Periodic Solutions for Systems of Forced Coupled Pendulum-Like Equations

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The existence of periodic solutions for systems of forced pendulum-like equations was studied in the papers by J. A. Marlin (Internat. J. Nonlinear Mech. 3 (1968), 439–447) and J. Mawhin (Internat. J. Nonlinear Mech. 5 (1970), 335–339). In both works some symmetry hypotheses on the forcing terms were considered. This paper discusses the existence and multiplicity of periodic solutions of systems under consideration without any requirement on the symmetry of the forcing terms. Note that as a model example it is possible to consider the motion of \( N \) coupled pendulums (see the already mentioned paper by J. A. Marlin) or the oscillations of an \( N \)-coupled point Josephson junction with external time-dependent disturbances studied in the autonomous case by M. Levi, F. C. Hoppensteadt, and W. L. Miranker (Quart. Appl. Math. 36 (1978), 167–198).

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1. INTRODUCTION

The solvability of the periodic boundary value problem (PBVP) for a scalar forced ODE of pendulum type has been studied by several authors \([3, 4, 6–8, 10, 12, 18–22]\). We refer to Mawhin [18] for an historical survey of classical results in this direction, and to Fournier–Mawhin [8], Mawhin–Willem [19], and Kannan–Ortega [11] for recent and more precise characterizations of the forcing terms for which the PBVP for the forced scalar pendulum equation admits one or more solutions.

The PBVP for a system of linearly or nonlinearly coupled pendulums with external time-dependent disturbances was considered in Marlin [14],

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but assuming the existence of particular symmetries. The results of [14] are
generalized in Mawhin [15], but nevertheless some symmetry of the
forcing terms is still required. These symmetries reduce the kernel of the
linear part of the system. Therefore they can be considered as nonresonance
conditions (see also [18] for a discussion of their role).

Motivated by this research, we study here the existence of solutions for
the PBVP
\[ x'' + \sigma x' + Ax + f(x) + g(x) = h(t), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^N, \]  
\[ x(T) = x(0), \quad x'(T) = x'(0). \]  

We assume throughout the paper that \( \sigma \) is a real number and that \( A \) is a
real \( N \times N \) constant matrix such that

(H1) 0 is an eigenvalue of \( A \) with geometric multiplicity 1, and no
other eigenvalue of \( A \) has the form \( m^2 \omega^2 - im\omega \) with \( m \in \mathbb{Z} \).

As usual, \( \omega := 2\pi/T \). Moreover we assume that

(H2) \( f: \mathbb{R}^N \to \mathbb{R}^N \) is continuous and bounded, say \( |f| \leq M_f \), and
there are functions \( f_j: \mathbb{R} \to \mathbb{R} \), \( j = 1, \ldots, N \), such that
\[ f(x) = (f_1(x_1), f_2(x_2), \ldots, f_N(x_N)) \]
for all \( x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \),

(g) \( g: \mathbb{R}^N \to \mathbb{R}^N \) is continuous and bounded, say \( |g| \leq M_g \),

(h) \( h \in L^\infty := L^\infty([0, T], \mathbb{R}^N) \).

For example, the symmetric tridiagonal matrix
\[ A = N^2 = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & \cdots & 0 & -1 & 1 \\
\end{bmatrix} \quad (1.3) \]

verifies (H1) (assuming \( T \) small enough when \( \sigma = 0 \)). The field \( f \) defined by
\[ f(x) = (\sin(x_1), \sin(x_2), \ldots, \sin(x_N)) \]  
(1.4)

verifies (H2)(f). We remark that, if (1.3), (1.4) hold and \( g \equiv 0 \), then (1.1) is
a model not only for the motion of a system of $N$ linearly coupled pendulums with external disturbances and a possible viscous damping, but also for the dynamics of an $N$-coupled point Josephson junction, which is important in the theory of superconductivity: see, e.g., Levi-Hoppensteadt-Miranker [13].

