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On almost sure convergence of the quadratic variation of Brownian motion

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Abstract

We study the problem of a.s. convergence of the quadratic variation of Brownian motion. We present some new sufficient and necessary conditions for the convergence. As a byproduct we get a new proof of the convergence in the case of refined partitions, a result that is due to Lévy. Our method is based on conversion of the problem to that of a Gaussian sequence via decoupling.

Keywords: Brownian motion; Quadratic variation; A.s. convergence

1. Introduction and main results

Let $H(t), 0 \leqslant t \leqslant 1$ represent a standard Brownian motion. In what follows, we let $H(A) = \int_0^1 1_A(t)\,\mathrm{d}H(t)$, $A \subset [0,1]$. Also $I_k^n \subset [0,1]$ will always represent an interval.

Let $I^n = \{I_k^n,\ k = 1,\dots,k_n\}, n = 1,2,\dots$ be a sequence of partitions of [0,1], i.e. for each n: $[0,1] = \bigcup_{k=1}^{k_n} I_k^n$ and $I_i^n \cap I_j^n = \emptyset$ whenever $i \neq j$.

Throughout we will assume the following (λ is Lebesgue measure)

Assumption. $\lambda_n = \max\{\lambda(I_k^n),\ k = 1,\dots,k_n\} \to 0$ as $n \to \infty$.

We would like to find necessary and sufficient conditions for almost sure convergence of $\sum_{k=1}^{k_n} H^2(I_k^n)$. Since $E(\sum_{k=1}^{k_n} H^2(I_k^n)) = 1$ and

$$\mathrm{Var}\left(\sum_{k=1}^{k_n} H^2(I_k^n)\right) = \mathrm{Var}(Z^2)\sum_{k=1}^{k_n} \lambda^2(I_k^n) \leqslant \mathrm{Var}(Z^2)\lambda_n \to 0$$

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by our assumption (where $Z \sim \mathrm{N}(0,1)$), it follows that we always have convergence in probability, i.e.

$$\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\to} 1 \quad \text{in probability.}$$

Our problem is, therefore, to characterize the sequences of partitions $\{I^n\}$ that will satisfy

$$\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\to} 1 \quad \text{a.s.}$$

Our main idea is to convert the problem via decoupling technique to a problem about convergence of Gaussian sequences. The advantage of this approach is that there is a rich theory on continuity of Gaussian processes that we can use. For example, there is an important necessary condition for continuity (boundedness) due to Sudakov ("Sudakov's minorization"), and there is an important sufficient condition due to Dudley formulated in term of finiteness of the "entropy integral." Both will be used here. For a reference see Jain and Marcus (1978) or Ledoux and Talagrand (1991).

To describe our decoupling result we let $H'(t)$, $0 \leqslant t \leqslant 1$, be a Brownian motion that is independent of $H(t)$, $0 \leqslant t \leqslant 1$. In Section 2 we prove:

Theorem 1. *The following are equivalent*:

(a) $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\to} 1$ *a.s.*,
(b) $\sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n) \underset{n\to\infty}{\to} 0$ *a.s.*,
(c) $\sum_{k=1}^{k_n} H^+(I_k^n)H'(I_k^n) \underset{n\to\infty}{\to} 0$ *a.s.* $(H^+(A) = \max\{H(A), 0\})$.

In Section 3 we apply Theorem 1 to obtain necessary conditions. Let Δ denote symmetric difference. For ease of notation we use throughout:

$$G_n = \sum_{k=1}^{k_n} \sqrt{\lambda(I_k^n)}H(I_k^n), \quad 1 \leqslant n,$$

$$G_{n,m} = \sum_{k,l} \sqrt{\lambda(I_k^n \Delta I_l^m)}H(I_k^n \cap I_l^m), \quad 1 \leqslant n \leqslant m < \infty.$$

Theorem 2. *If* $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\to} 1$ *a.s. then*

(a) $G_n \underset{n\to\infty}{\to} 0$, *a.s. and*
(b) $G_{n,m} \underset{m\to\infty}{\to} G_n$ *a.s. for each fixed* n.

In particular, $\lim_n \lim_m G_{n,m} = 0$ *a.s.*

De La Vega (1974) defines a sequence of partitions with $\lambda_n = \mathrm{O}(1/\log(n))$ for which $\sum_{k=1}^{k_n} H^2(I_k^n)$ does not converge a.s. Is there such a sequence of partitions for which the a.s. convergence of G_n to 0 also fails? The answer is positive. We present an example in which $\lambda_n = \mathrm{O}(1/\log(n))$. This condition is sharp in a sense since it is known that $\lambda_n = \mathrm{o}(1/\log(n))$ is sufficient for a.s. convergence of $\sum_{k=1}^{k_n} H^2(I_k^n)$ hence, by Theorem 2(a), it is also sufficient for a.s. convergence of G_n to 0.

We will also prove the following necessary condition:

Theorem 3. *If* $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ *a.s., then*

$$G_{n,m} \underset{m,n\to\infty}{\longrightarrow} 0 \quad a.s.$$

Finally in Section 4 we prove a sufficient condition:

Theorem 4. *If* $\int_0^1 \sqrt{\log(n_\varepsilon)}\,\mathrm{d}\varepsilon < \infty$ *then* $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ *a.s.*

The quantity n_ε (see (4.3) in Section 4) is a function of the relationship between the partitions and the Brownian modulus of continuity. In the case of a refined sequence of partitions (the set of points that generate the partition is increasing in n) Lévy (1940) proved that there is a.s. convergence. Lévy's result appears here as Corollary 7 and follows immediately from Theorem 4. This constitutes a new proof of Lévy's result. In Corollary 8 we formulate a sufficient condition in terms of the quantities $\{\sum_{k=1}^{k_n} \lambda^2(I_k^n)\}$ alone. This should be compared with a known result in which the condition (see (4.6) in Section 4) depends on $\{\lambda_n\}$ alone.

Some notation: $E(X|F)$, the conditional expectation of a random variable X given a σ-algebra F, is denoted by $E_F(X)$. Similarly, the conditional variance is denoted by $\mathrm{Var}_F(X)$.

