NORTH-HOLLAND

On the Structure of Commutative Matrices. II

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ABSTRACT

A finite set **A** of $N \times N$ nilpotent commutative matrices that have one-dimensional joint kernel is considered. The theorem (due to Suprunenko and Tyshkevich) that the algebra $\mathscr{A}$ generated by **A** and the identity matrix has dimension equal to N is proved. A canonical basis for $\mathscr{A}$ is given, and related structure constants are discussed. © Elsevier Science Inc., 1997

0. INTRODUCTION

In this article we continue to study the structure of commutative matrices that we began in [11]. Now, our main results are extensions of results of Kravchuk, Suprunenko, and Tyshkevich (see [18, §§2.6–7]). Our motivation comes from multiparameter spectral theory [1]. Similarly to the way results of [11, Section 2] are used to construct bases for root subspaces of nonderogatory eigenvalues in [12], the results of this paper are used to find the corresponding bases for simple eigenvalues (see [10]). We will present this application to multiparameter spectral theory separately.

In [11] we considered an n-tuple $\mathbf{A} = \{A_i, i = 1, 2, \ldots, n\}$ of commutative nilpotent $N \times N$ matrices over the complex numbers. Now we also

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consider the algebra $\mathscr{A}$ generated by **A** and the identity matrix. For the most part we make a further assumption that **A** is simple, i.e., that the joint kernel of matrices in **A** is one-dimensional. Then we show that the algebra $\mathscr{A}$ has (vector-space) dimension equal to N. This result is found in [18, p. 62, Theorem 13]. We also describe a canonical basis $\mathscr{T}$ for the algebra $\mathscr{A}$. When $n = 2$ the basis $\mathscr{T}$ coincides (possibly after a change of basis for $\mathbb{C}^N$) with bases given in [2, 13, 20].

In [11] we viewed **A** also as a cubic array. The matrices in an array were brought by a simultaneous similarity to a special block upper triangular form called the reduced form. The reduced form has two important properties: the column cross sections of the blocks on the first upper diagonal are linearly independent, and the products of row and column cross sections are symmetric. (See Proposition 1 and Corollary 1 of [11].) The main result of [11] tells us how to reconstruct a commutative array from two sets of matrices, one of which is a set of symmetric matrices. Now we show, that when **A** is simple, the symmetric matrices are determined by the canonical basis and their entries are precisely the structure constants for multiplication in $\mathscr{A}$.

We proceed with a brief overview of the setup of the paper. In the next section we recall notation from [11], and in Section 2 we discuss some further properties of the general commutative array **A**. We also obtain an upper bound for the dimension of the algebra $\mathscr{A}$ in terms of N and the dimension of the joint kernel of matrices in **A**. In the remaining Section 3–5 we study the simple case. In Section 3 we show that the dimension of $\mathscr{A}$ is equal to N. Next, in Section 4, we introduce a canonical basis for the algebra $\mathscr{A}$ and the associated set of structure constants. We show that a simple array **A** is determined by the structure constants and a set of coefficients that depend only on the joint kernel of A_i. This is a minimal set required to describe **A**. In Section 5 we illustrate the discussion with two examples, and we consider the relation of our results with [2, 13, 20].

We conclude the introduction with some remarks on related literature. Finite sets of commutative matrices, algebras they generate, and their reduced forms under simultaneous similarity were studied, among others, by Trump [19] and Rutherford [17]. (See [14] for earlier references.) It was show by Gel'fand and Ponomarev [5] that to find a canonical form for general n-tuples of commuting matrices is as hard as to find a canonical form for an arbitrary n-tuple of matrices. In Section 4 we briefly touch on this problem in the case when **A** is simple. While elementary properties of (nilpotent) commutative matrices are usually exhibited in monographs on linear algebra (e.g. [4, 6, 15]) our main reference is the book by Suprunenko and Tyshkevich [18].

