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On the chaotic behavior of a generalized logistic p-adic dynamical system

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Abstract

In the paper we describe basin of attraction p-adic dynamical system $G(x) = (ax)^2(x + 1)$. Moreover, we also describe the Siegel discs of the system, since the structure of the orbits of the system is related to the geometry of the p-adic Siegel discs.
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1. Introduction

Applications of p-adic numbers in p-adic mathematical physics [6,19,35,49,50], quantum mechanics [26] and many others [27,48] stimulated increasing interest in the study of p-adic dynamical systems. Note that the p-adic numbers were first introduced by the German mathematician K. Hensel. During a century after their discovery they were considered mainly objects of pure mathematics. Starting from 1980's various models described in the language of p-adic analysis have been actively studied.

On the other hand, the study of p-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic vertices over a number field, as in [14]. In [29,47] the p-adic field has arisen in physics in the theory of superstrings, promoting questions about their dynamics. Also some applications of p-adic dynamical

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systems to some biological, physical systems were proposed in [3,4,8,17,30,31]. In [9,32] dynamical systems (not only monomial) over finite field extensions of the p-adic numbers were considered. Other studies of non-Archimedean dynamics in the neighborhood of a periodic and of the counting of periodic points over global fields using local fields appeared in [20,21,33,34, 41]. Certain rational p-adic dynamical systems were investigated in [23,36,37], which appear from problems of p-adic Gibbs measures [24,38–40]. Note that in [12,13,43] a general theory of p-adic rational dynamical systems over complex p-adic filed $\mathbb{C}_p$ has been developed. In [10,11] the Fatou set of a rational function defined over some finite extension of $\mathbb{Q}_p$ has been studied. Besides, an analogue of Sullivan's no wandering domains theorem for p-adic rational functions, which have no wild recurrent Julia critical points, was proved.

The most studied discrete p-adic dynamical systems (iterations of maps) are the so-called monomial systems. In [5,28] the behavior of a p-adic dynamical system $f(x) = x^n$ in the fields of p-adic numbers $\mathbb{Q}_p$ and $\mathbb{C}_p$ was investigated. In [31] perturbated monomial dynamical systems defined by functions $f_q(x) = x^n + q(x)$, where the perturbation $q(x)$ is a polynomial whose coefficients have small p-adic absolute value, have been studied. It was investigated the connection between monomial and perturbated monomial systems. Formulas for the number of cycles of a specific length to a given system and the total number of cycles of such dynamical systems were provided. These investigations show that the study of perturbated dynamical systems is important. Even for a quadratic function $f(x) = x^2 + c$, $c \in \mathbb{Q}_p$, its chaotic behavior is complicated (see [4,46,47]). In [16,46] the Fatou and Julia sets of such a p-adic dynamical system were found. Certain ergodic and mixing properties of monomial and perturbated dynamical systems have been considered in [1,18].

The aim of this paper is to investigate the asymptotic behavior of a nonlinear p-adic dynamical system, especially a generalized p-adic logistic map $G(x) = (ax)^2(x + 1)$. Note that the logistic map $f(x) = Cx(1 + x)$ and generalized logistic maps are well known in the literature and it is of great importance in the study of dynamical systems (see [7,15,22]). Much is known about the behavior of the dynamics of the orbits of a p-adic analog of the logistic map (see [46,47]). On the other hand, our dynamical system is also a perturbated cubic dynamical system, since it can be reduced to the form $f(x) = x^3 + ax^2$. In the paper we will consider all possible cases of the perturbated term ax^2 with respect to the parameter a. Note that globally attracting sets play an important role in dynamics, restricting the asymptotic behavior to certain regions of the phase space. However, descriptions of the global attractor can be difficult as it may contain complicated chaotic dynamics. Therefore, in the paper we will investigate the basin of attraction of such a dynamical system. Moreover, we also describe the Siegel discs of the system, since the structure of the orbits of the system is related to the geometry of the p-adic Siegel discs (see [2]).

2. Preliminaries

2.1. *p-Adic numbers*

Let $\mathbb{Q}$ be the field of rational numbers. Throughout the paper p will be a fixed prime number. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer and p, n, m are relatively prime. The p-adic norm of x is given by $|x|_p = p^{-r}$ and $|0|_p = 0$. This norm satisfies so-called the strong triangle inequality

$$|x + y|_p \leqslant \max\{|x|_p, |y|_p\}.$$

From this inequality one can infer that

$$\text{if } |x|_p \neq |y|_p, \quad \text{then } |x-y|_p = \max\{|x|_p, |y|_p\}, \tag{2.1}$$

$$\text{if } |x|_p = |y|_p, \quad \text{then } |x-y|_p \leqslant |2x|_p. \tag{2.2}$$

This is a ultrametricity of the norm. The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$. Note that any p-adic number $x \neq 0$ can be uniquely represented in the canonical series:

$$x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right), \tag{2.3}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leqslant x_j \leqslant p-1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$ (see more detail [25,44,45]). Observe that in this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called *a quadratic residue modulo p* if the equation $x^2 \equiv a \pmod{p}$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. *(See [45,48].) In order that the equation*

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + \cdots), \quad 0 \leqslant a_j \leqslant p-1,\ a_0 > 0,$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are satisfied:

(i) *$\gamma(a)$ is even*;
(ii) *a_0 is a quadratic residue modulo p if $p \neq 2$, if $p = 2$ besides $a_1 = a_2 = 0$.*

For any $a \in \mathbb{Q}_p$ and $r > 0$ denote

$$\bar{B}_r(a) = \{x \in \mathbb{Q}_p\colon\ |x-a|_p \leqslant r\}, \qquad B_r(a) = \{x \in \mathbb{Q}_p\colon\ |x-a|_p < r\},$$
$$S_r(a) = \{x \in \mathbb{Q}_p\colon\ |x-a|_p = r\}.$$

A function $f : B_r(a) \to \mathbb{Q}_p$ is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n, \quad f_n \in \mathbb{Q}_p,$$

which converges uniformly on the ball $B_r(a)$.

Note the basics of p-adic analysis, p-adic mathematical physics are explained in [25,44,45, 48].

2.2. *Dynamical systems in $\mathbb{Q}_p$*

In this subsection we recall some standard terminology of the theory of dynamical systems (see for example [31,42]).

Consider a dynamical system (f, B) in $\mathbb{Q}_p$, where $f : x \in B \to f(x) \in B$ is an analytic function and $B = B_r(a)$ or $\mathbb{Q}_p$. Denote $x^{(n)} = f^n(x^{(0)})$, where $x^0 \in B$ and $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n}$.