The paper has five sections. In Section 2 we recall some basic facts concerning the linear part of the PBVP (1.1), (1.2). In Section 3 we prove some existence results for an abstract equation which contains (1.1), (1.2) as a very particular case. The technique we use is essentially the Alternative Method. The uniform boundedness of $f + g$, assumed in (H2), allows us to solve the auxiliary equation using simply the Schauder's fixed point principle, while the bifurcation equation is solved by the connectivity argument, based on the addition–excision property of Leray–Schauder degree, which is presented in Amann–Ambrosetti–Mancini [1] (see also Dancer [6]). This argument works well when the bifurcation equation is 1-dimensional: this is true in our case because of the simplicity of the eigenvalue 0 of $A$, assumed in (H1). Section 4 contains two theorems. The first one concerns the case of a field $f$ whose components $f_j$ verify some sign-conditions with respect to the components of fixed unit vectors spanning the kernels of $A$ and $A^*$. In the second theorem we prove the existence of at least two distinct solutions to (1.1), (1.2) if all $f_j$'s are periodic with the same period and some other technical assumptions are satisfied. Both theorems are applications of the abstract result of Sections 3. In Section 5 we give two examples illustrating the use of the results obtained in the preceding analysis.

Since (H1) implies $A = 0$ when $N = 1$, we will assume $N \geq 2$.

2. THE LINEAR EQUATION

Let $T > 0$, $N \geq 2$. We define on

$$\text{dom}(L) := \{ x \in H^{2,\infty}([0, T], \mathbb{R}^N) \mid x^{(k)}(T) = x^{(k)}(0), k = 0, 1 \}$$

the vector differential operator

$$L: \text{dom}(L) \subset L^\infty := L^\infty([0, T], \mathbb{R}^N) \rightarrow L^\infty$$

$$Lx := x'' + \sigma x' + Ax,$$

where $'=d/dt$, $\sigma$ is a real number and $A$ is a matrix which verifies (H1). The classical linear theory [9] shows that $L$ is a linear Fredholm operator with index 0, and that $\ker L = \ker A$, $\ker L^* = \ker A^*$. Since $\dim \ker A = 1$,
we find a unit vector $\phi$ (resp. $\phi^*$) in $\mathbb{R}^N$ spanning ker $A$ (resp. ker $A^*$). Choosing projectors $P$, $Q$ in $L^\infty$

$$PX := \left( T^{-1} \int_0^T x, \phi \right) \phi, \quad Qy := \left( T^{-1} \int_0^T y, \phi^* \right) \phi^*,$$

we denote by $K_{P,Q} : L^\infty \to L^\infty$ the generalized (right) inverse of $L$ relative to $P$ and $Q$ [17]. It is completely continuous.

Let us fix $x \in \text{dom}(L) \cap \text{ker} P$, and $y \in L^\infty$ such that $x = K_{P,Q} y$. Let $A^+$ be the (Moore–Penrose) pseudoinverse of $A$, and let $L_m^{-1}$ be the (two-sided) inverse of $L_m := (-m^2 \omega^2 + im \omega) I + A$, $m \in \mathbb{Z} \setminus \{0\}$. Then a well-known technique based on Fourier series and Parseval identity yields (see [5, 16])

$$|x|_\infty \leq \left( \| A^+ \|^2 + \sum_{m \neq 0} \| L_m^{-1} \|^2 \right)^{1/2} |y|_\infty. \quad (2.1)$$