It was pointed out by the referee that the results of Corollary 1 and Theorem 2 are related to the problem of finding good bounds for the dimension of the algebra $\mathscr{A}$. A satisfactory solution to the problem has not yet been found. Most authors have attempted to get a bound as a function of n and N. For instance, there are now several proofs (e.g. [2, 13, 20]) that if $n \leq 2$ the dimension of $\mathscr{A}$ is at most N and that, if the algebra $\mathscr{A}$ is maximal commutative subalgebra of the full matrix algebra, it has dimension exactly N. (This is the case in our setup when $\mathbf{A}$ is simple.) Our Corollary 1 provides a bound of a different type which involves N and the dimension d_1 of the joint kernel of $\mathbf{A}$; more precisely, we show that $\dim \mathscr{A} \leq 1 + d_1(N - d_1)$. This is closer to a result of Gustafson [8], who used the joint cokernel (rather then the joint kernel) of matrices in $\mathbf{A}$. The approach in [8] is module-theoretic; in the language of linear algebra the fact that θ in [8, §2] is a monomorphism implies that $\dim \mathscr{A} \leq 1 + r_1(N - r_1)$, where r_1 (denoted by n in [8]) is the dimension of the joint cokernel.

After the paper had been submitted, we came across another module-theoretic paper [16] by Neubauer and Saltman, where the structure of two generated commutative matrix algebras is studied and several characterizations of algebras for which $\dim \mathscr{A} = N$ are given.

1. COMMUTATIVE ARRAYS

We first recall notation and definitions from [11]. In addition, we now denote the set of integers $\{1, 2, \ldots, n\}$ by $\underline{n}$. A set of commutative nilpotent $N \times N$ matrices $\mathbf{A} = \{A_s, s \in \underline{n}\}$ is viewed also as a cubic array of dimensions $N \times N \times n$. Such an array is called commutative. For $i \geq 1$ we write

$$\ker \mathbf{A}^i = \bigcap_{k_1 + \cdots + k_n = i} \ker\left(A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} \right).$$

Suppose that $M = \min_i \{\ker \mathbf{A}^i = \mathbb{C}^N\}$. Then

$$\{0\} \subset \ker \mathbf{A}^1 \subset \ker \mathbf{A}^2 \subset \cdots \subset \ker \mathbf{A} = \mathbb{C}^N \tag{1}$$

is a filtration of the vector space $\mathbb{C}^N$. Further we write

$$D_i = \dim \ker \mathbf{A}^i \quad \text{and} \quad d_i = D_i - D_{i-1} \tag{2}$$

for $i \in \underline{M}$. Here $D_0 = 0$. Then there exists a basis

$$\mathscr{B} = \left\{ z_1^1, z_2^1, \dots, z_{d_1}^1; z_1^2, z_2^2, \dots, z_{d_2}^2; \cdots; z_1^M, z_2^M, \dots, z_{d_m}^M \right\}$$

for $\mathbb{C}^N$ such that for every $i \in \underline{M}$ the set

$$\mathscr{B}_i = \left\{ z_1^1, z_2^1, \dots, z_{d_1}^1; z_1^2, z_2^2, \dots, z_{d_2}^2; \cdots; z_1^i, z_2^i, \dots, z_{d_i}^i \right\}$$

is a basis for $\ker \mathbf{A}^i$. Such a basis $\mathscr{B}$ is said to be *filtered.* A set of commutative nilpotent matrices $\mathbf{A}$ is then simultaneously reduced to a special upper triangular form and viewed as a cubic array

