Sapozhenko [6] and is stated next.

Lemma 1.4 (*Sapozhenko*). *For $n \in \mathbb{N}$, let $G = (V, E)$ be a k-regular, n-vertex graph, with $k \geq \sqrt{n}$. Then there exists a family $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ of subsets of V such that:*

1. $m = 2^{O(n^{3/4} \ln n)}$;
2. *For all* $A_i \in \mathcal{F}$, $|A_i| \leq n/2 + O(n^{3/4} \ln n)$;
3. *For every independent set I of G there exists $A_i \in \mathcal{F}$ such that $I \subseteq A_i$.*

The proof of Theorem 1.1, given in Section 3 is based on a covering lemma for sum-free subsets of $[n]$. Such a covering lemma was proved by Green [3].

Lemma 1.5 (*Green*). *There exists a family $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ of subsets of $[n]$ such that:*

1. $m = 2^{o(n)}$;
2. *For all* $A_i \in \mathcal{F}$, $|A_i| \leq n/2 + o(n)$;
3. *For every sum-free subset S of $[n]$ there exists $A_i \in \mathcal{F}$ such that $S \subseteq A_i$.*

From Lemma 1.5 it follows that an upper bound on the number of maximal sum-free subsets of $[n]$ which are contained in an arbitrary set $A \subseteq [n]$ of size $n/2 + o(n)$, translates immediately to an upper bound on $f_{\max}(n)$, up to a multiplicative factor of $2^{o(n)}$. So, in order to prove Theorem 1.1, we fix an arbitrary subset A of $[n]$ of size $n/2 + o(n)$ and show that it cannot contain more than $2^{3n/8+o(n)}$ maximal sum-free subsets of $[n]$. This last task is achieved by a reduction to the problem of estimating the number of maximal (by inclusion) independent sets in triangle-free graphs, a problem which is completely solved by Hujter and Tuza [4]. For our purposes, the following upper bound of Hujter and Tuza will suffice.

Theorem 1.6 (*Hujter–Tuza*). *For $n \geq 4$, the number of maximal (by inclusion) independent sets in an n-vertex, triangle-free graph is at most $2^{n/2}$.*

2. The number of maximal product-free subsets of groups

In this section we prove Theorem 1.2. Let $(G, \cdot)$ be a group of order n. We are interested in estimating $f_{\max}(G)$ from above. For a product-free subset Q of G, let $H = H(G, Q)$ be the Cayley graph associated as usual with G and Q; that is, H is a graph with vertex set G and edge set E, where $(u, v) \in E$ if and only if there exists $s \in Q$ for which $u \cdot s = v$ or $v \cdot s = u$. It is not hard to see that H is k-regular, with $k = |Q \cup Q^{-1}|$ and that any product-free subset S of G for which $Q \subseteq S$, is an independent set of $H(G, Q)$.

For brevity, let us call a maximal product-free subset of G a good subset. Clearly there are at most $2^{o(n)}$ good subsets having size less than $\lceil\sqrt{n}\rceil$. So in order to prove Theorem 1.2, it is enough to count only good subsets having size at least $\lceil\sqrt{n}\rceil$. Each such good subset, given the discussion above, is an independent set in at least one of the graphs $H(G, Q)$, where Q ranges over all product-free subsets of G of size $\lceil\sqrt{n}\rceil$. Since there are at most $\binom{n}{\lceil\sqrt{n}\rceil} = 2^{o(n)}$ graphs $H(G, Q)$, with $Q \subseteq G$ being a product-free set of size $\lceil\sqrt{n}\rceil$, it would suffice for our purpose to fix a product-free subset $Q \subseteq G$ of size $\lceil\sqrt{n}\rceil$ and upper bound the number of independent sets S in $H(G, Q)$ that correspond to good subsets, that is to maximal product-free subsets of G. Therefore, proving the following lemma would imply Theorem 1.2.

Lemma 2.1. *Let G be a group of order n. Let $Q \subseteq G$ be product-free with $|Q| = \lceil\sqrt{n}\rceil$. Then the number of independent sets S in $H(G, Q)$ that are also maximal product-free subsets of G is at most* $2^{0.406n+o(n)}$.