Namely, let $\sum_{m \in \mathbb{Z}} \hat{x}_m e^{im\omega t}$ and $\sum_{m \in \mathbb{Z}} \hat{z}_m e^{im\omega t}$ be the Fourier series associated, respectively, to $x$ and $z := (I - Q) y$. Then $(\hat{z}_0, \phi^*) = 0$, $A \hat{x}_0 = \hat{z}_0$, with $(\hat{x}_0, \phi) = 0$, i.e., $\hat{x}_0 = A^+ \hat{z}_0$, and $\hat{x}_m = L_m^{-1} \hat{z}_m$ for $m \neq 0$. Therefore,

$$|x|_\infty \leq |\hat{x}_0| + \left| \sum_{m \neq 0} \hat{x}_m e^{im\omega t} \right|_\infty \leq |A^+| |\hat{z}_0| + \sum_{m \neq 0} \| L_m^{-1} \| \cdot |\hat{z}_m|$$

$$\leq \| A^+ \| |\hat{z}_0| + \left( \sum_{m \neq 0} \| L_m^{-1} \|^2 \right)^{1/2} \left( \sum_{m \neq 0} |\hat{z}_m|^2 \right)^{1/2}$$

$$\leq \left( \| A^+ \|^2 + \sum_{m \neq 0} \| L_m^{-1} \|^2 \right)^{1/2} \left( T^{-1} \int_0^T |(I - Q) y|^2 \right)^{1/2}$$

$$\leq \left( \| A^+ \|^2 + \sum_{m \neq 0} \| L_m^{-1} \|^2 \right)^{1/2} \left( T^{-1} \int_0^T |y|^2 \right)^{1/2}$$

being $I - Q$, $Q$ a pair of $L^2$-orthogonal projectors.

We will use in the sequel the following lemma.

**Lemma 1.** $\sum_{m \neq 0} \| L_m^{-1} \|^2 \downarrow 0$ as $T \downarrow 0$.

**Proof.** For $m \neq 0$, $\omega > 0$, define $z(m, \omega) := -m^2 \omega^2 + im \omega$. Fix $\omega_0$ so large that

$$\| A \| \leq |z(1, \omega)|/4 \quad (2.2)$$

for $\omega \geq \omega_0$. Let $\lambda_1(H)$ be the first eigenvalue of a selfadjoint matrix $H$, we have
\[ \|L_m^{-1}\|^2 = (\lambda_1(L^*_m L_m))^{-1} \]
\[ = ((z(m, \omega))^2 + \lambda_1(z(m, \omega) A + z(m, \omega) A^* A))^{-1} \]
\[ \leq ((z(m, \omega))^2 + \lambda_1(z(m, \omega) A + z(m, \omega) A^*))^{-1} \]
\[ \leq (|z(m, \omega)|^2 - 2 |z(m, \omega)| \cdot \|A\|)^{-1} \]
\[ \leq 2 |z(m, \omega)|^{-2} \]
\[ = 2(m^4 \omega^4 + m^2 \omega^2 \sigma^2)^{-1}. \]  

**Remark 1.** When \( A \) is symmetric and positive semidefinite, as in (1.3), then (2.2) is not needed. In fact, in this case,
\[ \lambda_1(z(m, \omega) A + z(m, \omega) A^*) = 2 \text{Re} z(m, \omega) \lambda_1(A) = 0, \]
and so
\[ \|L_m^{-1}\|^2 \leq (m^4 \omega^4 + m^2 \omega^2 \sigma^2)^{-1}. \]

**Remark 2.** An explicit upper bound for \( \sum_{m \neq 0} \|L_m^{-1}\|^2 \) can be easily derived from (2.3) or (2.4). Let \( \rho_m := m^4 \omega^4 + m^2 \omega^2 \sigma^2 \), and fix a positive integer \( m_0 \). Then
\[ \sum_{m \neq 0} \rho_m^{-1} \leq 2 \sum_{m > 0} m^{-4} \omega^{-4} - 2 \omega^{-2} \sum_{1 \leq m \leq m_0} \sigma^2 m^{-2} \rho_m^{-1} \]
\[ \leq \omega^{-4} \pi^4 / 45 - 2 \omega^{-2} \sum_{1 \leq m \leq m_0} \sigma^2 m^{-2} \rho_m^{-1}. \]