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{12} & \mathbf{A}^{13} & \cdots & \mathbf{A}^{1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{23} & \cdots & \mathbf{A}^{2,M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \tag{3}$$

where

$$\mathbf{A}^{kl} = \begin{bmatrix} \mathbf{a}_{11}^{kl} & \mathbf{a}_{12}^{kl} & \cdots & \mathbf{a}_{1,d_l}^{kl} \\ \mathbf{a}_{21}^{kl} & \mathbf{a}_{22}^{kl} & \cdots & \mathbf{a}_{2,d_l}^{kl} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{d_k,1}^{kl} & \mathbf{a}_{d_k,2}^{kl} & \cdots & \mathbf{a}_{d_k,d_l}^{kl} \end{bmatrix} \tag{4}$$

is a cubic array of dimensions $d_k \times d_l \times n$, and $\mathbf{A}_{ij}^{kl} \in \mathbb{C}^n$. The *row* and *column cross sections* of $\mathbf{A}^{kl}$ are

$$R_i^{kl} = \left[\mathbf{a}_{i1}^{kl} \quad \mathbf{A}_{i2}^{kl} \quad \cdots \quad \mathbf{a}_{i,dl}^{kl} \right], \qquad i \in \underline{d_k}, \tag{5}$$

and

$$\left(C_j^{kl} \right)^T = \left[\mathbf{a}_{1j}^{kl} \quad \mathbf{a}_{2j}^{kl} \quad \cdots \quad \mathbf{a}_{d_k,j}^{kl} \right], \qquad j \in \underline{d_l}. \tag{6}$$

These are matrices of dimensions $n \times d_l$ and $n \times d_k$, respectively.

The array $\mathbf{A}$ is the form (3) is called *reduced* if the matrices $C_j^{k,k+1}$, $j \in \underline{d_{k+1}}$, are linearly independent for $k \in \underline{M-1}$.

By [11, Proposition 1] it follows that the array (3) is reduced. Furthermore, a commutative cubic array (3) is reduced if and only if it is written in a filtered basis.

We call a matrix A *symmetric* if $A = A^T$. In [11, Corollary 1] we observed that $\mathbf{A}$ is commutative if and only if certain products of row and column cross sections are symmetric. The main result of [11], Theorem 3, tells us how to construct the column cross sections of $\mathbf{A}^{23}$ from the row cross sections of $\mathbf{A}^{12}$ and a set of symmetric matrices.

2. KRAVCHUK-TYPE THEOREM FOR A SET OF COMMUTATIVE MATRICES

For $k = 2, 3, \dots, M$ we denote by $\mathscr{S}_k$ the linear span of the set

$$\left\{\mathbf{a}_{ij}^{1l}; l = 2, 3, \dots, k, i \in \underline{d_1}, j \in \underline{d_l}\right\}.$$

PROPOSITION 1. *For* $k = 2, 3, \dots, M-1$, $l = k+1, k+2, \dots, M$, $i \in \underline{d_k}$, $j \in \underline{d_l}$, *one has* $\mathbf{a}_{ij}^{kl} \in \mathscr{S}_{l-k+1}$.

Proof. By the construction of column cross sections of the array $\mathbf{A}^{23}$ in the proof of [11, Theorem 3] (in particular see the first displayed formula in [11, p. 176]) it follows that $\mathbf{a}_{ij}^{23} \in \mathscr{S}_2$. In a similar way, we apply the construction of [11, Theorem 3] to the arrays $\mathbf{A}^{k-1,k}$ and $\mathbf{A}^{k,k+1}$, $k = 2, 3, \dots, M-1$, to obtain that

$$\mathbf{a}_{ij}^{k,k+1} \in \mathscr{S}_{2k},$$

where $\mathscr{S}_{2k} = \operatorname{Span}\{\mathbf{A}_{ij}^{k-1,k}; i \in d_{k-1}, j \in d_{k+1}\}$. Then it follows that

$$\mathbf{a}_{ij}^{k,k+1} \in \mathscr{S}_{2k} \subset \mathscr{S}_{2,k-1} \subset \cdots \subset \mathscr{S}_{21} = \mathscr{S}_2.$$

Next we apply the construction of [11, Theorem 3] to the arrays

$$\begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{13} \\ \mathbf{0} & \mathbf{A}^{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{A}^{24} \\ \mathbf{A}^{34} \end{pmatrix}$$