If $f(x^{(0)}) = x^{(0)}$ then $x^{(0)}$ is called a *fixed point*. A fixed point $x^{(0)}$ is called an *attractor* if there exists a neighborhood $U(x^{(0)})$ ($\subset B$) of $x^{(0)}$ such that for all points $y \in U(x^{(0)})$ it holds $\lim_{n\to\infty} y^{(n)} = x^{(0)}$, where $y^{(n)} = f^n(y)$. If $x^{(0)}$ is an attractor then its *basin of attraction* is

$$A\big(x^{(0)}\big) = \big\{y \in \mathbb{Q}_p\colon\ y^{(n)} \to x^{(0)},\ n \to \infty\big\}.$$

A fixed point $x^{(0)}$ is called *repeller* if there exists a neighborhood $U(x^{(0)})$ of $x^{(0)}$ such that $|f(x) - x^{(0)}|_p > |x - x^{(0)}|_p$ for $x \in U(x^{(0)})$, $x \neq x^{(0)}$. For a fixed point $x^{(0)}$ of a function $f(x)$ a ball $B_r(x^{(0)})$ (contained in B) is said to be a *Siegel disc* if each sphere $S_\rho(x^{(0)})$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x^{(0)})$ then all iterated points $x^{(n)} \in S_\rho(x^{(0)})$ for all $n = 1, 2, \ldots.$ The union of all Siegel discs with the center at $x^{(0)}$ is said to be *a maximum Siegel disc* and is denoted by $SI(x^{(0)})$.

Remark 2.1. In non-Archimedean geometry, a center of a disc is nothing but a point which belongs to the disc, therefore, in principle, different fixed points may have the same Siegel disc.

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$. Set

$$\lambda = \frac{d}{dx} f\big(x^{(0)}\big).$$

The point $x^{(0)}$ is called *attractive* if $0 \leqslant |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

3. A generalized logistic map and its fixed points

Our main interest is a p-adic generalized map, which is defined by

$$G(x) = (ax)^2(x+1),$$

where $x, a \in \mathbb{Q}_p$. Using a simple conjugacy $h(x) = ax$ we can reduce G to the following form $f = h \circ G \circ h^{-1}$, i.e.

$$f(x) = x^3 + ax^2, \quad a \in \mathbb{Q}_p. \tag{3.1}$$

Henceforth, we will deal with the function f. Direct checking shows that the fixed points of the function (3.1) are the following ones

$$x_1 = 0 \quad \text{and} \quad x_{2,3} = \frac{-a \pm \sqrt{a^2+4}}{2}. \tag{3.2}$$

Here $x_{2,3}$ are the solutions of

$$x^2 + ax - 1 = 0. \tag{3.3}$$

Note that these fixed points are formal, because, basically in $\mathbb{Q}_q$ the square root does not always exist. A full investigation of a behavior of the dynamics of the function needs the existence of the fixed points $x_{2,3}$. Therefore, we have to verify when $\sqrt{a^2+4}$ does exist.

3.1. Existence of fixed points

In this subsection, basically we are going to use Lemma 2.1 to show the existence of $\sqrt{a^2+4}$. Therefore, we consider several distinct cases with respect to the parameter a and the prime p.

Case $|a|_p < 1$.

In this case we can write a as follows

$$a = p^k \varepsilon, \quad k \geqslant 1, \quad |\varepsilon|_p = 1. \tag{3.4}$$

Let us represent a^2+4 in the canonical form (see (2.3))

$$a^2+4 = p^\gamma \bigl(a_0 + a_1 p + a_2 p^2 + \cdots\bigr). \tag{3.5}$$

Now first assume that $p=3$. Then from (3.4) we get

$$a^2+4 = 1+3+3^{2k}\varepsilon^2.$$

Hence from (3.5) we find that $\gamma = 0$ and $a_0 = 1$. According to Lemma 2.1 we have to solve the equation $x^2 \equiv 1 \pmod 3$. One can see that it has a solution $x = 3N+1$, $N \in \mathbb{Z}$. Therefore, in this setting $\sqrt{a^2+4}$ exists.

Let $p \geqslant 5$. Then we have

$$a^2+4 = 4 + p^{2k}\varepsilon^2.$$

Whence from (3.5) one sees that $\gamma = 0$ and $a_0 = 4$. The equation $x^2 \equiv 4 \pmod p$ has solution $x = pN+2$, $N \in \mathbb{Z}$, hence in this setting $\sqrt{a^2+4}$ also exists.

Note that the case $p=2$ is usually pathological, therefore it should be considered in more detail.

Now let $p=2$. Then from (3.4) we find

$$a^2+4 = 2^2\bigl(1+2^{2(k-1)}\varepsilon^2\bigr), \tag{3.6}$$

here $|\varepsilon|_2 = 1$. Using Lemma 2.1 and $|\varepsilon|_2 = 1$ one gets that

$$\varepsilon^2 = 1 + 2^m \varepsilon_1, \tag{3.7}$$

for some $m \geqslant 3$, $|\varepsilon_1|_2 = 1$. Now substituting (3.7) to (3.6) we obtain

$$a^2+4 = 2^2\bigl(1+2^{2(k-1)} + 2^{2(k-1)+m}\varepsilon_1\bigr). \tag{3.8}$$

If $k \geqslant 3$ then $\gamma = 2$ and $a_0 = 1$, $a_1 = a_2 = 0$ in terms of (3.5). Therefore, according to Lemma 2.1 we infer that $\sqrt{a^2+4}$ exists.

If $k=2$, then from (3.8) one yields

$$a^2+4 = 2^2\bigl(1+2^2+2^{2+m}\varepsilon_1\bigr). \tag{3.9}$$

Hence, from (3.9) we conclude that $a_1 = 1$ and therefore Lemma 2.1 implies that $\sqrt{a^2+4}$ does not exist.

Finally, if $k=1$, then we find that

$$a^2+4=2^3\left(1+2^{m-1}\varepsilon_1\right), \tag{3.10}$$

which with $m\geqslant 3$ implies that $\gamma=3$, and again using Lemma 2.1 we conclude that $\sqrt{a^2+4}$ does not exist.

Thus we can formulate

Proposition 3.1. *Let $|a|_p<1$, then the expression $\sqrt{a^2+4}$ exists in $\mathbb{Q}_p$ if and only if either $p\geqslant 3$ or $p=2$ and $|a|_p\leqslant 1/p^3$.*

Case $|a|_p>1$.

In this case we can write that

$$a=p^{-k}\varepsilon,\quad k\geqslant 1,\quad |\varepsilon|_p=1. \tag{3.11}$$

Then from (3.11) we get

$$a^2+4=p^{-2k}\left(\varepsilon^2+4p^{2k}\right),$$

which with Lemma 2.1 implies that $\sqrt{a^2+4}$ exists.

Proposition 3.2. *Let $|a|_p>1$, then the expression $\sqrt{a^2+4}$ exists in $\mathbb{Q}_p$.*

Case $|a|_p=1$.

This case is more complicated than others. So we will consider several subcases with respect to p. Before going to details from $|a|_p=1$ and (2.3) we infer that a can be represented as follows

$$a=a_0+a_1p+a_2p^2+\cdots \tag{3.12}$$

here $a_0\neq 0$.

First assume that $p=2$. Then taking into account Lemma 2.1 we can write

$$a^2=1+b_3 2^3+b_4 2^4+\cdots \tag{3.13}$$

whence one gets

$$a^2+4=1+2^2+b_3 2^3+b_4 2^4+\cdots.$$

But again according to Lemma 2.1 we conclude that $\sqrt{a^2+4}$ does not exist.