Let $X(t)$, $0 \leqslant t \leqslant 1$ be a stochastic process with continuous paths. We will denote its (random) modulus of continuity by

$$h_X(x) = \sup\{|X(t) - X(s)| : |t-s| \leqslant x,\ t,s \in [0,1]\}, \quad 0 \leqslant x \leqslant 1. \tag{1.1}$$

Obviously, the continuity of X implies that $h_X(x) \underset{x\to 0}{\longrightarrow} 0$ a.s.

2. Reformulations of a.s. convergence

Let $H(t)$, $0 \leqslant t \leqslant 1$ and $H'(t)$, $0 \leqslant t \leqslant 1$, be two independent Brownian motions. We start with the proof of Theorem 1.

Proof of Theorem 1. (a) $\Rightarrow$ (b): If $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ a.s. then the same holds for H', namely $\sum_{k=1}^{k_n} H'^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ a.s. By subtraction we get

$$\sum_{k=1}^{k_n} H^2(I_k^n) - H'^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{2.1}$$

Formula (2.1) is equivalent to $\sum_{k=1}^{k_n}(H(I_k^n)+H'(I_k^n))(H(I_k^n)-H'(I_k^n)) \underset{n\to\infty}{\to} 0$ a.s.

Since $(H(t)+H'(t))/\sqrt{2}$ and $(H(t)-H'(t))/\sqrt{2}$ are two independent Brownian motions, we get (b).

(b) ⇒ (a): From the proof of (a) ⇒ (b), we get that (b) is equivalent to (2.1). Now put $X_n=\sum_{k=1}^{k_n} H^2(I_k^n)-1$ and $X_n'=\sum_{k=1}^{k_n} H'^2(I_k^n)-1$. It is easy to see that (2.1) is equivalent to

$$\sup_{n\geqslant N}\{|X_n-X_n'|\} \underset{N\to\infty}{\to} 0 \quad \text{in probability.} \tag{2.2}$$

Let $\varepsilon>0$. Our assumption $\lambda_n\to 0$ implies that $X_n\to 0$ in probability. So there exists $\beta>0$ so that for all n

$$P(|X_n|<\varepsilon/2)\geqslant\beta.$$

It follows by symmetrization (see Pollard, 1984, p. 14) that for all N

$$P\left(\sup_{n\geqslant N}\{|X_n|\}>\varepsilon\right)\leqslant(1/\beta)P\left(\sup_{n\geqslant N}\{|X_n-X_n'|\}>\varepsilon/2\right). \tag{2.3}$$

From (2.2) and (2.3) we get $\sup_{n\geqslant N}\{|X_n|\} \underset{N\to\infty}{\to} 0$ in probability, which implies (a).

(b) ⇒ (c): Put $Y_n=\sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n), n\geqslant 1$. (b) implies that

$$P_H(Y_n\to 0)=1 \quad \text{a.s.,} \tag{2.4}$$

where the subscript H represents the σ-algebra generated by H. Since $\{Y_n\}$ is a Gaussian sequence given $\sigma(H)$, (2.4) implies by a basic result on Gaussian sequences (see Landau and Shepp, 1970) that $E_H(\sup|Y_n|)<\infty$ a.s.; so by dominated convergence

$$E_H\left(\sup_{n\geqslant N}|Y_n|\right) \underset{N\to\infty}{\to} 0 \quad \text{a.s.} \tag{2.5}$$

We next put $Z_n=\sum_{k=1}^{k_n} H^+(I_k^n)H'(I_k^n), n\geqslant 1$. We claim that

$$E_H(Y_n-Y_m)^2\geqslant E_H(Z_n-Z_m)^2, \quad 1\leqslant n\leqslant m. \tag{2.6}$$

Indeed

$$\begin{aligned} E_H(Y_n-Y_m)^2 &= \sum_{k,l}(H(I_k^n)-H(I_l^m))^2\lambda(I_k^n\cap I_l^m)\\ &\geqslant \sum_{k,l}(H^+(I_k^n)-H^+(I_l^m))^2\lambda(I_k^n\cap I_l^m)\\ &= E_H(Z_n-Z_m)^2, \end{aligned}$$

where the inequality follows from $|a-b|\geqslant|a^+-b^+|$, $a,b\in\mathbb{R}$.

It follows from (2.6) via another basic result in Gaussian processes (see Jain and Marcus, 1978, p. 102) that for each N

$$E_H\left(\sup_{n\geqslant N} Z_n\right) \leqslant E_H\left(\sup_{n\geqslant N} Y_n\right). \tag{2.7}$$

We claim that for each N:

$$E_H\left(\sup_{n\geqslant N} |Z_n|\right) \leqslant 2E_H\left(\sup_{n\geqslant N} Z_n\right), \tag{2.8}$$

Proof of (2.8): Fix $M \geqslant N$. We calculate

$$\begin{aligned} E_H\left(\sup_{n\geqslant N} |Z_n|\right) &\leqslant E_H|Z_M| + E_H\left(\sup_{n\geqslant N} |Z_n - Z_M|\right) \\ &\leqslant E_H|Z_M| + E_H\left(\sup_{n\geqslant N}(Z_n - Z_M)^+\right) + E_H\left(\sup_{n\geqslant N}(Z_n - Z_M)^-\right) \\ &= E_H|Z_M| + E_H\left(\sup_{n\geqslant N}(Z_n - Z_M)\right) + E_H\left(\sup_{n\geqslant N}(-1)(Z_n - Z_M)\right), \end{aligned}$$

where the last equality follows because the sequence $\{Z_n - Z_M : n \geqslant N\}$ contains 0. Since given H, the Gaussian sequences $\{Z_n - Z_M : n \geqslant N\}$ and $\{(-1)(Z_n - Z_M) : n \geqslant N\}$ are equal in distribution, we can continue the calculation as follows:

$$\begin{aligned} &= E_H|Z_M| + 2E_H\left(\sup_{n\geqslant N}(Z_n - Z_M)\right) \\ &= E_H|Z_M| + 2E_H\left(\sup_{n\geqslant N}(Z_n)\right), \end{aligned}$$

because $E_H(Z_M) = 0$. So we get that

$$E_H\left(\sup_{n\geqslant N} |Z_n|\right) \leqslant E_H|Z_M| + 2E_H\left(\sup_{n\geqslant N} Z_n\right). \tag{2.9}$$

Since $\{E_H|Z_n|\}^2 \leqslant E_H(Z_n^2) = \sum_{k=1}^{k_n} H^+(I_k^n)^2\lambda(I_k^n) \to 0$ a.s. and M is arbitrary, (2.8) follows easily from (2.9).

