The nearness problems for symmetric matrix with a submatrix constraint\[∗\]

Yongxin Yuan\[^{a,b,*}\], Hua Dai\[^{b}\]

\[^{a}\]Department of Mathematics, Jiangsu University of Science and Technology, Zhenjiang 212003, PR China
\[^{b}\]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China

Received 10 October 2006

Abstract

In this paper, we first give the representation of the general solution of the following least-squares problem (LSP): given a matrix $X \in \mathbb{R}^{n \times p}$ and symmetric matrices $B \in \mathbb{R}^{p \times p}$, $A_0 \in \mathbb{R}^{r \times r}$, find an $n \times n$ symmetric matrix $A$ such that $\|X^TA - B\| = \min$, s.t. $A([1, r]) = A_0$, where $A([1, r])$ is the $r \times r$ leading principal submatrix of the matrix $A$. We then consider a best approximation problem: given an $n \times n$ symmetric matrix $\tilde{A}$ with $\tilde{A}([1, r]) = A_0$, find $\hat{A} \in S_E$ such that $\|\tilde{A} - \hat{A}\| = \min_{A \in S_E} \|\tilde{A} - A\|$, where $S_E$ is the solution set of LSP. We show that the best approximation solution $\hat{A}$ is unique and derive an explicit formula for it.

© 2007 Elsevier B.V. All rights reserved.

MSC: 65F18; 15A24; 15A57

Keywords: Symmetric matrix; Singular value decomposition; Best approximation; Model updating

1. Introduction

Throughout this paper, we denote the real $m \times n$ matrix space by $\mathbb{R}^{m \times n}$, the set of all orthogonal matrices in $\mathbb{R}^{n \times n}$ by $\text{OR}^{n \times n}$, the set of all symmetric matrices in $\mathbb{R}^{n \times n}$ by $\text{SR}^{n \times n}$, the transpose and the Moore–Penrose generalized inverse of a real matrix $A$ by $A^T$ and $A^+$, respectively. $I_n$ represents the identity matrix of size $n$. For $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, $A \ast B$ represents the Hadamard product of the matrices $A$ and $B$, i.e., $A \ast B = (a_{ij}b_{ij}) \in \mathbb{R}^{m \times n}$. For $A, B \in \mathbb{R}^{m \times n}$, an inner product in $\mathbb{R}^{m \times n}$ is defined by $(A, B) = \text{trace}(B^TA)$, then $\mathbb{R}^{m \times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm.

The problem of finding symmetric solutions $A$ of the matrix equation $X^TA = B$ has been considered by several authors (see [6,7,14]) due to its some applications, especially in the design and analysis of the vibrating structures. Using the finite element technique, the dynamic analysis of a mechanical or civil structure is modelled by the generalized eigenvalue problem [8]

$$K_0x = \lambda M_0x,$$

\(^{\ast}\) Research supported by the National Natural Science Foundation of China (No.10271055).

\(^{*}\) Corresponding author. Department of Mathematics, Jiangsu University of Science and Technology, Zhenjiang 212003, PR China. Tel.: +86 0511 4401748.

E-mail addresses: yuanyx_703@163.com (Y. Yuan), hdai@nuaa.edu.cn (H. Dai).

0377-0427/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2007.01.033
where $K_a, M_a \in \mathbb{R}^{n \times n}$ are the analytical stiffness and mass matrices, respectively. High accuracy and large size structural applications require highly correlated finite element models to predict the system’s dynamic behavior. Very often natural frequencies and mode shapes (eigenvalues and eigenvectors) of a finite element model described by (1) do not match very well with experimentally measured frequencies and mode shapes obtained from a real-life vibrating structure. Thus, a vibration engineer needs to update the theoretical finite element model of the structure such that the updated model predicts the observed dynamic behavior. The improved model may be considered to be a better dynamic representation of the structure. This model can be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes.