Let $p=3$. Then in this case we have

$$a^2=1+b_1 3+b_2 3^2+\cdots \tag{3.14}$$

hence

$$a^2+4=2+(b_1+1)3+b_2 3^2+\cdots.$$

Consequently, by means of Lemma 2.1 one gets that $\sqrt{a^2+4}$ does not exist, since the equation $x^2 \equiv 2 \pmod 3$ has no solution in $\mathbb{Z}$.

Let $p \geqslant 5$. Then from (3.12) we find

$$a^2+4 = a_0^2+4+2a_0a_1p+\left(a_1^2+2a_0a_2\right)p^2+\cdots. \tag{3.15}$$

Let $a_0^2+4 \not\equiv 0 \pmod p$ then Lemma 2.1 implies that $\sqrt{a^2+4}$ exists if and only if the following relation holds

$$\frac{x^2-a_0^2-4}{p} \in \mathbb{Z} \tag{3.16}$$

for some $x \in \mathbb{Z}$.

Let $a_0^2+4 \equiv 0 \pmod p$, then we have $a_0^2+4 = Np$ for some $N \in \{1, \ldots, p-1\}$. From (3.15) one finds that

$$a^2+4 = p\left(N+2a_0a_1+\left(a_1^2+2a_0a_2\right)p+\cdots\right). \tag{3.17}$$

Taking into account Lemma 2.1 and (3.17) we can formulate the following condition: if

$$\begin{cases} a_0^2+4 \equiv 0 \pmod p, \\ \frac{a_0^2+4}{p}+2a_0a_1 \equiv 0 \pmod p, \\ a_1^2+2a_0a_2 \not\equiv 0 \pmod p, \end{cases} \tag{3.18}$$

then $\sqrt{a^2+4}$ exists.

For example, if $p=5$, then using the last condition we find that for the element

$$a = 1+2\cdot 5+2\cdot 5^2+\cdots$$

$\sqrt{a^2+4}$ exists.

Thus we have the following

Proposition 3.3. *Let $|a|_p = 1$, then the expression $\sqrt{a^2+4}$ does not exist in $\mathbb{Q}_p$ when $p=2$ or $p=3$. Let $p \geqslant 5$. If $|a^2+4|_p = 1$, then $\sqrt{a^2+4}$ exists in $\mathbb{Q}_p$ if and only if* (3.16) *is valid for some $x \in \mathbb{Z}$. If $|a^2+4|_p < 1$ and* (3.18) *holds, then $\sqrt{a^2+4}$ exists.*

4. Behavior of the fixed points

In this section we are going to calculate norms of the fixed points and their behavior.

Let us first note that the derivative of f is

$$f'(x) = 3x^2+2ax. \tag{4.1}$$

From this we immediately conclude that the fixed point x_1 is attractive. Therefore, furthermore we will deal with $x_{2,3}$.

Using (3.3) we find

$$x_2 + x_3 = -a, \qquad x_2 x_3 = -1, \tag{4.2}$$

and

$$f'(x_\sigma) = 3 - a x_\sigma, \quad \sigma = 2, 3. \tag{4.3}$$

Case $|a|_p < 1$.

Let $p \geqslant 3$, then from (3.2) and (4.2) one finds that $|x_\sigma|_p = 1$.

Let $p = 2$, then Proposition 3.1 implies that $a = p^k \varepsilon$ for some $k \geqslant 3$ with $|\varepsilon|_p = 1$. From this and taking into account (3.2) we have

$$\begin{aligned} |x_\sigma|_p &= p\left|a \pm \sqrt{a^2+4}\right|_p \\ &= p\left|p^k\varepsilon \pm \sqrt{p^{2k}\varepsilon^2 + p^2}\right|_p \\ &= \left|p^{k-1}\varepsilon \pm \sqrt{p^{2(k-1)}\varepsilon^2 + 1}\right|_p = 1, \end{aligned}$$

since $|p^{2(k-1)}\varepsilon^2 + 1|_p = 1$ and $k \geqslant 3$.

Now let us compute $|f'(x_\sigma)|_p$. Let $p \neq 3$, then using (4.3) one gets

$$\left|f'(x_\sigma)\right|_p = |3 - a x_\sigma|_p = 1, \quad \sigma = 2, 3.$$

This means that the fixed points x_σ ($\sigma = 2, 3$) are indifferent.

Now let $p = 3$, then we easily obtain that $|f'(x_\sigma)|_p < 1$, $\sigma = 2, 3$, which implies that the fixed points are attractive.

We summarize the considered case by the following

Lemma 4.1. *Let $|a|_p < 1$. The fixed point x_1 is attractive. If $p \neq 3$ then the fixed points $x_{2,3}$ are indifferent. If $p = 3$ then $x_{2,3}$ are attractive.*

Case $|a|_p = 1$.

In this case according to Proposition 3.3 we have to consider $p \geqslant 5$. If $|x_2|_p < 1$ then (4.2) implies that $|x_3|_p > 1$. This with (2.1) yields that $|x_2 + x_3|_p > 1$, which contradicts to the first equality of (4.2). Hence $|x_\sigma|_p = 1$, $\sigma = 2, 3$.

Now by means of (4.3) and (4.2) one finds that

$$\begin{aligned} \left|f'(x_2) f'(x_3)\right|_p &= \left|(3 - a x_2)(3 - a x_3)\right|_p \\ &= \left|9 - 3(x_2 + x_3)a + a^2 x_2 x_3\right| \\ &= \left|9 + 2a^2\right|_p. \end{aligned} \tag{4.4}$$

Analogously,

$$\left|f'(x_2) + f'(x_3)\right|_p = \left|6 + a^2\right|_p. \tag{4.5}$$

These two equalities (4.4), (4.5) imply that

$$\left|f'(x_2)f'(x_3)\right|_p \leqslant 1, \qquad \left|f'(x_2)+f'(x_3)\right|_p \leqslant 1.$$

Let us prove the following

Lemma 4.2. *Let $|a|_p = 1$, then the inequalities*

$$\left|f'(x_2)f'(x_3)\right|_p < 1, \qquad \left|f'(x_2)+f'(x_3)\right|_p < 1 \tag{4.6}$$

are not valid in the same time.

Proof. Let us assume that (4.6) is valid. According to (4.4), (4.5) from (4.6) we obtain $|9+2a^2|_p < 1$ and $|6+a^2|_p < 1$. The last ones are equivalent to the following equations

$$9+2x^2 \equiv 0 \pmod p, \qquad 6+x^2 \equiv 0 \pmod p.$$

From these relations we infer that $p = 3$, which is impossible thanks to Proposition 3.3. □

This lemma implies that the both fixed points cannot be simultaneously attractive. Now let us provide some examples for the occurrence of the other cases.

Example 4.1. Let $p = 5$. Then according to Proposition 3.3 we infer that $a_0 = 1$ or $a_0 = 4$ in representation (3.12). So we have $a^2 = 1 + p\varepsilon_1$ for some $|\varepsilon_1|_p = 1$. Consequently, (4.4) and (4.5) imply that

$$\left|f'(x_2)f'(x_3)\right|_p = 1, \qquad \left|f'(x_2)+f'(x_3)\right|_p = 1,$$

which means that $|f'(x_2)|_p = 1$, $|f'(x_3)|_p = 1$, i.e. both fixed points are indifferent.

