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A CHARACTERIZATION OF TOTALLY BALANCED HYPERGRAPHS

Jenö LEHEL

Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, Hungary

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A hypergraph is totally balanced if every non-trivial cycle has an edge containing at least three vertices of the cycle. Totally balanced hypergraphs are characterized here as special tree-hypergraphs. This approach provides a conceptually simpler proof of Anstee's related result and yields the structural description of totally balanced hypergraphs.

1. Introduction

Hypergraph structures corresponding to the notion of trees in graph theory have been examined by several authors. In particular, hypergraphs without cycles and hypergraphs satisfying certain inequalities between edge and vertex number were investigated. The first result of this nature incorporated into hypergraph theory by Berge is due to Lovász ([16], [4, p. 393]). Further research by Lovász, by Las Vergnas and others results in the notion of totally balanced hypergraphs ([17, p. 528]) and the notion of cyclomatic number for hypergraphs ([1, 12, 15]).

We follow here a different line by concentrating on the 'geometrical picture' of trees. Several authors found and examined in this way those structures which are called now tree-hypergraphs (e.g., [7, 10]). In this approach totally balanced hypergraphs appear as a subfamily of tree-hypergraphs closed under subhypergraph choice, furthermore, hypergraphs with cyclomatic number zero are the duals of tree-hypergraphs (see [1]).

Our object, totally balanced hypergraphs, occurs in various applications: in extremal combinatorial problems (Anstee [2], Füredi and Tuza [11]), in integer linear programming (Hoffman, Kolen and Sakarovitch [13], Lubiw [18]), in abstract convexity (Farber and Jamison [9], Lehel [14]) and of course in graph theory (Brouwer, Duchet and Schrijver [6], Farber [8]).

A hypergraph is called *totally balanced* if every cycle of length greater than two has an edge containing at least three vertices of the cycle.

In Section 2, we show that totally balanced hypergraphs are *tree-hypergraphs*, that is, one can define their hyperedges as subtrees of some basic tree (Theorem 2.1). Our main ideas concerning the tree representation of totally balanced hypergraphs are formulated in Lemmas 2.2 and 2.3. These observations are used in Section 3 to provide essentially simpler proof of an important property of totally balanced hypergraphs obtained by Anstee. In fact, our Theorem 3.4 contains the complete description of the structure of totally balanced hypergraphs. The same result is implicitly in [2], where it is actually proved that any totally balanced hypergraph on n vertices can be completed by new edges to a totally balanced hypergraph with maximal number of edges $m = \binom{n+1}{2}$.

Further properties and characterizations concerning totally balanced hypergraphs are mentioned also in Brouwer and Kolen [5], Farber [8], Lehel [14] and Anstee and Farber [3].

2. Tree representation

A *simple hypergraph* is a set of different non-empty subsets called edges from some finite underlying set. The edge set of H is denoted by $E(H)$ and the underlying set, called the vertex set of H, is denoted by $V(H)$.

The *subhypergraph* induced by a set $A \subseteq V(H)$ is a hypergraph H_A defined by vertex set A and edge set $E(H_A) = \{e \cap A : e \in E(H)\}$. A hypergraph H' with $E(H') \subseteq E(H)$ is called a *partial hypergraph* of H.

The cycle $(v_1, e_1, \ldots, v_k, e_k)$ in a hypergraph H is called a *special cycle* of H if $k \geq 3$ and $v_i \in e_j$ holds iff $j = i$, $j = i-1$ or $(i, j) = (1, k)$.

A hypergraph is called *balanced* if it has no special cycle of odd length, and a hypergraph is *totally balanced* if it has no special cycle at all.

Hypergraph notions not defined here are used in accordance with Berge's terminology ([4]).

First we make a list of elementary properties of totally balanced hypergraphs deduced from the definition. If H is a totally balanced hypergraph, then:

(i) The dual of H, any partial-subhypergraph of H are totally balanced;

(ii) H has the *Helly-property*, i.e., pairwise intersecting edges of H have a common vertex, since the Helly-property is satisfied by any hypergraph without special cycle of length three ([19]);

(iii) Its line graph, denoted by $L(H)$, is a *chordal graph*, i.e., $L(H)$ contains no induced circuit of length greater than 3.

A hypergraph H is called *tree-hypergraph* if one can give a tree T with vertex set $V(H)$ such that every hyperedge $e \in E(H)$ induces a subtree of T. We say that T is a *basic tree* of H. Several authors characterize tree-hypergraphs as follows:

Theorem (Flament [10], Duchet [7]). *H is a tree-hypergraph if and only if H has the Helly-property and $L(H)$ is a chordal graph.*

By this theorem and using properties (i)–(iii), we obtain

Theorem 2.1. *A hypergraph H is totally balanced if and only if every subhypergraph of H is a tree-hypergraph.*

Our aim is to characterize totally balanced hypergraphs in the family of tree-hypergraphs. For this purpose we associate to a tree-hypergraph H having basic tree T a new hypergraph H/T which is spanned by H on the edges of T. More formally, $V(H/T)=E(T)$ and $E(H/T)=\{e\subset V(H/T): \bigcup_{u\in e} u\in E(H)\}$.