Let $X \in \mathbb{R}^{n \times p}$ be the measured modal matrix, $A \in \mathbb{R}^{p \times p}$ the measured natural frequencies matrix, where $n \geq p$, and $A$ is diagonal. The most common approach in finite element model updating is first to modify the analytical mass or stiffness matrix to satisfy the following orthogonality conditions:

$$X^T M X = I_p, \quad X^T K X = A,$$

where $M, K \in \mathbb{R}^{n \times n}$ are symmetric matrices and represent the corrected mass and stiffness matrices, respectively. To date, some methods have been proposed to correct mass and stiffness matrices from measured response data [2,4,12,13,17]. However, the system mass and stiffness matrices are adjusted globally. From a practical viewpoint, a spatial representation of the structural-element property changes that resulted from the model errors is generally preferred for engineering applications. Model errors can be localized by using sensitivity analysis [10,15], residual noise, misinterpretation of data, etc. Thus, the problem of updating the mass or stiffness matrix can be mathematically formulated as follows.

**Problem I.** Given a matrix $X \in \mathbb{R}^{n \times p}$ and symmetric matrices $B \in \mathbb{R}^{p \times p}$, $A_0 \in \mathbb{R}^{r \times r}$, find an $n \times n$ symmetric matrix $A$ such that

$$\|X^T A X - B\| = \min_{A([-1, r])] = A_0, \quad \text{s.t.} \ A([-1, r])] = A_0,$$

where $A([-1, r])]$ is the $r \times r$ leading principal submatrix of the matrix $A$.

**Problem II.** Given an $n \times n$ symmetric matrix $\tilde{A}$ with $\tilde{A}([-1, r])] = A_0$, find $\tilde{A} \in S_E$ such that

$$\|\tilde{A} - \tilde{A}\| = \min_{\tilde{A} \in S_E} \|\tilde{A} - A\|, \quad (2)$$

where $S_E$ is the solution set of Problem I.

The paper is organized as follows. In Section 2, we give an expression of the general solution of Problem I using the generalized inverses and the singular value decompositions (SVDs) of matrices. As a by-product of our results on Problem I, we obtain a necessary and sufficient condition on $X, B, A_0$ for existence of $A \in S_R^{n \times n}$ such that

$$X^T A X = B, \quad A([-1, r])] = A_0, \quad \text{and a general form for all such } A.$$  

In Section 3, we show that there exists a unique solution to Problem II and present the expression of the solution $\tilde{A}$ of Problem II. Finally, in Section 4, a numerical algorithm to acquire the best approximation solution under the Frobenius norm sense is described and a numerical example is provided. Clearly, the results obtained are shown to include those given in [16] as particular cases.

2. The solution of Problem I

To begin with, we introduce a lemma (see [3]).

**Lemma 1.** If $Z \in \mathbb{R}^{r \times r}$, $Y \in \mathbb{R}^{r \times s}, E \in \mathbb{R}^{r \times s}$ then $Z Y E = E$ has a solution $F \in \mathbb{R}^{r \times q}$ if and only if $Z Z^+ E Y^+ Y = E$. In this case, the general solution of the equation can be described as $F = Z Z^+ E Y^+ + J (I_q - YY^+) + (I_r - Z^+ Z)L$, where $J, L \in \mathbb{R}^{r \times q}$ are arbitrary matrices.
Let the partition of the matrix $X$ be

$$
X = \begin{bmatrix} X_1 \\
X_2 
\end{bmatrix}, \quad X_1 \in \mathbb{R}^{r \times p}, \quad X_2 \in \mathbb{R}^{(n-r) \times p}.
$$

(3)

Write

$$
A = \begin{bmatrix} A_0 & F \\
F^T & H \end{bmatrix} \in \mathbb{R}^{r \times (n-r)},
$$

where $F \in \mathbb{R}^{r \times (n-r)}$ and $H \in \mathbb{R}^{(n-r) \times (n-r)}$ are yet to be determined. From (3) and (4) we have

$$
\|X^TAX - B\| = \|X_2^THX_2 + X_2^TF^TX_1 + X_1^TFX_2 - (B - X_1^T A_0X_1)\|.
$$

(5)