2.1. Proof of Lemma 2.1

Throughout the proof, we assume that n is sufficiently large. The starting point of the proof is Lemma 1.4, the covering lemma for independent sets in regular graphs. For a product-free subset Q of G of size $\lceil\sqrt{n}\rceil$, we want to count the number of independent sets S in $H(G, Q)$ that are also maximal product-free subsets of G. Since $H(G, Q)$ is k-regular with $k \geq \sqrt{n}$, by Lemma 1.4 there exists a family $\mathcal{F}$ consisting of $2^{O(n^{3/4}\ln n)}$ subsets of G, each of size at most $n/2 + O(n^{3/4}\ln n)$, which cover all independent sets in $H(G, Q)$. In particular, this family covers all independent sets in $H(G, Q)$ which are also maximal product-free subsets of G. Hence, since the size of $\mathcal{F}$ is $2^{o(n)}$, in order to prove Lemma 2.1 it is enough to fix a set $A \in \mathcal{F}$ and show that the number $f_{\max}(G, A)$ of maximal product-free subsets of G that are contained in A is at most $2^{0.406n+o(n)}$. Moreover, since any set A in $\mathcal{F}$ has size at most $n/2 + O(n^{3/4}\ln n)$, it is enough to fix an arbitrary subset A of G for which $n/2 + \sqrt{n} \leq |A| \leq n/2 + O(n^{3/4}\ln n)$ holds, and show that for such a set A, $f_{\max}(G, A) \leq 2^{0.406n+o(n)}$. This is exactly what we do; so let us fix for the rest of this section a subset $A \subseteq G$ satisfying the above size constraints. We shall assume for simplicity that G is not the trivial group, so that any maximal product-free subset of G has size at least 1.

Say that an element $x \in G$ is *forced* by $\{y, z\} \subseteq G$, if $\{x, y, z\}$ is not product-free. We consider the following randomized process. The process is given the cardinality n of G, a bijection $\pi : [n] \to G$, the subset A of G and a real $\epsilon \in (0, 1)$, and returns a subset of A.

$\mathbf{P}(n, \pi, A, \epsilon)$:
Let $R \leftarrow \emptyset$.
For $i = 1$ to n do the following:
If $\pi(i) \in A$ and $\pi(i)$ is not forced by any $\{y, z\} \subseteq R$ then let $R \leftarrow R \cup \{\pi(i)\}$ with probability ϵ.
Return R.

For the rest of this section, we let S be a maximal product-free subset of G that is contained in A. Denote by $p(\epsilon, S)$ the probability that $S = \mathbf{P}(n, \pi, A, \epsilon)$, given a uniformly random bijection $\pi : [n] \to G$. For $x \in A \setminus S$, let F_x be the set of all $\{y, z\} \subseteq S$ that force x. Observe that from the assumption that S is a maximal product-free subset of G, for $x \in A \setminus S$ we have $|F_x| \geq 1$ whereas for $x \in S$, there does not exist any $\{y, z\} \subseteq S$ that forces x. For a uniformly random bijection $\pi : [n] \to G$ and for $x \in A \setminus S$, define

$$B_x = \begin{cases} 1 & \pi^{-1}(y) < \pi^{-1}(x) \text{ and } \pi^{-1}(z) < \pi^{-1}(x), \quad \text{for some } \{y, z\} \in F_x, \\ 0 & \text{Otherwise,} \end{cases}$$

and let $B = \sum_x B_x$, where the sum ranges over all $x \in A \setminus S$.

In the analysis of the above randomized process, we make use of the following simple fact.

Proposition 2.2. *Let X be a random variable taking its values from $\{0, 1, \ldots, t\}$. Furthermore, assume that $\mathbb{E}(X) \geq c \geq 1$. Then:*

$$\Pr(X \geq c - 1) \geq \frac{1}{t - c + 1}.$$

Proof. Write $p = \Pr(X \geq c-1)$. Then we can upper bound $\mathbb{E}(X)$ as follows.

$$\begin{aligned}\mathbb{E}(X) &\leq \Pr(X < c-1)(c-1) + \Pr(X \geq c-1)t \\ &= (1-p)(c-1) + pt.\end{aligned}$$

Taking $p < \frac{1}{t-c+1}$, we obtain $\mathbb{E}(X) < c$ which is a contradiction to the assumption in the proposition. Hence $p \geq \frac{1}{t-c+1}$. □

Lemma 2.3. $p(\epsilon, S) \geq n^{-1}(1-\epsilon)^{|A\setminus S|-\mathbb{E}(B)+1}\epsilon^{|S|}$.