Lemma 2.2. *Let H be a tree-hypergraph with basic tree T. Then H is totally balanced iff H/T is totally balanced.*

Proof. Let $H'=H/T$ and denote by e' the edge of H' corresponding to $e\in E(H)$. Since T is a basic tree of H, e and e' is the vertex set and the edge set of a subtree of T, respectively.

Sufficiency. Let $C=(x_1, e_1, \ldots, x_k, e_k)$ be a cycle of H and denote by $T(X)$ the minimal subtree of T containing the set $X=\{x_1, \ldots, x_k\}$. If F is the set of vertices having degree one in $T(X)$ then obviously, $F\subseteq X$. Moreover, if C is a special cycle of H then $F=X$ and the pendant edges of $T(X)$ denoted by $u_1, \ldots, u_k$ together with the hyperedges $e'_1, \ldots, e'_k$ define a cycle $C'=(u_1, e'_1, \ldots, u_k, e'_k)$ of H'. Clearly, C' is a special cycle of H/T.

Necessity. Suppose that $C'=(u_1, e'_1, \ldots, u_k, e'_k)$ is a cycle of H' and denote by $T'(U)$ the minimal subtree of T containing the set $U=\{u_1, \ldots, u_k\}$. If F' is the set of pendant edges of $T'(U)$, then, obviously, $F'\subseteq U$. Moreover, if C' is a special cycle of H/T, then $F'=U$ and the endvertices $x_1, \ldots, x_k$ of $T'(U)$ together with the hyperedges $e_1, \ldots, e_k$ define a cycle $C=(x_1, e_1, \ldots, x_k, e_k)$ of H. Clearly, C is a special cycle of H. □

Lemma 2.3. *If H is a totally balanced hypergraph with basic tree T, then H/T has a basic tree T' such that the vertex set of any subtree of T' is the edge set of some subtree of T.*

Proof. For every $v\in V(H)$, define the set $N(v)=\{v\}\cup\{x\in V(H): \{v, x\}\in E(T)\}$ and denote by $H(v)$ and $T(v)$ the subhypergraph of H and the substar of T induced by $N(v)$, respectively. Every hypergraph $H(v)$ is totally balanced with basic tree $T(v)$ and thus by Theorem 2.1, via Lemma 2.2, $H(v)/T(v)$ is a tree-hypergraph with some basic tree $T'(v)$.

Obviously, the union T' of the trees $T'(v)$, $v\in V(H)$, is a basic tree of H/T. Moreover, if $\{u, v\}\in E(T')$, then u and v are adjacent edges of T. □

Among graphs (i.e., 2-uniform hypergraphs) only forests are totally balanced hypergraphs. By this observation and using Lemma 2.2, one can construct totally balanced 3-uniform hypergraphs as follows. Let T be a given tree and F' be a subforest of the line graph of T (see Fig. 1). Then the hypergraph H defined by $V(H)=V(T)$ and $E(H)=\{u\cup v: u, v\in E(T), \{u, v\}\in E(F')\}$ is totally balanced by Lemma 2.2; and since $\{u, v\}\in E(F')$ implies that u and v are adjacent edges of T, H is 3-uniform.

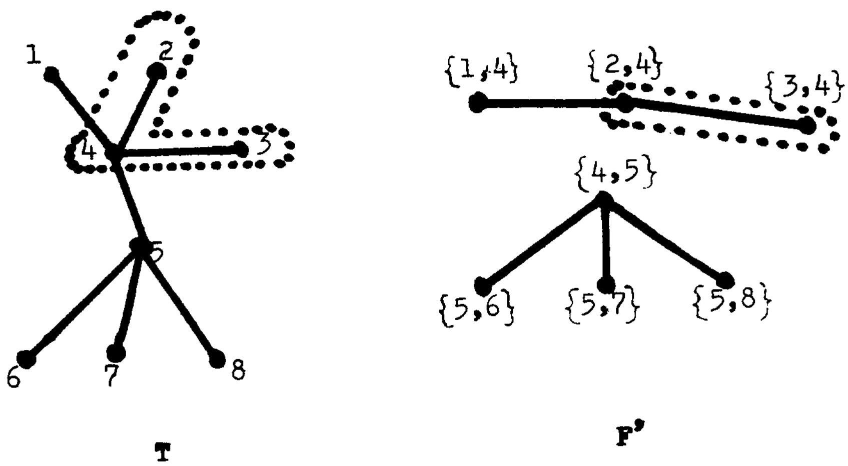

Fig. 1

On the other hand, every 3-uniform totally balanced hypergraph can be obtained by the above mentioned procedure, as is implicitly stated in Lemma 2.3.