Let the SVD of the matrix $X_2$ be

$$
X_2 = P \begin{bmatrix} \Omega & 0 \\
0 & 0 \end{bmatrix} Q^T.
$$

(6)

where $P = [P_1, P_2] \in \mathbb{O}^{(n-r)\times(n-r)}$, $V = [Q_1, Q_2] \in \mathbb{O}^{p \times p}$, $\Omega = \text{diag}(\omega_1, \ldots, \omega_s)$, $\omega_i > 0$ ($i = 1, \ldots, s$), $s = \text{rank}(X_2)$, $P_1 \in \mathbb{R}^{(n-r)\times s}$, $Q_1 \in \mathbb{R}^{p\times s}$, and let

$$
P^T H P = \begin{bmatrix} H_{11} & H_{12} \\
H_{12}^T & H_{22} \end{bmatrix} \in \mathbb{R}^{s \times (n-r-s)}.
$$

Then the equation of (5) is equivalent to

$$
\|X^T AX - B\|^2 = \|\Omega H_{11} \Omega + \Omega P_1^T F^T X_1 Q_1 + Q_1^T X_1^T F P_1 \Omega - Q_1^T (B - X_1^T A_0 X_1) Q_1\|^2
$$

$$
+ \|\Omega P_1^T F^T X_1 Q_2 - Q_1^T (B - X_1^T A_0 X_1) Q_2\|^2
$$

$$
+ \|Q_2^T X_1^T F P_1 \Omega - Q_2^T (B - X_1^T A_0 X_1) Q_1\|^2
$$

$$
+ \|Q_2^T (B - X_1^T A_0 X_1) Q_2\|^2.
$$

(7)

It follows from (7) that $\|X^T AX - B\| = \min$ if and only if

$$
H_{11} = \Omega^{-1} [Q_1^T (B - X_1^T A_0 X_1) Q_1 - \Omega P_1^T F^T X_1 Q_1 - Q_1^T X_1^T F P_1 \Omega] \Omega^{-1}
$$

and

$$
\|Q_2^T X_1^T F P_1 \Omega - Q_2^T (B - X_1^T A_0 X_1) Q_1\| = \min.
$$

(8)

Assume that the SVD of the matrix $X_1 Q_2$ is

$$
X_1 Q_2 = U \begin{bmatrix} \Sigma & 0 \\
0 & 0 \end{bmatrix} V^T.
$$

(10)

where $U = [U_1, U_2] \in \mathbb{O}^{r \times r}$, $V = [V_1, V_2] \in \mathbb{O}^{(p-s) \times (p-s)}$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_t)$, $\sigma_i > 0$ ($i = 1, \ldots, t$), $t = \text{rank}(X_1 Q_2)$, $U_1 \in \mathbb{R}^{r \times t}$, $V_1 \in \mathbb{R}^{(p-s) \times t}$. Solving the minimization problem (9) we obtain

$$
F = F_0 + U_2 L + J P_2^T,
$$

(11)

where

$$
F_0 = Q_2^T X_1^T + Q_2^T (B - X_1^T A_0 X_1) Q_1 \Omega^{-1} P_1^T,
$$

and $J \in \mathbb{R}^{r \times (n-r-s)}$, $L \in \mathbb{R}^{(r-t) \times (n-r)}$ are arbitrary matrices.
Substituting (11) into (8) yields
\[
H_{11} = H_{110} - P_1^T L^T U_2^T X_1 Q_1 \Omega^{-1} - \Omega^{-1} Q_1^T X_1^T U_2 L P_1,
\]
where
\[
H_{110} = \Omega^{-1} Q_1^T (B - X_1^T A_0 X_1) Q_1 \Omega^{-1} - P_1^T F_0^T X_1 Q_1 \Omega^{-1} - \Omega^{-1} Q_1^T X_1^T F_0 P_1.
\]

By now, we have proved the following result.