3. **Bounded Perturbations of the Linear Equation**

In this section \( S \) is an abstract continuous bounded map \( L^\infty \rightarrow L^\infty \), \( |S(x)|_\infty \leq M \) for all \( x \in L^\infty \). We consider the solvability of
\[ Lx + S(x) = h, \quad x \in \text{dom}(L), \]  
where \( L \) is as in Section 2, and \( h \in L^\infty \) is fixed. It is obvious that (3.1) is equivalent to the system
\[ u + K_{P, Q} S(r \phi + u) = K_{P, Q} h, \]  
\[ QS(r \phi + u) = Qh, \]
where \( r := (Px, \phi), \ u := x - r \phi \). Recall that \( (I - Q) h = (I - Q) k \) implies \( K_{P, Q} h = K_{P, Q} k \), by the definition of \( K_{P, Q} \).
Theorem 1. Let
\[ \Sigma := \{ (r, u) \in \mathbb{R} \times \ker P \mid (3.1a) \text{ holds} \}, \]
\[ p_1(\Sigma) := \{ r \in \mathbb{R} \mid \exists u \in \ker P \text{ such that } (r, u) \in \Sigma \}. \]
Then:

(i) \[ p_1(\Sigma) = \mathbb{R}, \]

(ii) for each \( e \in \ker Q \) there exists at least one \( \rho, -M \leq \rho \leq M \), such that \( (3.1) \) with \( h := \rho \phi^* + e \) is solvable.

Proof. Apply Schauder's fixed point principle. 

Theorem 1 gives very rough information about the right-hand side \( h \) for which \( (3.1) \) is solvable: to obtain more precise results we will use a simple and elegant argument due to Amann–Ambrosetti–Mancini [1] (see also Dancer [6]). Let \( \Sigma \) be as in Theorem 1, and define a map
\[ \Gamma: \Sigma \to \mathbb{R}, \quad \Gamma(r, u) := (Q(S(r\phi + u) - h), \phi^*). \]

Theorem 2. Suppose that there are real numbers \( r_-, r_+ \) such that
\[ Q(r_-, 0) \in \mathcal{D}(r_+, w) \]
for all \( v, w \in \ker P \) with \( (r_-, v), (r_+, w) \in \Sigma \). Then \( (3.1) \) is solvable.

Proof. Let \( \alpha := \max \{ |r_-|, |r_+| \} \). Using degree theory as in [1, 6], we find a connected subset \( \Sigma_\alpha \) of \( \Sigma \) such that
\[ [-\alpha, \alpha] \subset p_1(\Sigma_\alpha) := \{ r \in \mathbb{R} \mid \exists u \in \ker P \text{ such that } (r, u) \in \Sigma_\alpha \} . \]

The continuity of \( \Gamma \), the connectedness of \( \Sigma_\alpha \) and the inequalities \( (3.2) \) give the existence of a point \( (r, u) \in \Sigma \) (i.e., such that \( (3.1a) \) holds) where \( \Gamma \) vanishes (i.e., \( (3.1b) \) holds, too).

We will introduce, in the next section, some special nonlinearities \( S \) for which \( (3.2) \) holds.

4. Periodic Perturbations of the Linear Equation

We suppose here that \( A, f, g, h \) verify (H1), (H2). If we define \( S: L^{\infty} \to L^{\infty} \) as the Nemyckii operator induced by \( f + g \), then the PBVP \( (1.1), (1.2) \) is clearly equivalent to the abstract equation \( (3.1) \). To apply Theorem 2, that is to verify \( (3.2) \), we introduce the following sign-condition on the components \( f_j \) with respect to the components \( \phi_j \) and \( \phi_j^* \), \( j = 1, \ldots, N \):
(H3) there are real numbers $r_+, r_-, \delta > 0$, and $\tau \geq 0$ such that
\[ |s - r_\pm \phi_j| < \delta \quad \text{implies} \quad \pm f_j(s) \geq \tau \]
for all indices $j \in J_+ := \{ j \mid 1 \leq j \leq N, \phi_j^* > 0 \}$, and $|s - r_\mp \phi_j| \leq \delta$ implies $\pm f_j(s) \geq \tau$ for all indices $j \in J_- := \{ j \mid 1 \leq j \leq N, \phi_j^* < 0 \}$.