In the next section we give a general construction based on the same ideas.

3. Construction

Totally balanced hypergraphs have a remarkable 'overlay' structure as it is discovered by Anstee in proving that a totally balanced simple hypergraph on n vertices has at most $\binom{n+1}{2}$ edges (see [2]). We provide a simpler proof of his result by using the tree representation described in Section 2. In fact, we prove that every totally balanced hypergraph can be obtained by the following 'tree sequence' construction starting with a given basic tree T_0 with vertex set is identified with the set of singletons of $N=\{1,\dots,n\}$.

Construction 3.1. Create a sequence of trees $t=(T_0, T_1, \dots, T_{n-1})$ in such a way that for every $1 \leq i \leq n-1$:

(a) The vertices of T_i are $(i+1)$-element subsets of N;

(b) $V(T_i)=\{x \cup y\colon x, y \in V(T_{i-1}),\ \{x, y\} \in E(T_{i-1})\}$;

(c) $\{u, v\} \in E(T_i)$ implies that $u=x \cup y$ and $v=y \cup z$ hold for some $x, y, z \in V(T_{i-1})$.

Now define hypergraph H^t by $V(H^t)=N$ and

$$E(H^t)=\{x \subset N\colon x \in V(T_j) \quad \text{for some } 0 \leq j \leq n-1\}.$$

Proposition 3.2. *The hypergraph H^t given by Construction 3.1 is totally balanced.*

Proof. Let us define the hypergraph $H_k^t (0 \leq k \leq n-1)$ by $V(H_k^t)=V(T_k)$ and

$$E(H_k^t)=\left\{x \subset V(T_k)\colon \bigcup_{v \in x} v \in V(T_i) \quad \text{for some} \quad k \leq i \leq n-1\right\}.$$

Clearly, H^t_{n-1} is a trivial totally balanced hypergraph having only one vertex, furthermore, $H^t_0 = H^t$. We show by reverse induction on k that every hypergraph H^t_k is totally balanced. By (b) and (c), if $x_1 \cup y_1, \ldots, x_p \cup y_p$ are the vertices of a subtree of T_i then the corresponding edges $\{x_1, y_1\}, \ldots, \{x_p, y_p\}$ induce a subtree of T_{i-1}, $1 \leq i \leq n-1$. Consequently, every H^t_k is a tree-hypergraph with basic tree $T_k (0 \leq k \leq n-1)$. Observing that $H^t_{k-1}/T_{k-1} \cong H^t_k$ and assuming that H^t_k is totally balanced, the proposition follows by Lemma 2.2. □

Figure 2 shows a sequence of trees whose vertices define the edges of a totally balanced hypergraph on vertex set $N = \{1, 2, 3, 4, 5, 6\}$.

Proposition 3.3. *Every totally balanced hypergraph is isomorphic to a partial hypergraph of some hypergraph* H^t *given by Construction* 3.1.

Proof. Let H_0 be a totally balanced hypergraph on vertex set $N = \{1, \ldots, n\}$ and let T_0 be a basic tree of H_0. We have to create a sequence of trees T_0, $T_1, \ldots, T_{n-1}$ satisfying the conditions in Construction 3.1.

For every $1 \leq i \leq n-1$ define the hypergraph H_i by

$$V(H_i) = \{x \cup y : x, y \in V(T_{i-1}), \{x, y\} \in E(T_{i-1})\}$$

and

$$E(H_i) = \left\{ e \subset V(H_i) : \bigcup_{v \in e} v \in E(H_{i-1}) \right\}.$$