Proof. Observe first that $B \leq |A \setminus S| \leq n$. We also have that $|S| \leq n/2$. (Indeed, if we fix $x \in S$ and take any two distinct $y, z \in S$, we have $x \cdot y \neq x \cdot z$ and $x \cdot y, x \cdot z \notin S$. This implies $|S| \leq n - |S|$ and so $|S| \leq n/2$.) Hence $|A \setminus S| \geq \sqrt{n}$; since for all $x \in A \setminus S$ we have $\Pr(B_x = 1) \geq 1/3$, this implies that $\mathbb{E}(B) \geq \sqrt{n}/3 \geq 1$ for n sufficiently large. From these bounds on B and $\mathbb{E}(B)$, using Proposition 2.2 we conclude that $\Pr(B \geq \mathbb{E}(B) - 1) \geq n^{-1}$. Let C_i be the event that on the ith iteration of $\mathbf{P}(n, \pi, A, \epsilon)$ we have that $\pi(i) \in S$ if and only if $\pi(i) \in R$, where R is the set defined during the process. Let $C = \bigcap_{i=1}^{n} C_i$. Clearly $p(\epsilon, S) \geq n^{-1}\Pr(C|B \geq \mathbb{E}(B) - 1)$. We next lower bound $\Pr(C|B \geq \mathbb{E}(B) - 1)$.

Since for every $x \in S$ there does not exist any $\{y, z\} \subseteq S$ that forces x, we have

$$\Pr\Big(C_i | \pi(i) \in S, \bigcap_{j<i} C_j\Big) = \epsilon. \tag{1}$$

Also, from the definition of the randomized process we have that

$$\Pr\Big(C_i | \pi(i) \in A \setminus S, B_{\pi(i)} = 0, \bigcap_{j<i} C_j\Big) = 1 - \epsilon, \tag{2}$$

while

$$\Pr\Big(C_i | \pi(i) \in A \setminus S, B_{\pi(i)} = 1, \bigcap_{j<i} C_j\Big) = 1. \tag{3}$$

Lastly, it is easily verified given the definition of the randomized process that the following holds:

$$\Pr\Big(C_i | \pi(i) \in G \setminus A, \bigcap_{j<i} C_j\Big) = 1. \tag{4}$$

From (1)–(4) we get that

$$\Pr(C|B \geq \mathbb{E}(B) - 1) \geq (1-\epsilon)^{|A\setminus S|-\mathbb{E}(B)+1}\epsilon^{|S|}.$$

Since $p(\epsilon, S) \geq n^{-1}\Pr(C|B \geq \mathbb{E}(B) - 1)$, the assertion in the lemma is now proved. □

Recall that we need to give an upper bound on $f_{\max}(G, A)$, the number of maximal product-free subsets of G that are contained in A. Let $f_{\max}(G, A, \gamma)$ denote the number of maximal product-free subsets of G of size γn that are contained in A. Clearly, $f_{\max}(G, A, 1/2) \leq 2^{o(n)}$. Also, since G is not the trivial group, any maximal product-free subset of G has at least one element. From these two facts, since $|S| \leq n/2$ and since there are at most n possible choices for γ such that γn is an integer, we have

$$f_{\max}(G, A) \leq 2^{o(n)} + n \cdot \max_{\gamma} f_{\max}(G, A, \gamma), \tag{5}$$

where $\max_\gamma$ denotes the maximum over all $\gamma \in [1/n, 1/2 - 1/(2n)]$. Thus, it remains to upper bound $\max_\gamma f_{\max}(G, A, \gamma)$. We do that next using Lemma 2.3. Let us write $|S| = \gamma n$ and assume $\gamma \in [1/n, 1/2 - 1/(2n)]$. Define