If $p = 7$ we also similarly can get analogous result as the previous one, i.e. x_σ is indifferent.

Example 4.2. Let $p = 11$ and $a = 4$. Then Proposition 3.3 implies that $\sqrt{a^2+4}$ exists. On the other hand, from the equalities (4.4), (4.5) we find that

$$\left|f'(x_2)f'(x_3)\right|_p = 1, \qquad \left|f'(x_2)+f'(x_3)\right|_p < 1,$$

which yields that $|f'(x_2)|_p = 1$, $|f'(x_3)|_p = 1$, i.e. both fixed points are indifferent. Note also that in this case from (4.5) we get $|a^2+4|_p = 1$.

Example 4.3. Now let $p = 11$ and $a = 1$. Then Proposition 3.3 implies that $\sqrt{a^2+4}$ exists. Consequently, from (4.4) and (4.5) we infer that

$$\left|f'(x_2)f'(x_3)\right|_p < 1, \qquad \left|f'(x_2)+f'(x_3)\right|_p = 1,$$

which implies that either $|f'(x_2)|_p < 1$, $|f'(x_3)|_p = 1$ or $|f'(x_2)|_p = 1$, $|f'(x_3)|_p < 1$. Without loss of generality we can assume that $|f'(x_2)|_p < 1$, $|f'(x_3)|_p = 1$, which means that x_2 is attractive and x_3 is indifferent. Note that in this case from (4.4) we have

$$\left|2\left(a^2+4\right)+1\right|_p = \left|2a^2+9\right| < 1 \quad \Longrightarrow \quad \left|a^2+4\right|_p = 1. \tag{4.7}$$

Thus we can formulate the following

Lemma 4.3. *Let $|a|_p = 1$. The fixed point x_1 is attractive. For the other fixed points there are the following possibilities*:

1. *both fixed points x_2 and x_3 are indifferent*;
2. *the fixed point x_2 is attractive and x_3 is indifferent, respectively*;
3. *the fixed point x_3 is attractive and x_2 is indifferent, respectively.*

Case $|a|_p > 1$.

In this case from (4.2) one gets that $|x_2 + x_3|_p = |a|_p$, $|x_2|_p|x_3|_p = 1$. These imply that either $|x_2|_p > 1$ or $|x_3|_p > 1$. Without lose of generality we may assume that $|x_2|_p > 1$, which means that $|x_3|_p < 1$. From the condition $|a|_p > 1$ we find that $|x_2|_p = |a|_p$ and $|x_3|_p = 1/|a|_p$.

From (4.3) we obtain

$$\left|f'(x_2)\right|_p = |a|_p|x_2|_p = |a|_p^2 > 1 \tag{4.8}$$

which means that the point x_2 is repelling.

From (4.1) with $|x_3|_p = 1/|a|_p$ one gets that $|f'(x_3)|_p = 1$, hence x_3 is an indifferent point.

Lemma 4.4. *Let $|a|_p > 1$. The fixed point x_1 is attractive. For the other fixed points there are the following possibilities*:

1. *the fixed point x_2 is repelling and x_3 is indifferent, respectively*;
2. *the fixed point x_3 is repelling and x_2 is indifferent, respectively.*

5. Attractors and Siegel discs

In the previous section we have established behavior of the fixed points of the dynamical system. Using those results, in this section we are going to describe the size of attractors and Siegel discs of the system.

Before going to details let us formulate certain useful auxiliary facts.

Let us assume that $x^{(0)}$ is a fixed point of f. Then f can be represented as follows

$$f(x) = f\left(x^{(0)}\right) + f'\left(x^{(0)}\right)\left(x - x^{(0)}\right) + \frac{f''(x^{(0)})}{2}\left(x - x^{(0)}\right)^2 + \frac{f'''(x^{(0)})}{6}\left(x - x^{(0)}\right)^3. \tag{5.1}$$

From the above equality putting $\gamma = x - x_0$ we obtain

$$\left|f(x) - f\left(x^{(0)}\right)\right|_p = |\gamma|_p\left|f'\left(x^{(0)}\right) + \frac{f''(x^{(0)})}{2}\gamma + \frac{f'''(x^{(0)})}{6}\gamma^2\right|_p. \tag{5.2}$$

Lemma 5.1. *Let $x^{(0)}$ be a fixed point of the function f given by* (3.1)*. If for $\gamma = x - x^{(0)}$ the following inequality holds*

$$\max\{|3x^{(0)} + a|_p |\gamma|_p, |\gamma|_p^2\} < |f'(x^{(0)})|_p$$

then

$$|f(x) - f(x^{(0)})|_p = |\gamma|_p |f'(x^{(0)})|_p. \tag{5.3}$$

The proof immediately comes from (5.2) and the following ones

$$f''(x) = 6x + 2a, \qquad f'''(x) = 6.$$

From Lemma 5.1 we get

Corollary 5.2. *(See [5].) Let $x^{(0)}$ be a fixed point of the function f given by* (3.1)*. The following assertions hold*:

(i) *if $x^{(0)}$ is an attractive point of f, then it is an attractor of the dynamical system. If $r > 0$ satisfies the inequality*

$$\max\{|3x^{(0)} + a|_p r, r^2\} < 1 \tag{5.4}$$

then $B_r(x^{(0)}) \subset A(x^{(0)})$;

(ii) *if $x^{(0)}$ is an indifferent point of f then it is the center of a Siegel disc. If r satisfies the inequality* (5.4) *then $B_r(x^{(0)}) \subset SI(x^{(0)})$*;

(iii) *if $x^{(0)}$ is a repelling point of f then $x^{(0)}$ is a repeller of the dynamical system.*

Now as in the previous section we consider several distinct cases with respect to the parameter a.

Case $|a|_p > 1$.

In the previous section point out that the fixed point $x_1 = 0$ is attractive (see Lemma 4.4), therefore let us first investigate $A(x_1)$. To do it, denote

$$r_k = \frac{1}{|a|_p^k}, \quad k \geqslant 0.$$

Now consider several steps along the description of $A(x_1)$.

(I) From Corollary 5.2 and (5.4) we find that $B_{r_1}(0) \subset A(x_1)$. Now take $x \in S_{r_1}(0)$, i.e. $|x| = r_1$. Then one gets

$$|f(x)|_p = |x|_p^2 |x + a|_p = |x|_p^2 |a|_p = r_1,$$

whence we infer that $|f^{(n)}(x)|_p = r_1$ for all $n \in \mathbb{N}$. This means that $x \notin A(x_1)$, hence $A(x_1) \cap S_{r_1}(0) = \emptyset$. Moreover, we have $f(S_{r_1}(0)) \subset S_{r_1}(0)$.

In the sequel we will assume that

$$\sqrt{|a|_p} \notin \{p^k,\ k \in \mathbb{N}\}. \tag{5.5}$$

Denote

$$A(\infty) = \{x \in \mathbb{Q}_p\colon\ |f^{(n)}(x)|_p \to \infty \text{ as } n \to \infty\}.$$

It is evident that $A(x_1) \cap A(\infty) = \emptyset$.