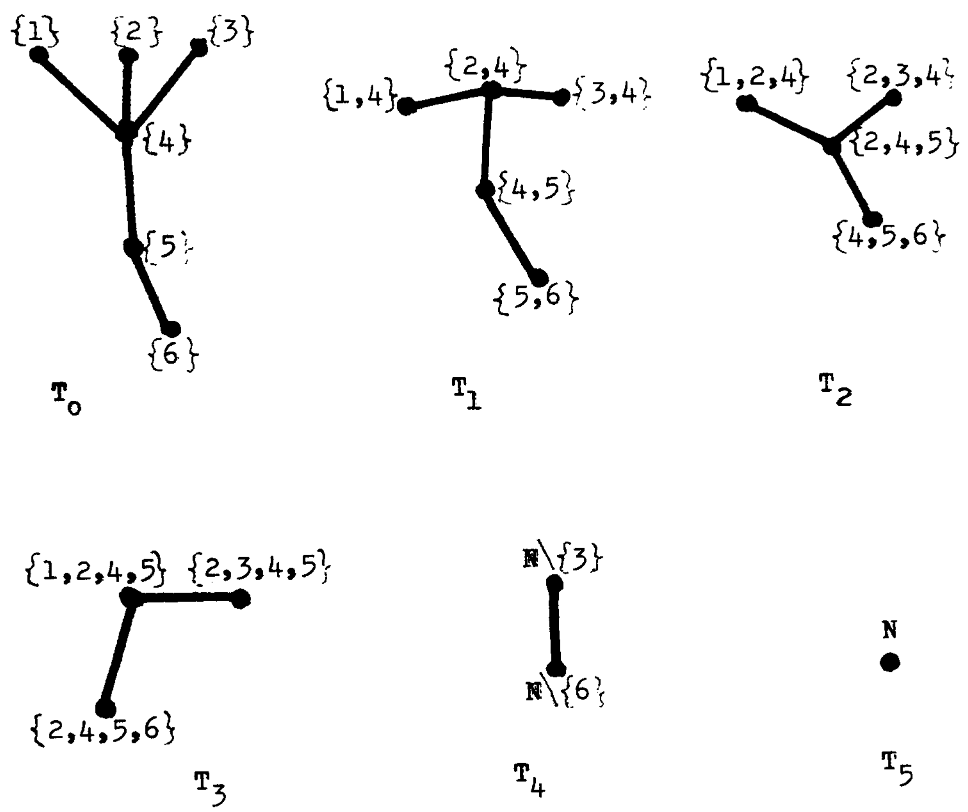

Fig. 2

Supposing that H_{i-1} is a totally balanced hypergraph with basic tree T_{i-1}, Lemma 2.2 implies that $H_{i-1}/T_{i-1} \cong H_i$ is totally balanced as well. Let T_i be a basic tree of H_i corresponding to the basic tree of H_{i-1}/T_{i-1} given by Lemma 2.3. The tree sequence $t=(T_0, \ldots, T_{n-1})$ obtained in this way satisfies (a), (b) and (c). Moreover, k-element edges of H_0 are vertices of T_{k-1}, $1 \leqslant k \leqslant n$, and hence $H_0 \subseteq H^t$. □

Now we summarize Propositions 3.2 and 3.3 by giving explicitely the structure of totally balanced hypergraphs.

Theorem 3.4. *The hypergraph H is totally balanced iff there is a sequence of trees* $T_0, \ldots, T_{n-1}$ *such that the vertices of* T_i, $1 \leqslant i \leqslant n-1$, *are* $(i+1)$*-element subsets of* $V(H)$ *and*

(i) $V(T_0) = V(H)$,
$V(T_i) = \{x \cup y : x, y \in V(T_{i-1}), \{x, y\} \in E(T_{i-1})\}$, $i = 1, \ldots, n-1$;

(ii) $\{u, v\} \in E(T_i)$ *implies that* $u = x \cup y$, $v = y \cup z$ *hold for some* x, y, $z \in V(T_{i-1})$, $i = 1, \ldots, n-1$;

(iii) *Every* $e \in E(H)$ *is a vertex of some* T_i $(0 \leqslant i \leqslant n-1)$. □

This structure theorem implies immediately the following result due to Anstee [2]: every totally balanced hypergraph on n vertices can be completed by new edges to a totally balanced simple hypergraph with maximal number of edges $m = \binom{n+1}{2}$ in which the number of k-element edges $(1 \leqslant k \leqslant n)$ is equal to $n-k+1$.

4. Concluding remarks

Totally balanced hypergraphs are characterized in Theorem 3.4 as tree-hypergraphs given by the tree sequence construction. In [5], Brouwer and Kolen extended a result of Lovász ([17 p. 77]) and characterized totally balanced hypergraphs in terms of chain point (nest vertex) as follows. A hypergraph H is totally balanced if and only if every subhypergraph H has a chain point, i.e., a vertex v such that the edges of the subhypergraph containing v are linearly ordered by inclusion. This theorem follows easily from our Theorem 3.4 and a similar proof (via Anstee's result) can be found also in [3].

We note finally that totally balanced hypergraphs or matrices, like interval hypergraphs, have advantageous algorithmic properties with regard to integer linear programming problems (see [13], [18]). Interval hypergraphs are obviously contained in the family of totally balanced hypergraphs. Their relationship is analyzed further in [14], where totally balanced hypergraphs are characterized as abstract interval structures extending Euclidean interval structures.

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