$$f(\epsilon) = (1-\epsilon)^{(1/2-\gamma)2n/3}\epsilon^{\gamma n}.$$

For every $x \in A \setminus S$, it holds that $\Pr(B_x = 1) \geq 1/3$ and so $\mathbb{E}(B) \geq |A \setminus S|/3$. Therefore, since $|A| \leq n/2 + O(n^{3/4} \ln n)$, we get that $|A \setminus S| - \mathbb{E}(B) + 1 \leq 2|A \setminus S|/3 + 1 = (1/2 - \gamma)2n/3 + O(n^{3/4} \ln n)$. Hence, from Lemma 2.3 it follows that for every $\epsilon \in (0, 1)$, $p(\epsilon, S) \geq n^{-1}(1-\epsilon)^{O(n^{3/4} \ln n)} f(\epsilon)$. It then follows, by Fact 1.3, that for every $\epsilon \in (0, 1)$,

$$f_{\max}(G, A, \gamma) \leq n(1-\epsilon)^{-O(n^{3/4} \ln n)}/f(\epsilon). \tag{6}$$

Define

$$\epsilon_\gamma = \frac{\gamma}{(1/2 - \gamma)2/3 + \gamma},$$

and observe that for $\gamma \in [1/n, 1/2 - 1/(2n)]$, $\epsilon_\gamma \in (0, 1 - O(1/n))$. Hence, for every $\gamma \in [1/n, 1/2 - 1/(2n)]$, $(1 - \epsilon_\gamma)^{-O(n^{3/4} \ln n)} = 2^{o(n)}$. This implies, using (6), that

$$\max_\gamma f_{\max}(G, A, \gamma) \leq 2^{o(n)} \cdot \max_\gamma 1/f(\epsilon_\gamma). \tag{7}$$

Define $g(\gamma) = f(\epsilon_\gamma)$. Then by (5) and (7), in order to give an upper bound on $f_{\max}(G, A)$ up to a subexponential factor, it remains to upper bound $\max_\gamma 1/g(\gamma)$. To do this, define $h(\gamma) = \ln g(\gamma)$ and take the derivative of $h(\gamma)$, which is

$$h'(\gamma) = -\frac{2}{3} \cdot \ln \frac{(1/2 - \gamma)2/3}{(1/2 - \gamma)2/3 + \gamma} - \ln \frac{(1/2 - \gamma)2/3 + \gamma}{\gamma}.$$

It is easy to see that $1/g(\gamma)$ is concave down on $[1/n, 1/2 - 1/(2n)]$ and that its maximum is achieved for γ satisfying $1/n < \gamma < 1/2 - 1/(2n)$. Hence, if γ^* is the solution to $h'(\gamma) = 0$, then $f_{\max}(G, A)$ is up to a subexponential factor at most $1/g(\gamma^*)$. Solving $h'(\gamma) = 0$ reduces to solving the cubic equation

$$23\gamma^3 + 3\gamma - 1 = 0. \tag{8}$$

The only real solution to (8) is $\gamma^* = w - 1/(23w) = 0.234\ldots$, where w satisfies $w^3 = 1/46 + \sqrt{1/46^2 + 1/23^3}$. Using this, we can deduce $1/g(\gamma^*) \leq 2^{0.406n}$. We thus conclude from the discussion above that $f_{\max}(G, A) \leq 2^{0.406n + o(n)}$. This gives us the validity of Lemma 2.1 and in turn, the validity of Theorem 1.2.

Using essentially the same arguments presented in this section, we could have proved that $f_{\max}(n) \leq 2^{0.406n + o(n)}$. However, as we show in the next section, we can do much better than this.

3. The number of maximal sum-free subsets of $[n]$

In this section we prove Theorem 1.1. Let n be a sufficiently large integer and let $l = \lceil n/2 \rceil$. For a sum-free subset $R \subseteq [l]$ and for $W \subseteq [n] \setminus [l]$, let $\mathcal{E}(R, W)$ be the family of all extensions $S \subseteq W$ of R to maximal sum-free subsets of $[n]$; In other words, $\mathcal{E}(R, W)$ is the family of all subsets S of W such that $R \cup S$ is a maximal sum-free subset of $[n]$.