Equipped with (2.8) and with the help of (2.5) and (2.7), we get

$$E_H\left(\sup_{n\geqslant N} |Z_n|\right) \underset{N\to\infty}{\longrightarrow} 0 \quad \text{a.s.};$$

so $\mathrm{P}_H(Z_n \to 0) = 1$ a.s., and (c) follows.

(c) $\Rightarrow$ (b): Since the process $-H$ is also a Brownian motion that is independent of H', it is easy to see that the sequence $\{\sum_{k=1}^{k_n} H^+(I_k^n)H'(I_k^n), n \geqslant 1\}$ is equal in distribution to $\{\sum_{k=1}^{k_n} H^-(I_k^n)H'(I_k^n),\ n \geqslant 1\}$. This implies that both sequences converge to 0 a.s.; hence their difference $\sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n)$, $n \geqslant 1$ converges to 0 as well. □

Remarks. (1) In the proof of (b) ⇒ (c) we established a conditional version of the following result: Let $\{A_n : 1 \leqslant n < \infty\}$ and $\{B_n : 1 \leqslant n < \infty\}$ be two centered Gaussian sequences so that $E(B_n^2) \to 0$, and

$$E(A_m - A_n)^2 \geqslant E(B_m - B_n)^2, \quad 1 \leqslant n \leqslant m.$$

Then $A_n \to 0$ a.s. implies $B_n \to 0$ a.s.

(2) A similar proof as in (c) ⇒ (b) (replace difference by sum) shows that (c) implies also

$$\sum_{k=1}^{k_n} |H(I_k^n)| H'(I_k^n) \underset{n\to\infty}{\to} 0 \quad \text{a.s.}$$

3. Necessary conditions for a.s. convergence

We will start with the following basic lemma.

Lemma 5. *Let $\{X_n\}$ be a sequence of random variables and let F be a σ-algebra of events.*

(a) *If* $\mathrm{Var}_F(X_n) \to 0$ *a.s. and* $P_F(|X_n| > \varepsilon) \to 0$ *a.s.* $\forall \varepsilon > 0$, *then* $E_F(X_n) \to 0$ *a.s.*

(b) *If* $\mathrm{Var}_F(X_n) \to 0$ *a.s. and* $X_n \to 0$ *a.s. then* $E_F(X_n) \to 0$ *a.s.*

Proof. *Part* (a): $\mathrm{Var}_F(X_n) \to 0$ a.s. implies that

$$P_F(|X_n - E_F(X_n)| > \varepsilon) \to 0 \quad \text{a.s. } \forall \varepsilon > 0.$$

When we put together this convergence and the assumption $P_F(|X_n| > \varepsilon) \to 0$ a.s. $\forall \varepsilon > 0$, we get that in fact

$$P_F(|E_F(X_n)| > \varepsilon) \to 0 \quad \text{a.s. } \forall \varepsilon > 0.$$

So we conclude that (sets are identified with their indicator functions): $\{|E_F(X_n)| > \varepsilon\} \to 0$ a.s. $\forall \varepsilon > 0$, and the result follows.

Part (b): $X_n \to 0$ a.s. implies that $\forall \varepsilon > 0$ we have $P_F(|X_n| > \varepsilon) \to 0$ a.s. by dominated convergence for conditional expectations. The result follows now from part (a). □

Proof of Theorem 2. *Part* (a): The proof is based on Lemma 5. From Theorem 1 we get that $\sum_{k=1}^{k_n} H^+(I_k^n) H'(I_k^n) \underset{n\to\infty}{\to} 0$ a.s. We have

$$E_{H'}\left(\sum_{k=1}^{k_n} H^+(I_k^n) H'(I_k^n)\right) = C \sum_{k=1}^{k_n} \sqrt{\lambda(I_k^n)} H'(I_k^n),$$

where $C = E(Z)^+$ and $Z \sim \mathrm{N}(0,1)$. From Lemma 5 all we need to prove is $\mathrm{Var}_{H'}(\sum_{k=1}^{k_n} H^+(I_k^n)H'(I_k^n)) \underset{n\to\infty}{\to} 0$ a.s. But

$$\mathrm{Var}_{H'}\left(\sum_{k=1}^{k_n} H^+(I_k^n)H'(I_k^n)\right) = \mathrm{Var}(Z^+)\sum_{k=1}^{k_n} \lambda(I_k^n){H'}^2(I_k^n)$$
$$\leqslant \mathrm{Var}(Z^+)h_{H'}^2(\lambda_n) \to 0 \quad \text{a.s.},$$

where $h_{H'}$ is the modulus of continuity of H' (see (1.1)).

Part (b): We will in fact prove that $G_n \underset{n\to\infty}{\to} 0$ a.s. implies $G_{n,m} \underset{m\to\infty}{\to} G_n$ a.s. for each fixed n.

Fix n. Let $J^m = \{I_k^n \cap I_l^m\}$, $m > n$, be the sequence of partitions generated by merging the original n-partition into the m-partitions. Now we define

$$Y_{n,m} = \sum_{k,l} \sqrt{\lambda(I_k^n \Delta J_l^m)} H(I_k^n \cap J_l^m), \quad n < m,$$

$$Y_m = \sum_k \sqrt{\lambda(J_k^m)} H(J_k^m), \quad n < m.$$

There is an $M > n$ so that the Gaussian process $\{Y_m : M \leqslant m\}$ dominates $\{Y_{n,m} - G_n : M \leqslant m\}$ in L_2 distance, i.e.

$$E(Y_{n,m} - Y_{n,p})^2 \leqslant E(Y_m - Y_p)^2, \quad M \leqslant m < p. \tag{3.1}$$

In fact, for $n < m \leqslant p < \infty$, we have

$$E(Y_m - Y_p)^2 = \sum_{l,j}\left(\sqrt{\lambda(J_l^m)} - \sqrt{\lambda(J_j^p)}\right)^2 \lambda(J_l^m \cap J_j^p)$$
$$= \sum_{k,l,j}\left(\sqrt{\lambda(J_l^m)} - \sqrt{\lambda(J_j^p)}\right)^2 \lambda(I_k^n \cap J_l^m \cap J_j^p),$$

while

$$E(Y_{n,m} - Y_{n,p})^2 = \sum_{k,l,j}\left(\sqrt{\lambda(I_k^n \Delta J_l^m)} - \sqrt{\lambda(I_k^n \Delta J_j^p)}\right)^2 \lambda(I_k^n \cap J_l^m \cap J_j^p)$$
$$= \sum_{k,l,j}\left(\sqrt{\lambda(I_k^n) - \lambda(J_l^m)} - \sqrt{\lambda(I_k^n) - \lambda(J_j^p)}\right)^2 \lambda\left(I_k^n \cap J_l^m \cap J_j^p\right).$$