Remark that (H3) does not concern indices in $\{1, 2, \ldots, N\} \setminus (J_+ \cup J_-)$.

**THEOREM 3.** Assume (H1), (H2), (H3). Moreover, suppose that

(i) $\|K_{p,q}\|_\infty (M_f + M_g + \| (I - Q) h \|_\infty ) \leq \delta$,

(ii) $M_g \leq \tau$,

(iii) $Qh = 0$.

Then the PBVP (1.1), (1.2) is solvable.

**Proof.** We have only to show that (3.2) is fulfilled. The assumption (iii)
implies that
\[ f(r, u) = (Qs(r + u), \phi^*) \]
\[ = T^{-1} \int_0^T \sum_{j=1,N} (f_j(r\phi_j + u_j) + g_j(r\phi + u)) \phi_j^*. \]
If we take $r_{\pm}$ from (H3), then it is easy to check (using (i) and (ii)) that
\[ \Gamma(r_-, v) \leq 0 \leq \Gamma(r_+, w) \]
for all $v$ and $w$ in ker $P$ with $(r_-, v)$ and $(r_+, w)$ in the set $\Sigma$.

Theorem 3 is a natural one. It states that the range of $L$ is contained in the range of $L + S$ when some a priori estimates and some sign-conditions hold. Note that trivial generalizations of Theorem 3 can be obtained assuming, e.g., that the map $g$ depends also on $t$, $0 \leq t \leq T$, and on $x'$. In this case we have only to assume that $g = g(t, x, y)$ is Carathéodory and bounded, and work in $H^{1,\infty}$ instead of $L^\infty$. Namely, the Nemyckii operator induced by $g$ can be substituted with any continuous one $L^\infty \to L^\infty$, or $H^{1,\infty} \to L^\infty$, with range contained in the ball centered at 0 with radius $M_g$.

However, in spite of its generality, Theorem 3 can give more precise informations on the range of $x \to Lx + f(x)$ looking to $g$ as to a perturbation of $h \in \text{Im} L$ instead of a perturbation of $f$. To do this, motivated by the applications (see next section), we restrict the classes of the matrices $A$ and of the fields $f$ under consideration assuming that:

(H1') $A = (a_{ij})$ verifies (H1); moreover, for $i = 1, 2, \ldots, N$,
\[ \sum_{j=1,N} a_{ij} = 0, \quad \phi_i^* \geq 0. \]
(H2') there are numbers $b_j > 0$, $j = 1, 2, ..., N$, and a $p$-periodic ($p > 0$) continuous map $\Psi : \mathbb{R} \to \mathbb{R}$ such that
\[ |\Psi(s)| \leq 1 \quad \text{for all } s \in \mathbb{R}, \]
\[ \Psi(s) > 0 \ (\text{resp. } < 0) \text{ for } p_0 < s < p_1 \ (\text{resp. } p_1 < s < p_0 + p), \]
\[ p_0 < p_1 < p_0 + p, \]
\[ f(x) := (b_1 \Psi(x_1), b_2 \Psi(x_2), ..., b_N \Psi(x_N)) \]
\[ \text{for all } x = (x_1, ..., x_N) \in \mathbb{R}^N. \]

Obviously (4.1) → (4.2) when $A$ is symmetric, because (4.1) implies that $\ker A$ is spanned by $\phi = N^{-1/2}(1, 1, ..., 1)$. A model example for $\Psi$ is $\Psi = \sin$, with $p_0 = 0$, $p_1 = p/2 = \pi$, like in the pendulum-type equations, but other periodic maps can be considered as well.

We can now state the main result of the paper.