(II) Let us take $x \in S_r(0)$ with $r > |a|_p$. Then we have

$$|f(x)|_p = |x|_p^2 |x + a|_p = |x|_p^2 |x|_p = |x|_p^3,$$

which means that $x \in A(\infty)$, i.e. $S_r(0) \subset A(\infty)$ for all $r > |a|_p$.

(III) Now assume that $x \in S_r(0)$ with $r \in (r_1, r_0) \cup (r_0, |a|_p)$. Then we have $f(S_r(0)) \subset S_{r^2|a|_p}(0)$. If $r \in (r_0, |a|_p)$ then $r^2|a|_p > |a|_p$, hence we have $S_r(0) \subset A(\infty)$. If $r \in (r_1, r_0)$ then according to our assumption (5.5) we have $r^{2^n}|a|_p^{1+2+\cdots+2^{n-1}} \neq 1$ for every $n \in \mathbb{N}$, hence there is $n_0 \in \mathbb{N}$ such that $f^{(n_0)}(S_r(0)) \subset A(\infty)$, from this one concludes that $S_r(0) \subset A(\infty)$. Consequently, we have $S_r(0) \subset A(\infty)$ for all $r \in (r_1, r_0) \cup (r_0, |a|_p)$.

(IV) If $x \in S_{r_0}(0)$, then one gets $f(S_{r_0}(0)) \subset S_{|a|_p}(0)$.

(V) Therefore, we have to consider $x \in S_{|a|_p}(0)$. From (3.1) we can write

$$|f(x)|_p = |a|_p^2 |x + a|_p. \tag{5.6}$$

From this we conclude that we have to investigate behavior of $|x + a|_p$ ($\leqslant |a|_p$). It is clear the following decomposition

$$S_{|a|_p}(0) = \bigcup_{r=0}^{|a|_p} S_r(-a). \tag{5.7}$$

(VI) Now if $|x + a|_p < r_3$ then from the last equality we get $|f(x)|_p < r_1$, this yields that $x \in A(x_1)$. Hence $B_{r_3}(-a) \subset A(x_1)$. Moreover, taking into account (I) we have $f(S_{r_3}(-a)) \subset S_{r_1}(0)$.

(VII) If $x \in S_{r_1}(-a)$ then from (5.6) we find that $f(x) \in S_{|a|_p}(0)$.

(VIII) If $x \in S_{r_2}(-a)$ then again using (5.6) one gets that $f(x) \in S_{r_0}(0)$. This with (IV) implies that $f^{(2)}(S_{r_1}(-a)) \subset S_{|a|_p}(0)$.

(IX) If $x \in S_r(-a)$ with $r \in (r_3, r_2) \cup (r_2, r_1) \cup (r_1, |a|_p]$. Then from (5.6) we obtain that $f(x) \in S_\rho(0)$, $\rho \in (r_1, r_0) \cup (r_0, |a|_p) \cup (|a|_p, |a|_p^3]$. Hence thanks to (II) and (III) we infer that $f(x) \in A(\infty)$.

Let us introduce some more notations. Given sets $A, B \subset \mathbb{Q}_p$ put

$$T_{f,A,B}(x) = \min\{k \in \mathbb{N}\colon\ f^{(k)}(x) \in B\}, \quad x \in A, \tag{5.8}$$

$$D[A, B] = \{x \in A\colon\ T_{f,A,B}(x) < \infty\}. \tag{5.9}$$

Taking into account (II)–(IX) and (5.8)–(5.9) we can define $D[S_{r_0}(0) \cup S_{|a|_p}(0), B_{r_3}(-a)]$, which is nonempty, since from (VI) one sees that $B_{r_3}(-a) \subset D[S_{r_0}(0) \cup S_{|a|_p}(0), B_{r_3}(-a)]$.

Thus taking into account (I)–(IX) we have the following equality $A(x_1) = B_{r_1}(0) \cup D[S_{r_0}(0) \cup S_{|a|_p}(0), B_{r_3}(-a)]$.

Now turn to the other fixed points. According to Lemma 4.4 without loss of generality we may assume that x_2 is repelling and x_3 is indifferent. In this case we know that $|x_2|_p = |a|_p$ and $|x_3|_p = r_1$. Now Corollary 5.2 with (5.4) yields that $B_{r_1}(x_3) \subset SI(x_3)$. It is clear that $0 \notin SI(x_3)$, therefore $SI(x_3) = B_{r_1}(x_3)$. From (I) we infer that $SI(x_3) \subset S_{r_1}(0)$.

Thus we have proved the following

Theorem 5.3. *Let $|a|_p > 1$ and* (5.5) *be satisfied. Then $A(x_1) = B_{r_1}(0) \cup D[S_{r_0}(0) \cup S_{|a|_p}(0), B_{r_3}(-a)]$. For the indifferent fixed point x_3 we have $SI(x_3) = B_{r_1}(x_3)$.*

Case $|a|_p < 1$.

From Lemma 4.1 we know that x_1 is attractive. So according to Corollary 5.2 we immediately find that $B_1(0) \subset A(x_1)$. Take $x \in S_1(0)$ then $|f(x)|_p = |x|_p^2 |x + a|_p = |x|^3 = 1$, hence $|f^{(n)}(x)|_p = 1$ for all $n \in \mathbb{N}$. This means that $x \notin A(x_1)$, hence $A(x_1) = B_1(0)$.

Now turn to the fixed points x_2 and x_3. According to Lemma 4.1 we consider two possible situations $p \neq 3$ and $p = 3$.

First assume $p \neq 3$. In this case x_2 and x_3 are indifferent, so Corollary 5.2 again implies that $B_1(x_\sigma) \subset SI(x_\sigma)$, $\sigma = 2, 3$.

Let us take $x \in S_r(x_\sigma)$, $r \geqslant 1$, then put $\gamma = x - x_\sigma$. It is clear that $|\gamma|_p = r$. By means of (5.2) and (3.3) we find

$$\begin{aligned} \left|f(x) - f(x_\sigma)\right|_p &= |\gamma|_p \left|3x_\sigma^2 + 2ax_\sigma + (3x_\sigma + a)\gamma + \gamma^2\right|_p \\ &= r\left|\gamma^2 + 3x_\sigma\gamma + 3 + a(\gamma - x_\sigma)\right|_p. \end{aligned} \tag{5.10}$$

If $r > 1$ then from (5.10) we easily obtain that

$$\left|f(x) - f(x_\sigma)\right|_p = |\gamma|_p^3$$

since $|\gamma^2 + 3x_\sigma\gamma + 3|_p = |\gamma|_p^2$, $|a(\gamma - x_\sigma)|_p = |a|_p|\gamma|_p$. This implies that $SI(x_\sigma) \subset \bar{B}_1(x_\sigma)$.

Lemma 5.4. *Let $|a|_p < 1$ and $p \neq 3$. The equality $SI(x_\sigma) = \bar{B}_1(x_\sigma)$ holds if and only if the equality holds*

$$\left|\gamma^2 + 3x_\sigma\gamma + 3\right| = 1 \tag{5.11}$$

for every $\gamma \in S_1(0)$.