The concavity of the square root function implies

$$(\sqrt{x} - \sqrt{y})^2 \geqslant (\sqrt{c-y} - \sqrt{c-x})^2, \quad 0 \leqslant x, y < c/2, \ c > 0.$$

We use this inequality to compare corresponding terms in the summations. Domination (3.1) follows since $\lambda(I_k^n)/2 > \lambda_m \vee \lambda_p$ for m and p large and fixed k. We can also show that $E(Y_{n,m} - G_n)^2 \underset{m\to\infty}{\to} 0$. From the result mentioned in Remark 1 in Section 2 we now get that $Y_m \underset{m\to\infty}{\to} 0$ a.s. will imply $Y_{n,m} \underset{m\to\infty}{\to} G_n$ a.s. To complete the argument observe that the summands in $Y_{n,m}$ and $G_{n,m}$ agree except for intervals I_j^m containing in their interiors a boundary point of the n-partition, of which there are at most k_n. Hence, $|Y_{n,m} - G_{n,m}| \leqslant 4k_n h_H(\lambda_m) \underset{m\to\infty}{\longrightarrow} 0$ a.s. Exactly the same argument shows that $|Y_m - G_m| \underset{m\to\infty}{\longrightarrow} 0$ a.s. Now we are done: $G_n \underset{n\to\infty}{\longrightarrow} 0$ a.s. implies $Y_m \underset{m\to\infty}{\longrightarrow} 0$ a.s., hence $Y_{n,m} \underset{m\to\infty}{\longrightarrow} G_n$ a.s. and part (b) follows. □

Proof of Theorem 3. Let us define

$$\tilde{Y}_{n,m} = \sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n) - \sum_{l=1}^{k_m} H(I_l^m)H'(I_l^m)$$
$$= \sum_{k,l} (H(I_k^n) - H(I_l^m))H'(I_k^n \cap I_l^m), \quad 1 \leqslant n \leqslant m.$$

From Theorem 1 we get that

$$\tilde{Y}_{n,m} \underset{m,n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{3.2}$$

Now put $Z_{n,m} = \sum_{k,l} |H(I_k^n) - H(I_l^m)|H'(I_k^n \cap I_l^m), 1 \leqslant n \leqslant m$. Upon conditioning on the σ-algebra generated by H we get that both $\{\tilde{Y}_{n,m} : 1 \leqslant n \leqslant m\}$ and $\{Z_{n,m} : 1 \leqslant n \leqslant m\}$ are centered Gaussian processes.

In addition, given H, the process $\{\tilde{Y}_{n,m} : 1 \leqslant n \leqslant m\}$ dominates $\{Z_{n,m} : 1 \leqslant n \leqslant m\}$ in L_2 distance, i.e.

$$E_H(Z_{n,m} - Z_{s,t})^2 \leqslant E_H(\tilde{Y}_{n,m} - \tilde{Y}_{s,t})^2, \quad 1 \leqslant n \leqslant m, \ 1 \leqslant s \leqslant t. \tag{3.3}$$

The calculations involved in proving (3.3) are similar to those involved in proving (2.6) and are skipped. We also get easily

$$E_H(Z_{n,m}^2) \underset{m,n\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

Using (3.3) and the convergence above we get

$$Z_{n,m} \underset{m,n\to\infty}{\longrightarrow} 0 \quad \text{a.s.,} \tag{3.4}$$

via an extension of Remark 1 following the proof of Theorem 1 to sequences with two indices. Next, we observe

$$E_{H'}(Z_{n,m}) = \sum_{k,l} \sqrt{\lambda(I_k^n \Delta I_l^m)}H'(I_k^n \cap I_l^m), \tag{3.5}$$

where $C = E(|Z|) < 1$, $Z \sim \mathrm{N}(0,1)$. By a straightforward extension of Lemma 5(b) to sequences with two indices, we get Theorem 3 if in addition to (3.4) and (3.5), we show

$$\mathrm{Var}_{H'}(Z_{n,m}) \underset{m,n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{3.6}$$

Proof of (3.6): We represent $A = \{(k,l): k = 1,\ldots k_n,\ l = 1,\ldots,l_m\}$ as a disjoint union $A = \bigcup_{i=1}^{5} A_i$ where

$$A_1 = \{(k,l): I_k^n \supset I_l^m \text{ and } I_k^n \neq I_l^m\},$$

$$A_2 = \{(k,l): I_k^n \subset I_l^m \text{ and } I_k^n \neq I_l^m\},$$

$$A_3 = \{(k,l): I_k^n = [a,b], I_l^m = [c,d],\ a < c < b < d\},$$

$$A_4 = \{(k,l): I_k^n = [a,b], I_l^m = [c,d],\ c < a < d < b\},$$

$$A_5 = \{(k,l): \lambda(I_k^n \cap I_l^m) = 0 \text{ or } I_k^n = I_l^m\}.$$

To prove (3.6) it is enough to prove for each $i = 1,\ldots,5$

$$\mathrm{Var}_{H'}\left(\sum_{A_i} |H(I_k^n) - H(I_l^m)| H'(I_k^n \cap I_l^m)\right) \underset{m,n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{3.7}$$

The case of A_1: The LHS of (3.7) has the form

$$\sum_{k=1}^{k_n} \mathrm{Var}_{H'}\left(\sum_{l\in B_k} |H(I_k^n) - H(I_l^m)| H'(I_l^m)\right), \tag{3.8}$$

where for each k we let $B_k = \{l: (k,l) \in A_1\}$. In (3.8) we used the independent increments property of H. Next, we write

$$\begin{aligned}
&\mathrm{Var}_{H'}\left(\sum_{l\in B_k} |H(I_k^n) - H(I_l^m)| H'(I_l^m)\right)\\
&\quad \leqslant 2\,\mathrm{Var}_{H'}\left(\sum_{l\in B_k} |H(I_k^n)| H'(I_l^m)\right)\\
&\qquad + 2\,\mathrm{Var}_{H'}\left(\sum_{l\in B_k} (|H(I_k^n) - H(I_l^m)| - |H(I_k^n)|) H'(I_l^m)\right) = (*) + (**).
\end{aligned}$$