**Theorem 4.** Assume (H1'), (H2'). Let $e \in L^\infty$ be given. Suppose that there exists a number $\delta$ such that
\[ 0 < \delta < \min\{(p_1 - p_0)/2, (p_0 + p - p_1)/2\}, \tag{4.3} \]
\[ \|A^+\left(\left(\sum_{j=1,N} b_j^2\right)^{1/2} + |(I - Q)e|_\infty\right)\| < \delta, \tag{4.4} \]
and that, for all $j = 1, ..., N$,
\[ |(Qe)_j| < \tau, \tag{4.5} \]
where $\tau$ is a positive number defined by
\[ \tau := \min\{\{|\Psi(s)| : |s - r_1| \leq \delta\} \cup \{-|\Psi(s)| : |s - r_2| \leq \delta\}\} \cdot \min\{b_j : j = 1, ..., N\}, \]
with
\[ r_1 := (p_0 + p_1)/2, \quad r_2 := r_1 + p/2, \quad r_0 := r_1 - p/2. \]

Then there exists an explicitly computable number $T_0 > 0$ such that, for $T \leq T_0$, the PBVP
\[ x_i'' + \sigma x_i' + \sum_{j=1,N} a_{ij} x_j + b_i \Psi(x_i) = e_i(t), \quad 0 \leq t \leq T \]
\[ x_i(T) = x_i(0), \quad x_i'(T) = x_i'(0) \]
has at least two solutions which do not differ by an integer multiple of the vector $p(1, 1, ..., 1) = (p, p, ..., p)$. 
**Proof.** We use twice our Theorem 3, letting
\[ g := -Qe, \quad h := (I - Q)e. \]
Note that in this case Theorem 3 holds when (i) is substituted by the weaker assumption
\[ (i') \|K_{P,Q}\|_\infty (M_I + \|I - Q\|h\|_\infty) \leq \delta, \]
and (ii) by
\[ (ii') \sup \{|g_i(x)| \mid i = 1, \ldots, N, x \in \mathbb{R}^N\} \leq \tau. \]
By (4.4) and Lemma 1 there is \( T_0 > 0 \) such that
\[ (\left\| A^+ \right\|^2 + \sum_{m \neq 0} \|L_m^{-1}\|^2 \right)^{1/2} \left( \left( \sum_{j=1,N} b_j^2 \right)^{1/2} + \|I - Q\|e\|_\infty \right) \leq \delta \quad (4.7) \]
whenever \( T \leq T_0 \). Using (2.1) we see that (4.7) implies the inequality (i').
Inequality (4.5) implies that assumption (ii') holds with strict inequality sign. Assumption (iii) holds by the definition of \( h \). It remains only to check (H3). But choosing
\[ r_+ := N^{1/2}r_1, \quad r_- \in \{N^{1/2}r_0, N^{1/2}r_2\} \]
we trivially get that
\[ |s - r_\pm N^{-1/2}| \leq \delta \Rightarrow \pm b_j \mathcal{P}(s) \geq \tau \]
for all \( j \in J_+ \) (recall that \( J_- = \emptyset \), by (4.2)). Thus (H3) holds in two different settings, namely \((r_+, r_-) = (N^{1/2}r_1, N^{1/2}r_0), (r_-, r_+) = (N^{1/2}r_1, N^{1/2}r_2)\). Moreover, by the strict inequality sign in (ii'), we have
\[ (-1)^l \Gamma(N^{1/2}r_l, u) < 0 \]
for all \((N^{1/2}r_l, u) \in \Sigma, l = 0, 1, 2\). Thus Theorem 3 gives the existence of solutions \( x^{(l)} := r^{(l)}(1, 1, \ldots, 1) + u^{(l)}, l = 0, 1, \) of the PBVP (4.6), with \( u^{(0)} \) and \( u^{(1)} \) in ker \( P \) and \( r_0 < r^{(0)} < r_1 < r^{(1)} < r_2 \).
If \( x^{(1)} - x^{(0)} = \mu \rho(1, 1, \ldots, 1) \) for some \( \mu \in \mathbb{Z} \), then \( u^{(1)} - u^{(0)} = 0 \) and \( r^{(1)} - r^{(0)} = \mu \rho \). Since \( r^{(1)} - r^{(0)} < r_2 - r_0 = \rho \), we must have \( \mu = 0 \).