Lemma 3.1. *For any sum-free subset $R \subseteq [l]$, and for any $W \subseteq [n] \setminus [l]$, we have $|\mathcal{E}(R, W)| \leq 2^{|W|/2}$.*

Proof. Let R be an arbitrary sum-free subset of $[l]$ and let W be an arbitrary subset of $[n] \setminus [l]$. As usual, for $X \subseteq \mathbb{N}$, let $2X = \{a + b : a, b \in X\}$. If S is a subset of W such that $R \cup S$ is sum-free then for every $x \in 2R$, we have $x \notin S$. Hence, $\mathcal{E}(R, W) = \mathcal{E}(R, W \setminus 2R)$. Defining $V = W \setminus 2R$, it thus suffices to give an upper bound on the cardinality of $\mathcal{E}(R, V)$. We do that next.

Let $H = (V, E)$ be the graph over the vertex set V and with the edge set E, where $(y, z) \in E$ if and only if for some $x \in R$, $|y - z| = x$.

Claim 3.2. *Let $I \subseteq V$. Then I is an independent set in H if and only if $R \cup I$ is a sum-free subset of $[n]$.*

Proof. First assume that for $I \subseteq V$, $R \cup I$ is not sum-free. Then there exists a non-sum-free set $\{x, y, z\} \subseteq R \cup I$ with $x \le y \le z$. Since (i) $V = W \setminus 2R$, (ii) R is sum-free, and (iii) $I \subseteq V \subseteq [n] \setminus [l]$, it follows that $x \in R$, $y, z \in I$ and $z - y = x$. This implies, by the definition of H, that I is not an independent set in H. Next assume that for $I \subseteq V$, I is not an independent set in H. Then by definition, there are $y, z \in I$ such that $z - y = x \in R$. Hence $R \cup I$ is not sum-free. □

Suppose that $S \subseteq V$ and that $R \cup S$ is a maximal sum-free subset of $[n]$. By Claim 3.2, S is an independent set in H. More than that, we claim that S is a maximal independent set in H. Indeed, assume that S is not a maximal independent set and let S' be an independent set in H which strictly contains S. Then by Claim 3.2, $R \cup S'$ is a sum-free subset of $[n]$, contradicting the fact that $R \cup S$ is a maximal sum-free subset of $[n]$. Thus, if $S \subseteq V$ and $R \cup S$ is a maximal sum-free subset of $[n]$ then S is a maximal independent set in H. It then follows that $|\mathcal{E}(R, V)|$ is at most the number of maximal independent sets in H.

Now, note that H is triangle-free. For otherwise, there are $x, y, z \in V$ such that $z - y = a \in R$, $y - x = b \in R$ and $z - x = c \in R$. But then $a + b = z - x = c$, which contradicts the fact that R is sum-free. By Theorem 1.6, the number of maximal independent sets in a triangle-free graph of order $|V|$ is at most $2^{|V|/2}$, and so

$$|\mathcal{E}(R, W)| = |\mathcal{E}(R, V)| \le 2^{|V|/2} \le 2^{|W|/2},$$

as required. □

To prove Theorem 1.1, we make use of Lemma 1.5, Green's covering lemma for sum-free subsets of $[n]$. Fix an arbitrary subset A of $[n]$ such that $|A| = n/2 + o(n)$. With Lemma 1.5 in hand, $f_{\max}(n)$ is at most, up to a multiplicative factor of $2^{o(n)}$, the number of subsets of A that are also maximal sum-free subsets of $[n]$. Now, from Lemma 3.1 and since the number of sum-free subsets of $[l]$ is at most $2^{l/2+O(1)}$, it follows that the number of subsets of A that are maximal sum-free subsets of $[n]$ cannot be larger than

$$\min\left\{2^{|A\cap[l]|}, 2^{l/2+O(1)}\right\} \cdot 2^{(|A|-|A\cap[l]|)/2}.$$

This last quantity is, since $|A| = n/2 + o(n)$, at most $2^{3n/8+o(n)}$. With that we complete the proof of Theorem 1.1.

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