We calculate

$$\begin{aligned}
(*) &= 2\,\mathrm{Var}_{H'}\left(|H(I_k^n)| H'\left(\bigcup_{l\in B_k} I_l^m\right)\right)\\
&= 2\left(H'\left(\bigcup_{l\in B_k} I_l^m\right)\right)^2 \mathrm{Var}(|H(I_k^n)|) \leqslant 2 h_{H'}^2(\lambda_n)\lambda(I_k^n).
\end{aligned}$$

$$(**) \leqslant 2E_{H'}\left(\sum_{l\in B_k}(|H(I_k^n)-H(I_l^m)|-|H(I_k^n)|)H'(I_l^m)\right)^2$$

$$\leqslant 2E_{H'}\left(\sum_{l\in B_k}|H(I_l^m)||H'(I_l^m)|\right)^2$$

$$\leqslant 2E_{H'}\left(\sum_{l\in B_k}{H'}^2(I_l^m)\cdot\sum_{l\in B_k}H^2(I_l^m)\right) \quad \text{(by Cauchy–Schwarz on the sum)}$$

$$\leqslant 2\left(\sum_{l\in B_k}{H'}^2(I_l^m)\right)\lambda(I_k^n)$$

$$\leqslant 2\left(\sum_{l\in B_k}{H'}^2(I_l^m)\right)\lambda_n.$$

Go back to (3.8) and get

$$\mathrm{Var}_{H'}\left(\sum_{A_1}|H(I_k^n)-H(I_l^m)|H'(I_k^n\cap I_l^m)\right)$$

$$\leqslant \sum_{k=1}^{k_n}2h_{H'}^2(\lambda_n)\lambda(I_k^n)+\sum_{k=1}^{k_n}2\left(\sum_{l\in B_k}{H'}^2(I_l^m)\right)\lambda_n$$

$$\leqslant 2h_{H'}^2(\lambda_n)+2\lambda_n\sum_{l=1}^{k_m}{H'}^2(I_l^m)\underset{m,n\to\infty}{\longrightarrow}0 \quad \text{a.s.}$$

as follows from the assumption of the theorem.

The case of A_2: This case is similar to the case of A_1 and is skipped.

The case of A_3: For ease of notation let us denote the intervals

$$\{I_k^n, I_l^m, I_k^n\cap I_l^m : (k,l)\in A_3\} \quad \text{by } \{J_i^n, J_i^m, J_i^{m,n}\},$$

respectively. We obviously get

$$\mathrm{Var}_{H'}\left(\sum_{A_3}|H(I_k^n)-H(I_l^m)|H'(I_k^n\cap I_l^m)\right)$$

$$\leqslant 2\,\mathrm{Var}_{H'}\left(\sum_{i\text{ odd}}|H(J_i^n)-H(J_i^m)|H'(J_i^{m,n})\right)$$

$$+2\,\mathrm{Var}_{H'}\left(\sum_{i\text{ even}}|H(J_i^n)-H(J_i^m)|H'(J_i^{m,n})\right)=(\sim)+(\sim\sim).$$

The point is that the two sums above are sums of independent random variables (given H') because $J_i^n \cup J_i^m$ and $J_{i+2}^n \cup J_{i+2}^m$ are "separated" by $J_{i+1}^{m,n}$.

We estimate $(\sim)$ by

$$(\sim) \leqslant 2 \sum_{i \text{ odd}} H'^2(J_i^{m,n}) \operatorname{Var}_{H'}(|H(J_i^n) - H(J_i^m)|)$$

$$\leqslant 2h_{H'}^2(\lambda_{m,n}) \sum_{i \text{ even}} \lambda(J_i^n) + \lambda(J_i^m)$$

$$\leqslant 4h_{H'}^2(\lambda_{m,n}) \underset{m,n\to\infty}{\to} 0 \quad \text{a.s.}$$

where $\lambda_{m,n} = \max\{\lambda(I_k^n \cap I_l^m)\}$. We estimate $(\sim\sim)$ in a similar way.

The case of A_4: This case is similar to the case of A_3 and is skipped.

The case of A_5: The terms in the sum are all 0 so the LHS of (3.7) is identically 0. This ends the proof of Theorem 2. □

Example. We present an example where $\lambda_n = O(1/\log(n))$ and G_n does not converge to 0 a.s.

Let $n_k \to \infty$ be a monotone sequence of integers so that $n_{k+1} - n_k \leqslant 2^k$. For $n_k \leqslant n < n_{k+1}$, we select $C_n \subset \{1, 2, \ldots, k\}$ so that $n \to C_n$ is one to one.

We define a sequence of partitions I^n, $n = 1, 2, \ldots$ by taking intervals of the form $((i-1)/k, i/k]$ for $i \in C_n$ and "much shorter" intervals of length $1/kn$ from the rest of [0,1]. Formally for $n_k \leqslant n < n_{k+1}$, we define

$$I^n = \left\{ \left(\frac{i-1}{k}, \frac{i}{k} \right] : i \in C_n \right\}$$

$$\cup \left\{ \left(\frac{(i-1)n + j - 1}{kn}, \frac{(i-1)n + j}{kn} \right] : i \notin C_n,\ 1 \leqslant j \leqslant n \right\}.$$

It easy to see that the condition $n_{k+1} - n_k \leqslant 2^k$ implies that $\lambda_n = O(1/\log(n))$. Let $A_n = \bigcup \{((i-1)/k, i/k] : i \in C_n\}$ be the union of the "large" intervals and let A_n^{C} denote its complement in $[0, 1]$. Since

$$G_n = \sqrt{1/k}H(A_n) + \sqrt{1/kn}H(A_n^{\mathrm{C}}), \quad n_k \leqslant n < n_{k+1}, \tag{3.9}$$

it follows that

$$G_n \underset{n\to\infty}{\to} 0 \text{ a.s. iff } \sqrt{1/k}H(A_n) \underset{n\to\infty}{\to} 0 \text{ a.s.,} \tag{3.10}$$

where in (3.10) and throughout $k = k(n)$ satisfies $n_k \leqslant n < n_{k+1}$. To see (3.10) observe that $\sqrt{1/kn}H(A_n^{\mathrm{C}})$ is normally distributed with variance smaller than $1/n$. By the tail inequality for Gaussian distributions

$$\sqrt{1/kn}H(A_n^{\mathrm{C}}) \underset{n\to\infty}{\to} 0 \quad \text{a.s.}$$

and (3.10) follows from (3.9).