5. Applications

As a first application of Theorem 4, we consider the PBVP on the interval \([0, T]\) for the 2-dimensional system
\[
\begin{align*}
x''_1 + \sigma x'_1 + a_1(x_1 - x_2) + b_1 \sin(x_1) &= e_1(t), \\
x''_2 + \sigma x'_2 + a_2(x_2 - x_1) + b_2 \sin(x_2) &= e_2(t),
\end{align*}
\tag{5.1}
\]
where \( a_i, b_i \) \((i = 1, 2)\) are strictly positive numbers and \( e = (e_1, e_2) \) is in \( L^\infty \).

See Marlin [14] for a mechanical interpretation of (5.1). The matrix

\[
A = \begin{bmatrix}
  a_1 & -a_1 \\
  -a_2 & a_2
\end{bmatrix}
\]

verifies (H1'). Namely, its eigenvalues are \( \lambda_1 = 0, \lambda_2 = a_1 + a_2 \). If \( \sigma = 0 \) we have to choose \( T \) so small that \( \lambda_2 < \omega^2 \). The kernel of \( A \) (resp. of \( A^* \)) is spanned by \( \phi = 2^{-1/2}(1, 1) \) (resp. by \( \phi^* = (a_1^2 + a_2^2)^{-1/2}(a_2, a_1) \)). The norm of \( A \) is \( \| A \| = 2^{1/2}(a_1^2 + a_2^2)^{1/2} \). The pseudoinverse of \( A \) is \( A^+ = 2^{-1}(a_1^2 + a_2^2)^{-1} A^* \), so that \( \| A^+ \| = 2^{-1/2}(a_1^2 + a_2^2)^{-1/2} \). Inequality (4.3) is \( 0 < \delta < \pi/2 \), and \( \tau = \cos(\delta) \cdot \min\{b_1, b_2\} \). Let \( \tilde{e} := T^{-1} \int_0^T e \). Thus \( Qe = (\tilde{e}, \phi^*) \phi^* \). A direct application of Theorem 4 gives the

**Corollary 1.** Suppose that there is a number \( \delta, 0 < \delta < \pi/2 \) such that

\[
2^{-1/2}(a_1^2 + a_2^2)^{-1/2}((b_1^2 + b_2^2)^{1/2} + \sup_{0 \leq t \leq T} (\tilde{e}_1^2 + \tilde{e}_2^2)^{1/2}) < \delta,
\]

\[
\frac{\max\{a_1, a_2\}}{(a_1^2 + a_2^2)} \left| a_2 \tilde{e}_1 + a_1 \tilde{e}_2 \right| < \cos(\delta) \cdot \min\{b_1, b_2\},
\]

where

\[
\tilde{e}_i = T^{-1} \int_0^T e_i, \quad i = 1, 2,
\]

\[
\tilde{e}_i - e_i - (\tilde{e}_i a_j^2 + \tilde{e}_j a_i a_j)/(a_i^2 + a_j^2), \quad \{i, j\} = \{1, 2\}.
\]

Then the PBVP on \([0, T]\) for (5.1) has at least two solutions which do not differ by an integer multiple of \( 2\pi(1, 1) \), provided \( T \) is small enough, say \( T \leq T_0 \).

**Remark 3.** The computation of \( T_0 \) is easy. Here, and in each other particular case of Theorem 4, we have only to recall the upper bound of \( \sum_{m \neq 0} \| L_m^{-1} \|^2 \) derived in Remark 2 in terms of integer powers of \( T \), and apply it in (4.7).