Proof. If (5.11) is satisfied for all $\gamma \in S_1(0)$ then from (5.10) we infer that $f(S_1(x_\sigma)) \subset S_1(x_\sigma)$, since $|a(\gamma - x_\sigma)|_p < 1$. This proves the assertion. Now suppose that $SI(x_\sigma) = \bar{B}_1(x_\sigma)$ holds. Assume that (5.11) is not valid, i.e. there is $\gamma_0 \in S_1(0)$ such that

$$\left|\gamma_0^2 + 3x_\sigma\gamma_0 + 3\right| < 1. \tag{5.12}$$

The last one with (5.10) implies that $|f(x_0) - x_\sigma|_p < 1$ for an element $x_0 = x_\sigma + \gamma_0$. But this contradicts to $SI(x_\sigma) = \bar{B}_1(x_\sigma)$. This completes the proof. □

From the proof of Lemma 5.4 we immediately obtain that if there is $\gamma_0 \in S_1(0)$ such that (5.12) is satisfied then $SI(x_\sigma) = B_1(x_\sigma)$. Moreover, we can formulate the following

Lemma 5.5. *Let $|a|_p < 1$ and $p \neq 3$. The following conditions are equivalent*:

(i) $SI(x_\sigma) = B_1(x_\sigma)$;
(ii) *there is $\gamma_0 \in S_1(0)$ such that* (5.12) *is satisfied*;
(iii) *$\sqrt{-3}$ exists in $\mathbb{Q}_p$.*

Proof. The implication (i) $\Leftrightarrow$ (ii) immediately follows from the proof of Lemma 5.11. Consider the implication (ii) $\Rightarrow$ (iii). The condition (5.12) according to the Hensel Lemma (see [25]) yields that the existence of a solution $z \in \mathbb{Q}_p$ of the following equation

$$z^2 + 3x_\sigma z + 3 = 0 \tag{5.13}$$

such that $|z - \gamma_0| < 1$ which implies that $|z|_p = 1$.

Now assume that there is a solution $z_1 \in \mathbb{Q}_p$ of (5.13). Then from Vieta's formula we conclude the existence of the another solution $z_2 \in \mathbb{Q}_p$ such that

$$z_1 + z_2 = -3x_\sigma, \qquad z_1 z_2 = 3.$$

From these equalities one gets that $|z_1 + z_2|_p = 1$, $|z_+ z_-|_p = 1$ which imply $z_1, z_2 \in S_1(0)$. So putting $\gamma_0 = z_1$ we find (5.12).

Let us now analyze when (5.13) has a solution belonging to $\mathbb{Q}_p$. We know that a general solution of (5.13) is given by

$$z_{1,2} = \frac{-3x_\sigma \pm \sqrt{-3 - 9ax_\sigma}}{2}, \tag{5.14}$$

here we have used (3.3). But it belongs to $\mathbb{Q}_p$ if $\sqrt{-3 - 9ax_\sigma}$ exists in $\mathbb{Q}_p$. Since $|-9ax_\sigma|_p = |a|_p < 1$ implies that $-9ax_\sigma = p^k \varepsilon$ for some $k \geqslant 1$ and $|\varepsilon|_p = 1$. Hence, $-3 - 9ax_\sigma = -3 + p^k \varepsilon$. Therefore according to Lemma 2.1 we conclude that $\sqrt{-3 - 9ax_\sigma}$ exists if and only if $\sqrt{-3}$ exists in $\mathbb{Q}_p$. The implication (iii) $\Rightarrow$ (ii) can be proven along the reverse direction in the previous implication. $\square$

Let us consider some concrete examples when $\sqrt{-3}$ exists with respect to p.

Example 5.1. Let $p = 2$, then -3 can be rewritten as follows

$$-3 = 1 + 2^2 + 2^3 + \cdots$$

so according to Lemma 2.1 we concludes that $\sqrt{-3}$ does not exist in $\mathbb{Q}_2$.

Analogously reasoning we may establish that when $p = 5, 11$ we find that $\sqrt{-3}$ does not exist. If $p = 7, 13$ then $\sqrt{-3}$ exists.

From Lemmas 5.4 and 5.5 we conclude that $SI(x_\sigma)$ is either $B_1(x_\sigma)$ or $\bar{B}_1(x_\sigma)$. The equality (3.2) yields that

$$|x_2 - x_3|_p = \left|\sqrt{a^2 + 4}\right|_p = |2|_p, \tag{5.15}$$

which implies that $SI(x_2) \cap SI(x_3) = \emptyset$ when $p \geqslant 5$ and $SI(x_2) = SI(x_3)$ when $p = 2$, since any point of a ball is its center.

Now consider the case $p = 3$. According to Lemma 4.1 we see that the both fixed points x_2 and x_3 are attractive. Taking into account $|x_\sigma|_p = 1$ and $|a|_p < 1$ from Corollary 5.2 one finds that $B_1(x_\sigma) \subset A(x_\sigma)$, $\sigma = 2, 3$. From the equality (5.15) we have $|x_2 - x_3|_p = 1$ which implies that $S_1(x_\sigma) \nsubseteq A(x_\sigma)$.

Let us take $x \in S_r(x_\sigma)$ with $r \geqslant 1$, then putting $\gamma = x - x_\sigma$ from (5.10) with $|3x_\sigma\gamma + 3 + a(\gamma - x_\sigma)|_p < |\gamma|_p$ we get

$$\left|f(x) - x_\sigma\right|_p = |\gamma|\left|\gamma^2 + 3x_\sigma\gamma + 3 + a(\gamma - x_\sigma)\right|_p = r^3,$$

which implies that $f(S_r(x_\sigma)) \subset S_{r^3}(x_\sigma)$ for every $r \geqslant 1$. Hence, in particular, we obtain $f(S_1(x_\sigma)) \subset S_1(x_\sigma)$.

Consequently we have the following

Theorem 5.6. *Let $|a|_p < 1$. The following assertions hold*:

(i) *Basin of attraction of x_1 is $B(x_1)$, i.e. $A(x_1) = B_1(0)$.*
(ii) *Let $p \neq 3$. Then $SI(x_\sigma) = B_1(x_\sigma)$ is valid if and only if $\sqrt{-3}$ exists in $\mathbb{Q}_p$. Otherwise $SI(x_\sigma) = \bar{B}_1(x_\sigma)$ holds.*
(iii) *If $p \geqslant 5$ then $SI(x_2) \cap SI(x_3) = \emptyset$, if $p = 2$ then $SI(x_2) = SI(x_3)$.*
(iv) *Let $p = 3$. Then $A(x_\sigma) = B_1(x_\sigma)$ $(\sigma = 2, 3)$.*

Note that if we consider our dynamical system over p-adic complex field $\mathbb{C}_p$ we will obtain different result from the formulated theorem, since $\sqrt{-3}$ always exists in $\mathbb{C}_p$.

Case $|a|_p = 1$.

In this case according to Proposition 3.3 we have to consider $p \geqslant 5$.