Claim. $\sqrt{1/k}H(A_n) \underset{n\to\infty}{\to} 0$ *a.s. iff* $\log(n_{k+1}-n_k)/k \to 0$.

The claim, together with (3.10), implies that

$$G_n \underset{n\to\infty}{\to} 0 \text{ a.s. iff } \log(n_{k+1}-n_k)/k \to 0. \tag{3.11}$$

Formula (3.11) gives us a method to produce sequences of partitions in which G_n does not converge to 0 a.s. All we need is that $\log(n_{k+1}-n_k)/k$ does not converge to 0.

Proof of the claim. $\Leftarrow$: For any $\varepsilon > 0$

$$\sum_k \sum_{n=n_k+1}^{n_{k+1}} P(\sqrt{1/k}H(A_n) > \varepsilon) \leqslant \sum_k \exp\left(\left(\frac{\log(n_{k+1}-n_k)}{k} - \frac{\varepsilon^2}{2}\right)k\right),$$

and, by the assumption $\log(n_{k+1}-n_k)/k \to 0$, the series on the right converges and the result follows from the Borel–Cantelli Lemma. Alternatively, it is easy to see that $\log(n_{k+1}-n_k)/k \to 0$ implies $\lambda_n = o(1/\log(n))$ which is known to imply $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\to} 1$ a.s.; the result follows now from Theorem 2.

$\Rightarrow$: Assume that $\log(n_{k+1}-n_k)/k$ does not converge to 0. Let d denote the L_2 distance and let $N(\delta)$ denote the δ-covering number of $\bigcup_{k\geqslant 1}\tilde{A}_k$ with respect to d, where $\tilde{A}_k \equiv \{\sqrt{1/k}H(A_n) : n_k \leqslant n < n_{k+1}\}$. This means that $N(\delta)$ is the minimal number of balls with radius $\delta > 0$, in the d metric, that covers $\bigcup_{k\geqslant 1}\tilde{A}_k$. We prove that

$$\limsup_{\delta\to 0}\{\delta^2 \log N(\delta)\} > 0. \tag{3.12}$$

Formula (3.12), together with Sudakov's minorization Theorem (see Ledoux and Talagrand, 1991, Corollary 3.19, p. 81), implies that we cannot have $\sqrt{1/k}H(A_n) \underset{n\to\infty}{\to} 0$ a.s. and we are done. □

To start the proof of (3.12), let $N_k(x)$ denote the x-covering number of $\tilde{A}_k$. Our purpose is to get a lower bound for $N_k(x)$, which obviously will be also a lower bound for $N(x)$.

Let $1/k < \varepsilon < 1/2$ be fixed. Since for $n_k \leqslant n < m < n_{k+1}$

$$d^2(\sqrt{1/k}H(A_n), \sqrt{1/k}H(A_m)) = \#(C_n \Delta C_m)/k^2,$$

it follows that

$$d^2(\sqrt{1/k}H(A_n), \sqrt{1/k}H(A_m)) \leqslant \varepsilon k, \tag{3.13}$$

where # denotes the cardinality of a set. Since for fixed n the mapping $C_m \to C \equiv C_n \Delta C_m$ is one to one ($C = C_n\Delta C_m$ iff $C_m = C\Delta C_n$), we get

$$\#\{C_m : \#(C_n\Delta C_m) \leqslant \varepsilon k\} \leqslant m_k(\varepsilon), \tag{3.14}$$

where $m_k(\varepsilon)$ denotes the number of subsets of $\{1,\dots,k\}$ with cardinality smaller or equal to εk. We conclude from (3.13) and (3.14) that

$B(\sqrt{1/k}H(A_n),\sqrt{\varepsilon/k})$, a ball with radius $\sqrt{\varepsilon/k}$ and center at $\sqrt{1/k}H(A_n)$, contains at most $m_k(\varepsilon)$ members of $\tilde{A}_k$. Let [ ] denote the integer part. There exists $k(\varepsilon)$ so that

$$m_k(\varepsilon)=\sum_{j=0}^{[\varepsilon k]}\binom{k}{j}\leqslant(\varepsilon k+1))\binom{k}{[\varepsilon k]}\leqslant 2^{\beta(\varepsilon)k},\quad k>k(\varepsilon), \tag{3.15}$$

where $\beta(\varepsilon)\equiv\varepsilon-\log_2(\varepsilon^{\varepsilon}(1-\varepsilon)^{1-\varepsilon})\underset{\varepsilon\to 0}{\to}0$.

The first inequality in (3.15) is because $\binom{k}{j}$ is increasing on $0\leqslant j\leqslant[k/2]$, while the second follows from Stirling's formula. Alternatively, (3.15) follows from large deviation theory since $2^{-k}m_k(\varepsilon)=P(S_k\leqslant k\varepsilon)$, where S_k is distributed Binomial$(k,1/2)$. From (3.15) and the explanation that precedes it, we get that for $k>k(\varepsilon)$ the ball $B(\sqrt{1/k}H(A_n),\sqrt{\varepsilon/k})$ contains at most $2^{k\beta(\varepsilon)}$ members of $\tilde{A}_k$. The conclusion is that

$$N_k(\sqrt{\varepsilon/k}/2)\geqslant\frac{n_{k+1}-n_k}{2^{\beta(\varepsilon)k}},\quad k>k(\varepsilon), \tag{3.16}$$

because in $\tilde{A}_k$ there are $n_{k+1}-n_k$ members and each $\sqrt{\varepsilon/k}$-ball with center in $\tilde{A}_k$ contains at most $2^{\beta(\varepsilon)k}$ of them and so does an $\sqrt{\varepsilon/k}/2$-ball with arbitrary center. We now recall that $\log(n_{k+1}-n_k)/k$ does not converge to 0. This gives us the existence of $\alpha>0$ and a sequence $k_i\uparrow\infty$ so that $n_{k_i+1}-n_{k_i}\geqslant 2^{\alpha k_i}$, $i\geqslant 1$. From (3.16) we get now

$$N_{k_i}(\sqrt{\varepsilon/k_i}/2)\geqslant 2^{(\alpha-\beta(\varepsilon))k_i},\quad k_i>k(\varepsilon). \tag{3.17}$$

Since obviously $N(\sqrt{\varepsilon/k_i}/2)\geqslant N_{k_i}(\sqrt{\varepsilon/k_i}/2)$, we get from (3.17)

$$\frac{\varepsilon}{k_i}\log_2 N(\sqrt{\varepsilon/k_i}/2)\geqslant\varepsilon(\alpha-\beta(\varepsilon),\quad k_i>k(\varepsilon). \tag{3.18}$$

Now we fix $0<\varepsilon<1/2$ small enough so that $\alpha>\beta(\varepsilon)$ and (3.12) follows from (3.18).