**Remark 4.** If \( e \) is the restriction on the interval \([0, T]\) of an odd \( T \)-periodic function, then \( \tilde{e} = 0, \tilde{e} = e \). More generally, when \( \tilde{e} = 0 \) the inequality (5.3) holds for any \( \delta, 0 < \delta < \pi/2 \), and so the conclusion of Corollary 1 still holds with the unique assumption that

\[
2^{-1/2}(a_1^2 + a_2^2)^{-1/2}((b_1^2 + b_2^2)^{1/2} + |e|_{\infty}) < \pi/2.
\]
A second application of Theorem 4 concerns the PBVP (4.6) with $A = (a_{ij})$ defined in (1.3), $\Psi = \sin$, $b_j = 1$, $j = 1, \ldots, N$. We have recalled in the Introduction that this particular form of (4.6) represents, for example, the oscillations of an $N$-coupled point Josephson junction with external time-dependent disturbances. Now the matrix $A$ is symmetric, and $\phi = \phi^* = N^{-1/2}(1, 1, \ldots, 1)$. Inequality (4.3) is $0 < \delta < \pi/2$, and $\tau = \cos(\delta)$. Moreover

$$Qe = \left( \sum_{j=1}^{N} \tilde{e}_j N^{-1/2} \right) \phi^* = \left( N^{-1} \sum_{j=1}^{N} \tilde{e}_j \right) (1, 1, \ldots, 1),$$

and

$$(I - Q)e = e - \left( N^{-1} \sum_{j=1}^{N} \tilde{e}_j \right) (1, 1, \ldots, 1).$$

Therefore we have from Theorem 4 the

**COROLLARY 2.** The PVBP on $[0, T]$ for the system

$$x'' + \sigma x' + N^2 \begin{bmatrix} 1 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 \end{bmatrix} x + \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \\ \vdots \\ \sin(x_N) \end{bmatrix} = e$$

has at least two solutions not differing by an integer multiple of the vector $2\pi(1, 1, \ldots, 1)$, when $T$ is sufficiently small and the inequalities

$$\|A^+\left(1 + \sup_{0 \leq t \leq T} \left( \sum_{j=1}^{N} \tilde{e}_j^2 \right)^{1/2} \right) < \delta,$$

$$|\bar{e}| < \cos(\delta),$$

hold for some $\delta$, $0 < \delta < \pi/2$, with

$$\bar{e} := N^{-1} \sum_{j=1}^{N} T^{-1} \int_{0}^{T} e_j,$$

$$\tilde{e} := e_j \bar{e}, \quad j = 1, \ldots, N,$$

$A^+ := the pseudoinverse of (1.3)$. 

Remark 5. Obviously (5.5) would be inconsistent if \( \|A^+\| \geq \pi/2 \). Let us compute \( \|A^+\| \). Define \( J := N^{-2}A \). By elementary linear algebra we get that

1. \( \|A^+\| = \|(N^2J)^+\| = \|N^{-2}J^+\| = N^{-2} \|J^+\| \),
2. the eigenvalues of \( J^+ \) are \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N \) where \( 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N \) are the eigenvalues of \( J \),
3. \( J^+ \) is symmetric, so that \( \|J^+\| = \lambda_2^{-1} \).

Now \( J \) being symmetric and tridiagonal, its first non-zero eigenvalue \( \lambda_2 \) can be computed with high accuracy by the bisection method [2]. We do not give here numerical details, like errors estimates. We report only the following results, obtained by the bisection method:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( |A^+| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.125</td>
</tr>
<tr>
<td>3</td>
<td>0.111 111 ...</td>
</tr>
<tr>
<td>4</td>
<td>0.106 694 ...</td>
</tr>
<tr>
<td>5</td>
<td>0.104 741 ...</td>
</tr>
<tr>
<td>10</td>
<td>0.102 158 ...</td>
</tr>
<tr>
<td>20</td>
<td>0.101 529 ...</td>
</tr>
</tbody>
</table>

Recall that \( \pi/2 = 1.570 \ldots \).

Remark 6. Note that the sufficient conditions on the forcing terms derived here in order to obtain existence and multiplicity results for the preceding PBVPs are analogous to these imposed in [3, 4, 6-8, 10-12, 18-21] for the scalar case \( N = 1 \).

References