**Theorem 1.** Suppose that \( X \in \mathbb{R}^{n \times p} \), \( B \in \mathbb{S} \mathbb{R}^{p \times p} \) and \( A_0 \in \mathbb{S} \mathbb{R}^{R \times R} \). Let the partition of the matrix \( X \) be (3), and the SVDs of the matrices \( X_2 \) and \( X_1 Q_2 \) be given by (6) and (10), respectively. Then the solution set \( S_E \) of Problem 1 can be expressed as
\[
S_E = \left\{ A \in \mathbb{R}^{n \times n} \mid A = \begin{bmatrix} A_0 \\ F^T \\ P \begin{bmatrix} H_{11} \\ H_{12} \end{bmatrix} P^T \end{bmatrix} \right\},
\]
where \( F, H_{11} \) are given by (11) and (13), respectively, and \( L, J, H_{12}, H_{22} \) with \( H_{22} = H_{22}^T \) are arbitrary matrices.

From (7), Lemma 1 and Theorem 1, we can easily obtain the following result.

**Corollary 1.** Under the same assumptions as in Theorem 1. Then the matrix equation
\[
X^T A X = B, \quad A([1, r]) = A_0
\]
have a solution \( A \in \mathbb{S} \mathbb{R}^{n \times n} \) if and only if
\[
Q_2^T (B - X_1^T A_0 X_1) Q_2 = 0, \quad V_2 V_2^T (B - X_1^T A_0 X_1) = 0,
\]
in which case, the general solution of Eq. (15) is
\[
A = \begin{bmatrix} A_0 \\ F^T \\ P \begin{bmatrix} H_{11} \\ H_{12} \end{bmatrix} \end{bmatrix},
\]
where \( F, H_{11} \) are given by (11) and (13), respectively, and \( L, J, H_{12}, H_{22} \) with \( H_{22} = H_{22}^T \) are arbitrary matrices.

3. The solution of Problem II

It is easy to verify that \( S_E \) is a closed convex subset of \( \mathbb{S} \mathbb{R}^{n \times n} \). From the best approximation theorem (see [1]), we know there exists a unique solution \( \tilde{A} \) in \( S_E \) such that (2) holds.

We now focus our attention on seeking the unique solution \( \tilde{A} \) in \( S_E \). For the given matrix \( \tilde{A} \in \mathbb{S} \mathbb{R}^{n \times n} \) with \( \tilde{A}([1, r]) = A_0 \), write
\[
\tilde{A} = \begin{bmatrix} A_0 \\ F^T \\ H \end{bmatrix},
\]
and
\[
P^T \tilde{H} P = \begin{bmatrix} \tilde{H}_{11} \\ \tilde{H}_{12} \end{bmatrix},
\]
where \( \tilde{H}_{ij} = P_1^T \tilde{H} P_j \) (\( i, j = 1, 2 \)).