Let us first describe the basin of attraction of the fixed point $x_1 = 0$. Analogously, reasoning as in the previous cases we may find that $B_1(0) \subset A(x_1)$. Now if $x \in S_r(0)$ with $r > 1$ one gets that $|f(x)|_p = |x|_p^3 = r^3$, which implies that $x \notin A(x_1)$ and $A(x_1) \subset \bar{B}_1(0)$.

Suppose that $x \in S_1(0)$. From (3.1) we find

$$\left|f(x)\right|_p = |x + a|_p. \tag{5.16}$$

From this we conclude that we have to investigate behavior of $|x + a|_p$. It is clear the following decomposition

$$S_1(0) = \bigcup_{r=0}^{1} S_r(-a). \tag{5.17}$$

Let $|x + a|_p \leqslant r$ with $r < 1$, then from (5.16) we get $|f(x)|_p < 1$, this yields that $x \in A(x_1)$. Hence $B_1(-a) \subset A(x_1)$.

So we can define the set $D[S_1(0), B_1(-a)]$, which is not empty since $B_1(-a) \subset D[S_1(0), B_1(-a)]$.

We are going to show that $S_1(0) \setminus D[S_1(0), B_1(-a)]$ is open. Now assume that $y \in S_1(0) \setminus D[S_1(0), B_1(-a)]$. Then this means that

$$\left|f^{(n)}(y) + a\right|_p = 1 \quad \text{for all } n \in \mathbb{N}. \tag{5.18}$$

For any $r < 1$ one can be established that $B_r(y) \subset S_1(0)$. We will show that $B_r(y) \subset S_1(0) \setminus D[S_1(0), B_1(-a)]$, which would be the assertion. Take $x \in B_r(y)$. To show $x \in S_1(0) \setminus D[S_1(0), B_1(-a)]$ it is enough to prove $|f^{(n)}(x) + a|_p = 1$ for all $n \in \mathbb{N}$. To do end, consider

$$\begin{aligned} \left|f(y) - f(x)\right|_p &= |y - x|_p \left|y^2 + xy + x^2 - a(y + x)\right|_p \\ &\leqslant |y - x|_p \leqslant r \end{aligned}$$

which implies that $f(x) \in B_r(f(y))$. By means of the induction we find that $f^{(n)}(x) \in B_r(f^{(n)}(y))$ for all $n \in \mathbb{N}$. This with (5.18) implies the assertion.

Thus we have the following

Theorem 5.7. *Let $|a|_p = 1$. Then the fixed point x_1 is attractor and $A(x_1) = B_1(0) \cup D[S_1(0), B_1(-a)]$.*

Let us turn to the fixed points x_2 and x_3. Note that furthermore, we always assume that these fixed points exist, which in accordance with Section 3 is equivalent to the existence of $\sqrt{a^2 + 4}$. Now according to Lemma 4.3 we will consider three different cases.

Case 1. In this case x_σ ($\sigma = 2, 3$) is indifferent. By means of Corollary 5.2 and using the same procedure as in the previous cases we immediately derive that $B_1(x_\sigma) \subset SI(x_\sigma)$.

Let $x \in S_r(x_\sigma)$ with $r > 1$. It then follows from (5.10) that

$$\left|f(x) - f(x_\sigma)\right|_p = |\gamma|_p \left|\gamma^2 + 3x_\sigma\gamma + 3 + a(\gamma - x_\sigma)\right|_p = r^3, \tag{5.19}$$

since $r^2 = |\gamma^2 + 3x_\sigma\gamma + 3|_p > |a(\gamma - x_\sigma)|_p = r$, where as before $\gamma = x - x_\sigma$. This shows that $f(S_r(x_\sigma)) \subset S_{r^3}(x_\sigma)$, from which one concludes that $SI(x_\sigma) \subset \bar{B}_1(x_\sigma)$.

Now let $x \in S_1(x_\sigma)$. Then from (5.19) we obtain

$$\left|f(x) - f(x_\sigma)\right|_p = \left|\gamma^2 + \gamma(3x_\sigma + a) + 3 - ax_\sigma\right|_p. \tag{5.20}$$

From this we derive the following

Lemma 5.8. *Let $|a|_p = 1$ and x_σ be an indifferent fixed point of f. Then the following assertions equivalent*:

(i) $SI(x_\sigma) = \bar{B}_1(x_\sigma)$;
(ii) *the equality*

$$\left|z^2 + z(3x_\sigma + a) + 3 - ax_\sigma\right|_p = 1 \tag{5.21}$$

holds for all $z \in S_1(0)$.

If $|a^2 + 4| < 1$ then the last condition equivalent to

(iii) *the equality*

$$\left|2z^2 - az + 6 + a^2\right|_p = 1 \tag{5.22}$$

holds for all $z \in S_1(0)$.

Proof. The implication (i) $\Leftrightarrow$ (ii) can be proved along the same line of the proof of Lemma 5.4. Now assume that $|a^2 + 4| < 1$. So we have to prove the implication (ii) $\Leftrightarrow$ (iii). From (3.2) one gets

$$3x_\sigma + a = \frac{-a \pm 3\sqrt{a^2+4}}{2}, \tag{5.23}$$

$$3 - ax_\sigma = \frac{6 + a^2 \mp \sqrt{a^2+4}}{2}. \tag{5.24}$$

Using (5.23)–(5.24) from (5.21) we obtain

$$\begin{aligned}\left|z^2 + z(3x_\sigma + a) + 3 - ax_\sigma\right|_p &= \left|2z^2 + z\left(-a \pm 3\sqrt{a^2+4}\right) + 6 + a^2 \mp \sqrt{a^2+4}\right|_p \\ &= \left|\left(2z^2 - az + 6 + a^2\right) \pm (3z-1)\sqrt{a^2+4}\right|_p. \end{aligned} \tag{5.25}$$

Taking in account our assumption from (5.25) we conclude that (5.21) holds if and only if (5.22) holds. □

Lemma 5.9. *Let* $|a|_p = 1$ *and* x_σ $(\sigma = 2, 3)$ *be an indifferent fixed point of* f. *Then the following assertions equivalent*:

(i) $SI(x_\sigma) = B_1(x_\sigma)$;
(ii) $\sqrt{\frac{a^2-6+a\sqrt{a^2+4}}{2}}$ *exists in* $\mathbb{Q}_p$.

Proof. From Lemma 5.8 we find that (i) is valid if and only if there is $z_0 \in S_1(0)$ such that

$$\left|z_0^2 + z_0(3x_\sigma + a) + 3 - ax_\sigma\right|_p < 1. \tag{5.26}$$

This thanks to the Hensel Lemma implies that the existence a solution $b \in \mathbb{Q}_p$ with $|b - z_0| < 1$ of the following equation

$$z_0^2 + z_0(3x_\sigma + a) + 3 - ax_\sigma = 0. \tag{5.27}$$

On the other hand, the condition $|3 - ax_\sigma|_p = 1$ with Vieta's formulas implies that if a solution $b_1 \in \mathbb{Q}_p$ of the equation exists, then $|b_1|_p = 1$ holds. Putting $z_0 = b_1$ we get (5.26). So (5.26) and (5.27) are equivalent. Hence, (5.27) has solution if and only if

$$\sqrt{\frac{a^2 - 6 + a\sqrt{a^2+4}}{2}}$$

exists in $\mathbb{Q}_p$, since every solution of (5.27) has a form

$$b_{1,2} = \frac{-(3x_\sigma + a) \pm \sqrt{\frac{a^2-6+a\sqrt{a^2+4}}{2}}}{2}. \qquad \square$$

Let us consider more special cases.