4. Sufficient conditions for a.s. convergence

$$\text{Let } g(x)=\begin{cases}3x\log(1/x), & 0<x<\mathrm{e}^{-1},\\ 3\mathrm{e}^{-1}, & \mathrm{e}^{-1}\leqslant x\leqslant 1,\end{cases}$$

and let $h_H(x)$ be the (random) modulus of continuity of the Brownian motion H as defined in (1.1). It follows from a well-known result on Brownian motion modulus of continuity that there exists a random $\delta_H>0$, so that

$$P(h_H^2(x)\leqslant g(x),0<x<\delta_H)=1. \tag{4.1}$$

Next, we define

$$Y_n=\sum_{k=1}^{k_n}H(I_k^n)H'(I_k^n),\quad n\geqslant 1.$$

When we condition on H, $\{Y_n\}$ becomes a Gaussian process. We will soon estimate the L_2 distance of the process $\{Y_n\}$,

$$d_H^2(n,m) = E_H\{(Y_n - Y_m)^2\}.$$

We define first the functions $f_n : [0,1] \to [0,1]$

$$f_n(t) = \sum_{k=1}^{k_n} \sqrt{\lambda(I_k^n)} I_k^n(t), \quad n \geqslant 1,$$

where $I_k^n(t)$ is the indicator function of the interval I_k^n; similarly we define the functions that are based on all intersections of intervals in partitions n and m,

$$f_{n,m}(t) = \sum_{k,l} \sqrt{\lambda(I_k^n \cap I_l^m)} (I_k^n \cap I_l^m)(t), \quad 1 \leqslant n \leqslant m.$$

With the above notation $f_{n,n} = f_n$. The functions just defined are in $L_2[0,1]$ and the norm notation $\| \ \|$ will relate to that space.

Also, we say that the sequence of the partitions is "refined" if the set of points that generate the intervals is increasing in n, or more formally if $\{b_k^n\} \subset \{b_k^m\}$, $n \leqslant m$, where $I_k^n = [b_k^n, b_{k+1}^n]$, $1 \leqslant k \leqslant k_n$, $1 \leqslant n$.

We are ready to state the following

Lemma 6. *Assume* $\lambda_n \vee \lambda_m \leqslant \delta_H$. *Then*

(a) $d_H^2(n,m) \leqslant 4g(\|f_n\|^2 + \|f_m\|^2 - 2\|f_{n,m}\|^2)$, $m,n \geqslant 1$.
(b) *If in addition we assume that the sequence of partitions is refined then*

$$d_H^2(n,m) \leqslant 4g(\|f_n\|^2 - \|f_m\|^2), \quad 1 \leqslant n \leqslant m.$$

Proof of Lemma 2. *Part* (a): A simple calculation shows that

$$Y_n - Y_m = \sum_{k,l} \{H(I_k^n) - H(I_l^m)\} H'(I_k^n \cap I_l^m);$$

so we get

$$\begin{aligned} d_H^2(n,m) &= \sum_{k,l} \{H(I_k^n) - H(I_l^m)\}^2 \lambda(I_k^n \cap I_l^m) \\ &\leqslant \sum_{k,l} \{h_H(\lambda(I_k^n \setminus I_l^m)) + h_H(\lambda(I_l^m \setminus I_k^n))\}^2 \lambda(I_k^n \cap I_l^m) \\ &\leqslant 4 \sum_{k,l} g(\lambda(I_k^n \Delta I_l^m)) \lambda(I_k^n \cap I_l^m) \\ &\leqslant 4g\left(\sum_{k,l} \lambda(I_k^n \Delta I_l^m) \lambda(I_k^n \cap I_l^m) \right), \end{aligned}$$

where the second inequality follows from (4.1) and the assumption $\lambda_n \vee \lambda_m \leqslant \delta_H$ and the last inequality follows from Jensen's inequality as $g(x)$ is concave.

Part (a) now follows from the simple calculation

$$\sum_{k,l} \lambda(I_k^n \Delta I_l^m)\lambda(I_k^n \cap I_l^m) = \sum_{k,l} \{\lambda(I_k^n) + \lambda(I_l^m) - 2\lambda(I_k^n \cap I_l^m)\}\lambda(I_k^n \cap I_l^m)$$
$$= \|f_n\|^2 + \|f_m\|^2 - 2\|f_{n,m}\|^2.$$

Part (b): This follows immediately from part (a) because in the refined case

$$f_m = f_{n,m}, \quad n \leqslant m. \qquad \square$$

In order to set up the statement of the next theorem we define

$$a_n = 2 \sup_{m_2 \geqslant m_1 \geqslant n} \{\|f_{m_2}\|^2 - \|f_{m_1,m_2}\|^2\}. \tag{4.2}$$

Obviously $\{a_n\}$ converges to 0 monotonically, i.e. $a_n \downarrow 0$. We also define for $0 < \varepsilon < 1$

$$n_\varepsilon = \min\{k : a_k < g^{-1}(\varepsilon^2/4)/2\}, \tag{4.3}$$

where g^{-1} is the inverse function of g.

Proof of Theorem 4. We will work here given $\sigma(H)$, the σ-algebra generated by H. So we may and will assume, without loss of generality, that $\lambda_n \leqslant \delta_H,\ n \geqslant 1$. This allows us to use Lemma 6(a) without restrictions.