For any matrix \( A \in S_E \), by using (11), (13), (17) and (18) we have
\[
\| \tilde{A} - A \| = 2\| F - \tilde{F} \| + \| H - \tilde{H} \| = 2\| P_1^T L^T U_2^T X_1 Q_1 \Omega^{-1} + \Omega^{-1} Q_1^T X_1^T U_2 L P_1 - (H_{110} - \tilde{H}_{11}) \| + 2\| U_2 L + J P_2^T - (\tilde{F} - F_0) \| + 2\| H_{12} - \tilde{H}_{12} \| + 2\| H_{22} - \tilde{H}_{22} \|.
\]
Notice that
\[
\|U_2L + JP_2^T - (\tilde{F} - F_0)\|^2 = \left\|U_2L + [0, J]\begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} - (\tilde{F} - F_0)\right\|^2
\]
\[
= \|U_2LP_1 - (\tilde{F} - F_0)P_1\|^2 + \|U_2LP_2 + J - (\tilde{F} - F_0)P_2\|^2. \tag{20}
\]
It follows from (19) and (20) that \(\|\tilde{A} - A\| = \min\) if and only if
\[
H_{12} = \tilde{H}_{12}, \quad H_{22} = \tilde{H}_{22}, \quad J = (\tilde{F} - F_0)P_2 - U_2LP_2 \tag{21}
\]
and
\[
f(L) = 2\|U_2LP_1 - \tilde{C}\|^2 + \|P_1^T L^T S + S^T LP_1 - \tilde{D}\|^2 = \min,
\tag{22}
\]
where
\[
\tilde{C} = (\tilde{F} - F_0)P_1, \quad \tilde{D} = H_{110} - \tilde{H}_{11}, \quad S = U_2^TX_1Q_1\Omega^{-1}.
\tag{23}
\]
From (22), we have
\[
f(L) = 2\text{trace}(P_1^T L^T LP_1) - 4\text{trace}(P_1^T L^T U_2^T \tilde{C}) + 2\text{trace}(\tilde{C}^T \tilde{C})
+ \text{trace}(P_1^T L^T SP_1^T L^T S) + 2\text{trace}(P_1^T L^T SS^T LP_1) + \text{trace}(S^T LP_1 S^T LP_1)
- 2\text{trace}(P_1^T L^T S\tilde{D}) - 2\text{trace}(S^T LP_1 \tilde{D}) + \text{trace}(\tilde{D}^2).
\]
Consequently,
\[
\frac{\partial f(L)}{\partial L} = 4LP_1P_1^T + 4SP_1^T L^T SP_1 + 4SS^T LP_1 P_1^T - 4U_2^T \tilde{C} P_1^T - 4S \tilde{D} P_1^T.
\]
Setting \(\frac{\partial f(L)}{\partial L} = 0\), we obtain
\[
LP_1 + SP_1^T L^T S + SS^T LP_1 = U_2^T \tilde{C} + S \tilde{D}. \tag{24}
\]
Let the SVD of the matrix \(S\) be
\[
S = G\begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}W^T,
\tag{25}
\]
where \(G = [G_1, G_2] \in \text{OR}^{(r-t)\times(r-t)}, W = [W_1, W_2] \in \text{OR}^{s\times s}, \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_f), \gamma_i > 0 (i = 1, \ldots, f), f = \text{rank}(S), G_1 \in \text{R}^{(r-t)\times f}, W_1 \in \text{R}^{s\times f}.\) Put
\[
G^T LP_1 W = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} f \begin{bmatrix} r - t - f \end{bmatrix}.
\tag{26}
\]
After some algebraic manipulation, the equation of (24) is equivalent to
\[
L_{11} + \Gamma L_{11}^T \Gamma + \Gamma^2 L_{11} = G_1^T (U_2^T \tilde{C} + S \tilde{D}) W_1, \tag{27}
\]
\[
L_{12} + \Gamma^2 L_{12} = G_1^T (U_2^T \tilde{C} + S \tilde{D}) W_2, \tag{28}
\]
\[
L_{21} = G_2^T (U_2^T \tilde{C} + S \tilde{D}) W_1, \tag{29}
\]
\[
L_{22} = G_2^T (U_2^T \tilde{C} + S \tilde{D}) W_2. \tag{30}
\]
Let \(L_{11} = [l_{ij}] \in \text{R}^{f\times f}\) and \(G_1^T (U_2^T \tilde{C} + S \tilde{D}) W_1 = [g_{ij}] \in \text{R}^{f\times f}.\) From (27) we have
\[
l_{ij} + \gamma_i l_{ij} \gamma_j + \gamma_i^2 l_{ij} = g_{ij} \quad \text{for } i, j = 1, \ldots, f.
\]
Solving these linear equations with respect to \( l_{ij}, \ i, j = 1, \ldots, f \), we obtain

\[
l_{ij} = \frac{1}{1 + \gamma_i^2 + \gamma_j^2} (g_{ij} + \gamma_i \gamma_j g_{ji} - \gamma_i \gamma_j g_{ji}) \quad \text{for } i, j = 1, \ldots, f.
\]