Corollary 5.10. *Let $|a|_p = 1$, $|a^2+4| < 1$ and x_σ be an indifferent fixed point of f. Then the following assertions equivalent*:

(i) $SI(x_\sigma) = B_1(x_\sigma)$;
(ii) *there is $z_0 \in S_1(0)$ such that*

$$\left|2z_0^2 - az_0 + 6 + a^2\right|_p < 1 \tag{5.28}$$

holds;
(iii) $\sqrt{-5}$ *exists in $\mathbb{Q}_p$ at $p \geqslant 7$.*

Proof. The implication (i) $\Leftrightarrow$ (ii) is a direct consequence of Lemma 5.8. Consider the implication (ii) $\Leftrightarrow$ (iii). In this case (5.28) thanks to the Hensel Lemma implies that the existence a solution $b \in \mathbb{Q}_p$ with $|b - z_0| < 1$ of the following equation

$$2z^2 - az + 6 + a^2 = 0. \tag{5.29}$$

According to Lemma 4.2 we get $|a^2 + 6| = 1$, which with $|a|_p = 1$ and Vieta's formulas implies that equivalence of (5.29) and (5.28). Hence, the existence of (5.29) equivalent to the existence of the squire root of the discriminant of (5.29), namely $\sqrt{-48 - 7a^2} \in \mathbb{Q}_p$. Rewrite $-48 - 7a^2$ as follows

$$-48 - 7a^2 = -20 - 7\left(a^2 + 4\right). \tag{5.30}$$

From $|a^2 + 4|_p < 1$ and the existence of $\sqrt{a^2+4}$ we infer that $a^2 + 4 = p^{2n}\varepsilon$ can be written for some $n \in \mathbb{N}$ and $|\varepsilon|_p = 1$. From (5.30) we have

$$-48 - 7a^2 = -20 - p^{2n}\varepsilon. \tag{5.31}$$

Hence, if $p \geqslant 7$, then keeping in mind (5.31) and Lemma 2.1 we deduce that $\sqrt{-48 - 7a^2} \in \mathbb{Q}_p$ if and only if $\sqrt{-5}$ exists in $\mathbb{Q}_p$. If $p = 5$ then from (5.31) we find that $-48 - 7a^2 = p(-4 - p^{2n-1}\varepsilon)$ which according to Lemma 2.1 yields that $\sqrt{-48 - 7a^2}$ does not exist in $\mathbb{Q}_p$. $\square$

Remark 5.1. If $p = 5$ and $|a^2 + 4|_p < 1$ then from the proof of the last lemma and Lemma 5.8 we immediately obtain that $SI(x_\sigma) = \bar{B}_1(x_\sigma)$.

Let us turn to the case $|a^2 + 4|_p = 1$. Note that this case is a rather tricky. Therefore, we will provide that some more sufficient conditions for fulfilling the equality $SI(x_\sigma) = B_1(x_\sigma)$.

Using (5.25) we assume that $|3z_0 - 1|_p < 1$ for some $z_0 \in S_1(0)$. Then from the equality

$$\left|2z_0^2 - az_0 + 6 + a^2\right|_p = \left|\frac{2}{3}(3z_0 - 1)^2 + \frac{4-a}{3}(3z_0 - 1) + \frac{1}{3}\left(3a^2 - a + 20\right)\right|_p \tag{5.32}$$

we conclude that $|2z_0^2 - az_0 + 6 + a^2|_p < 1$ if and only if $|3a^2 - a + 20|_p < 1$. By means of (5.25) and the last condition, we can formulate the following

Corollary 5.11. *Let $|a|_p = 1$, $|a^2 + 4| = 1$ and x_σ $(\sigma = 2, 3)$ be an indifferent fixed point of f. If $|3a^2 - a + 20|_p < 1$ holds, then $SI(x_\sigma) = B_1(x_\sigma)$.*

Summarizing the obtained we formulate

Theorem 5.12. *Let $|a|_1 = 1$ and x_2, x_3 be indifferent fixed points of f. The following assertions hold*:

(i) *If $\sqrt{\frac{a^2-6+a\sqrt{a^2+4}}{2}}$ exists in $\mathbb{Q}_p$ then $SI(x_\sigma) = B_1(x_\sigma)$. Otherwise $SI(x_\sigma) = \bar{B}_1(x_\sigma)$.*

(ii) *Let $|a^2 + 4|_p < 1$. If $\sqrt{-5}$ exists in $\mathbb{Q}_p$ at $p > 5$, then $SI(x_\sigma) = B_1(x_\sigma)$. Otherwise $SI(x_\sigma) = \bar{B}_1(x_\sigma)$. Moreover, we have $SI(x_2) = SI(x_3)$.*

Note that the last assertion immediately follows from (5.15).

Note that the case 3 similar to the case 2, therefore we will consider only case 2.

Case 2. In this setting we will suppose that the fixed point x_2 is attractive and x_3 is indifferent, respectively. Recall that later according to (4.7) means that $|a^2 + 4|_p = 1$. For the point x_3 the Siegel discs would the same as in the previous case. So we have to investigate only x_2.

We can easily show that $B_1(x_2) \subset A(x_2)$. By means of (5.10) we can also establish that $A(x_2) \cap S_r(x_2) = \emptyset$ for all $r > 1$.

Let $x \in S_1(x_2)$. Then from (5.20) one holds

$$\left|f(x) - f(x_2)\right|_p = \left|\gamma^2 + 3x_2\gamma + a\gamma + 3 - ax_2\right|_p. \tag{5.33}$$

Attractivity of the point x_2 means that $|3 - ax_2|_p < 1$ therefore if

$$\begin{aligned}\left|\gamma^2 + 3x_2\gamma + a\gamma\right|_p &= |\gamma + 3x_2 + a| \\ &= |x + 2x_2 + a|_p < 1\end{aligned} \tag{5.34}$$

then from (5.33) we get that $x \in A(x_2)$, i.e. $B_1(-2x_2 - a) \subset A(x_2)$. Here we have used the notation $\gamma = x - x_2$.

If $|\gamma^2 + 3x_2\gamma + a\gamma|_p = 1$ then $f(x) \in S_1(x_2)$. So we may again repeat the above procedure. Hence this leads that we can define the set $D[S_1(x_2), B_1(-2x_2 - a)]$, which is non-empty. Using the same argument as above cases (i.e. $|a|_p < 1$) one can show that $A(x_2) = B_1(x_2) \cup D[S_1(x_2), B_1(-2x_2 - a)]$.

Theorem 5.13. *Let $|a|_1 = 1$. Assume that x_2 is attractive and x_3 is indifferent fixed points of f. The following assertions hold*:

(i) *$A(x_2) = B_1(x_2) \cup D[S_1(x_2), B_1(-2x_2 - a)]$.*

(ii) *The Siegel disc of x_3 would be the same as in Theorem* 5.12.

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