It follows from Lemma 6 and definition (4.2) that for $m \geqslant n$

$$d_H^2(n,m) \leqslant 4g(\|f_n\|^2 - \|f_m\|^2 + 2[\|f_m\|^2 - \|f_{n,m}\|^2])$$
$$\leqslant 4g(\|f_n\|^2 - \|f_m\|^2 + a_n).$$

We conclude that if $d_H(n,m) \geqslant \varepsilon$ then necessarily

$$\|f_n\|^2 - \|f_m\|^2 \geqslant g^{-1}(\varepsilon^2/4) - a_n.$$

From that and definition (4.3) we get that, if $d_H(n,m) \geqslant \varepsilon, n_\varepsilon \leqslant n < m$, then

$$\|f_n\|^2 - \|f_m\|^2 \geqslant (1/2)g^{-1}(\varepsilon^2/4). \tag{4.4}$$

Next we will estimate N_ε, the ε-covering number of the positive integers with respect to the random metric d_H. Recall that the ε-covering number is the minimal number of ε-radius balls that cover the space, i.e.

$$N_\varepsilon = \min\left\{ j : \exists n_1 < n_2 < \cdots < n_j \text{ so that } \bigcup_{1 \leqslant k \leqslant j} B(n_k, \varepsilon) = \{n \geqslant 1\} \right\},$$

where $B(n,\varepsilon) = \{m : d_H(n,m) \leqslant \varepsilon\}$. From (4.4) and the fact that $\|f_n\| \leqslant 1, \forall n$, it follows that for $\varepsilon > 0$ small enough the number of ε-balls needed to cover $\{n_\varepsilon \leqslant n\}$ is smaller than or equal to $2/g^{-1}(\varepsilon^2/4) \leqslant 9/\varepsilon^4$ (for the last inequality observe that $g^{-1}(x) \geqslant 4x^2$ for x small enough). From that we get for ε small enough the estimate

$$N_\varepsilon \leqslant n_\varepsilon + 9/\varepsilon^4. \tag{4.5}$$

Finally, it follows from Dudley's Theorem (see Jain and Marcus, 1978, p. 160) that given $\sigma(H)$, $\int_0^\infty \sqrt{\log(N_\varepsilon)}\,d\varepsilon < \infty$ is a sufficient condition for continuity of the sequence $Y_n = \sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n)$, $n \geqslant 1$, because $\{Y_n\}$ is Gaussian given $\sigma(H)$. Due to (4.5), and $N_\varepsilon = 1$ for $\varepsilon \geqslant 2$, the condition $\int_0^\infty \sqrt{\log(N_\varepsilon)}\,d\varepsilon < \infty$ follows from our assumption $\int_0^1 \sqrt{\log(n_\varepsilon)}\,d\varepsilon < \infty$. The continuity of the sequence $\{Y_n\}$ given $\sigma(H)$, implies that $P_H(Y_n \to 0) = 1$, which in turn implies $P(Y_n \to 0) = 1$. The theorem follows now from Theorem 1. □

The following two corollaries follow from Theorem 4. The first one was proved first by Lévy (1940). In current textbooks it is proved using reversed martingales, a method that is completely different from the one that we are using here.

Corollary 7. *If the sequence of partitions is refined then* $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ *a.s.*

Proof. In this case $a_n = 0$, $\forall n$ which implies $n_\varepsilon = 1$, $\forall \varepsilon > 0$. So $\int_0^1 \sqrt{\log(n_\varepsilon)}\,d\varepsilon < \infty$ is fulfilled in a trivial way and we use Theorem 4. □

For the next corollary of Theorem 4, we will assume, without loss of generality, that $\{\|f_n\|^2\}$ is a non-increasing sequence (that converges to 0). Also we let α denote the inverse function of $n \to \|f_n\|^2$, i.e.

$$\alpha(x) = \inf\{n : \|f_n\|^2 \leqslant x\}, \quad 0 < x \leqslant 1.$$

Finally we put $\rho_\varepsilon = \alpha(g^{-1}(\varepsilon^2/4)/4)$.

Corollary 8. *If* $\int_0^1 \sqrt{\log(\rho_\varepsilon)}\,d\varepsilon < \infty$, *then* $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ *a.s.*

Proof. Since $\{\|f_n\|^2\}$ is non-increasing it follows from (4.2) that $a_n \leqslant 2\|f_n\|^2$. From (4.3) it follows now that

$$n_\varepsilon \leqslant \alpha(g^{-1}(\varepsilon^2/4)/4),$$

and the result follows from Theorem 4. □

Remark 1. It is essentially known (see Dudley, 1973) that

$$\sum_{n=1}^{\infty} \exp(-c/\lambda_n) < \infty \quad \forall c > 0, \tag{4.6}$$

is a sufficient condition for $\sum_{k=1}^{k_n} H^2(I_k^n) \underset{n\to\infty}{\longrightarrow} 1$ a.s. To see it, for example, we may use Hanson and Wright (1971) to estimate $P(\sum_{k=1}^{k_n} H(I_k^n)H'(I_k^n) > \varepsilon)$ and then use the Borel–Cantelli lemma (and Theorem 1). Condition (4.6) should be compared with Corollary 8. In that regard observe that $\lambda_n^2 \leqslant \|f_n\|^2 \leqslant \lambda_n$.

Remark 2. Let I^n and $J^n, n = 1, 2, \ldots$, be two sequences of partitions of $[0, 1]$. Define $\rho_n = \{\sum_{k,l} \lambda(I_k^n \Delta J_l^n)\lambda(I_k^n \cap J_l^n)\}^{1/2}$. Observe that, in fact,

$$\rho_n = \left\| \sum_k H^2(I_k^n) - \sum_k H^2(J_k^n) \right\| \Big/ \sqrt{2}.$$

It follows from (11.23) in Ledoux and Talagrand (1991) that, for a universal constant $K > 0$, we have

$$P\left(\left| \sum_k H^2(I_k^n) - \sum_k H^2(J_k^n) \right| > c \right) \leqslant K \exp(-c/K\rho_n), \quad c > 0.$$

We conclude that if $\sum_{n=1}^{\infty} \exp(-c/\rho_n) < \infty$, $\forall c > 0$ then $\sum_k H^2(I_k^n)$ and $\sum_k H^2(J_k^n)$ converge to 1 a.s. or not, together. This remark is useful in construction of examples where the partition sequence is not refined, the mesh λ_n converges as slowly as we want to 0, but nonetheless there is a.s. convergence.

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