(29)

Let \( \Phi = [1/(1 + \gamma_i^2 + \gamma_j^2)] \in \mathbb{R}^{f \times f} \), then (29) may be expressed as

\[
L_{11} = \Phi * (G_1^T(U_2^T \tilde{C} + S \tilde{D})W_1 + G_1^T(U_2^T \tilde{C} + S \tilde{D})W_1 I^2 - I W_1^T(U_2^T \tilde{C} + S \tilde{D})G_1 I).
\]

(30)

From (28), we have

\[
L_{12} = (I_f + I^2)^{-1} G_1^T(U_2^T \tilde{C} + S \tilde{D})W_2.
\]

Thus, from (26) we have

\[
LP_1 = G \begin{bmatrix} L_{11} & (I_f + I^2)^{-1} G_1^T(U_2^T \tilde{C} + S \tilde{D})W_2 \\ G_1^T(U_2^T \tilde{C} + S \tilde{D})W_1 & G_2^T(U_2^T \tilde{C} + S \tilde{D})W_2 \end{bmatrix} W^T,
\]

(31)

where \( L_{11} \) is given by (30).

Inserting \( J \) in (21) into (11), we obtain

\[
F = F_0 P_1 P_1^T + \tilde{F} P_2 P_2^T + U_2 L P_1 P_1^T.
\]

Summing up above discussion, we have proved the following result.

**Theorem 2.** For the given matrix \( \tilde{A} \in \mathbb{S}^{n \times n} \) with \( \tilde{A}([1, r]) = A_0 \), then the matrix best approximation problem (2) has a unique solution \( \hat{A} \in \mathbb{S}_k \). Furthermore, let the partition of \( \tilde{A} \) be (17), \( \tilde{H}_{ij} = P_i^T H_j P_i \) (i, j = 1, 2). Then the unique solution of Problem II can be expressed as

\[
\hat{A} = \begin{bmatrix} A_0 & F_0 P_1 P_1^T + \tilde{F} P_2 P_2^T + U_2 L P_1 P_1^T \\ (F_0 P_1 P_1^T + \tilde{F} P_2 P_2^T + U_2 L P_1 P_1^T)^T & P \begin{bmatrix} H_{11} & H_{12} \\ \tilde{H}_{12} & \tilde{H}_{22} \end{bmatrix} \end{bmatrix} P^T,
\]

(32)

where \( LP_1 \) and \( H_{11} \) are given by (31) and (13), respectively.

### 4. A numerical example

Based on Theorems 1 and 2 we can describe an algorithm for solving Problems I and II as follows.

**Algorithm 1.**

1. Input matrices \( X, B, A_0 \), and \( \tilde{A} \).
2. Form the matrix \( X_1, X_2 \) according to (3).
3. Compute the SVD (6) of the matrix \( X_2 \) and then compute the SVD (10) of \( X_1 Q_2 \).
4. Compute \( F_0 \) and \( H_{110} \) by (12) and (14), respectively.
5. Partition matrix \( \tilde{A} \) as in (17) to get \( \tilde{F}, \tilde{H} \).
6. Compute \( \tilde{H}_{ij}, \ i, j = 1, 2 \) by (18).
7. Compute the matrices \( \tilde{C}, \tilde{D} \) and \( S \) by (23).
8. Compute the SVD (25) of the matrix \( S \).
9. Compute \( L_{11} \) by (30).
10. Compute \( LP_1 \) by (31) and then compute \( H_{11} \) by (13).
11. Compute the unique solution \( \hat{A} \) of Problem II according to (32).
Example 1 (An example for updating the mass matrix of a vibrating system described in (1)). Let $A_0$, $X$, $B$ and $\tilde{A}$ be given by

$$
A_0 = \begin{bmatrix}
0.3333 & 0.1667 & 0 & 0 \\
0.1667 & 0.6667 & 0.1667 & 0 \\
0 & 0.1667 & 0.6667 & 0.1667 \\
0 & 0 & 0.1667 & 0.6667 \\
\end{bmatrix}, \quad X = \begin{bmatrix}
39.230 & 347.67 & -45.500 \\
-22.312 & -289.62 & 54.211 \\
68.151 & 640.05 & -90.167 \\
-61.888 & -640.58 & 100.18 \\
206.84 & 491.56 & 141.90 \\
-427.7 & -1474.9 & -161.45 \\
315.58 & 1375.1 & 36.539 \\
-184.24 & -993.56 & 33.587 \\
\end{bmatrix},
$$

$$
B = I_3 \quad \text{and} \quad \tilde{A} = \begin{bmatrix}
0.3333 & 0.1667 & 0 & 0 & 0.1000 & 0.3500 & 0.6000 & 0.2000 \\
0.1667 & 0.6667 & 0.1667 & 0 & 0.4500 & 0.7000 & 0.8000 & 0.2800 \\
0 & 0.1667 & 0.6667 & 0.1667 & 0.1800 & 0.2100 & 0.3900 & 0.4700 \\
0 & 0 & 0.1667 & 0.6667 & 0.4000 & 0.3000 & 0.2000 & 0.1000 \\
0.1000 & 0.4500 & 0.1800 & 0.4000 & 0.1200 & 0.4800 & 0.4680 & 0.3600 \\
0.3500 & 0.7000 & 0.2100 & 0.3000 & 0.4800 & 0.8400 & 0.6060 & 0.3480 \\
0.6000 & 0.8000 & 0.3900 & 0.2000 & 0.4680 & 0.6060 & 0.4680 & 0.4020 \\
0.2000 & 0.2800 & 0.4700 & 0.1000 & 0.3600 & 0.3480 & 0.4020 & 0.1200 \\
\end{bmatrix}.
$$

According to Algorithm 1 we obtain the unique solution of Problem II as follows:

$$
\tilde{A} = \begin{bmatrix}
0.3333 & 0.1667 & 0 & 0 & 0.1224 & 0.3447 & 0.5783 & 0.2297 \\
0.1667 & 0.6667 & 0.1667 & 0 & 0.3938 & 0.7322 & 0.8287 & 0.2283 \\
0 & 0.1667 & 0.6667 & 0.1667 & 0.2359 & 0.1893 & 0.3459 & 0.5353 \\
0 & 0 & 0.1667 & 0.6667 & 0.3212 & 0.3376 & 0.2507 & 0.0182 \\
0.1224 & 0.3938 & 0.2359 & 0.3212 & 0.0684 & 0.5068 & 0.4982 & 0.3091 \\
0.3447 & 0.7322 & 0.1893 & 0.3376 & 0.5068 & 0.8209 & 0.5974 & 0.3681 \\
0.5783 & 0.8287 & 0.3459 & 0.2507 & 0.4982 & 0.5974 & 0.4406 & 0.4404 \\
0.2297 & 0.2283 & 0.5353 & 0.0182 & 0.3091 & 0.3681 & 0.4404 & 0.0622 \\
\end{bmatrix}.
$$

Although we do not need to verify the consistency condition (16), we note that the condition (16) does hold for this example. Furthermore, we can figure out

$$
\|X^T \tilde{A} X - I_3\| = 7.8233e - 010, \quad \|\tilde{A} - \hat{A}\| = 0.3031.
$$

References