(see [11, p. 177]). This shows that $\mathbf{a}_{ij}^{24} \in \mathcal{S}_3$. As in the case $\mathbf{a}_{ij}^{k,k+1}$ we show inductively that $\mathbf{a}_{ij}^{k,k+2} \in \mathcal{S}_3$ for $k \geq 2$. Proceeding in the above manner for $l - k + 1 = 3, 4, \ldots, M - 1$, we obtain that $\mathbf{a}_{ij}^{kl} \in \mathcal{S}_{l-k+1}$ for all possible choices of i, j, k, and l. ■

Suppose that $M_N(\mathbb{C})$ is the algebra of all $N \times N$ matrices over $\mathbb{C}$ and that $\mathscr{A}$ is the subalgebra generated by the set of commutative matrices $\mathbf{A}$ and the identity matrix $I = I_N$. As a vector space, $\mathscr{A}$ is spanned by I and all the products of elements of $\mathbf{A}$, and in particular every element in $\mathscr{A}$ is of the form $A = \alpha I + B$, where $\alpha \in \mathbb{C}$ and B is nilpotent. Furthermore, A has a block upper triangular form $A = [A^{kl}]_{k,l=1}^{M}$, where A^{kl} is a $d_k \times d_l$ matrix block, $A^{kk} = \alpha I_{d_k}$, and $A^{kl} = 0$ for $k > l$.

The following is a version of Kravchuk's theorem (see [18, p. 57]).

THEOREM 1. *If* $A = [A^{kl}]_{k,l=1}^{M} \in \mathscr{A}$ *is such that* $A^{1l} = 0$ *for* $l \in \underline{M}$, *then* $A = 0$.

Proof. Since $A^{11} = 0$, it follows that $A^{kk} = 0$ for all $k \in M$, and so A is nilpotent. Let $A_{n+1} = A$ and $\hat{\mathbf{A}} = \{A_i;\ i \in \overline{N+1}\}$. Then $\hat{\mathbf{A}}$ can be viewed as a commutative cubic array of dimensions $N \times N \times (n+1)$. Since $A_{n+1} \in \mathscr{A}$ it follows that $\hat{\mathbf{A}} = [\hat{\mathbf{A}}^{kl}]_{k,l=1}^{M}$ is in the reduced form (3). Proposition 1 applied to $\hat{\mathbf{A}}$ implies that each entry of the block arrays $\hat{\mathbf{A}}^{kl}$ is in the linear span of the entries of $\hat{\mathbf{A}}^{1l}$. Since $A_{n+1}^{1l} = A^{1l} = 0$, it follows that $A_{n+1}^{kl} = 0$ for all k and l, and so $A_{n+1} = A = 0$. ■

The next result follows immediately from Theorem 1.

COROLLARY 1. *Each element* $A = [A^{kl}]_{k,l=1}^{M}$ *in* $\mathscr{A}$ *is uniquely determined by its first (block) row, i.e. by the entries in* $A^{1,l}$, $l \in \underline{M}$. *Furthermore,* $\dim \mathscr{A} \leq 1 + d_1(N - d_1)$.

3. THE SIMPLE CASE

As we already mentioned in Section 1, we view $\mathbf{A}$ as a set of commutative matrices and also as a commutative array. A commutative array $\mathbf{A}$ is called *simple* if $d_1 = 1$, i.e., if $\dim \bigcap_{i=1}^{N} \ker A_i = 1$.

The result of this and the next section are a generalization of results in [18, §2.7]. The authors in [18] study maximal commutative algebras of nilpotent matrices, whereas we arrive at these results while studying n-tuples of nilpotent matrices. Also we work with the complete filtration (1).

THEOREM 3. *If the array* $\mathbf{A}$ *is simple, then* $\dim \mathscr{A} = N$.

Proof. Since $d_1 = 1$, it follows by Corollary 1 that

$$\dim \mathscr{A} \leq N. \tag{7}$$

To prove the converse inequality, we consider, for $j \in \underline{M-1}$, the set $\mathbf{A}_j$ of all products of j elements of $\mathbf{A}$ as a cubic array $\mathbf{A}_j = [\mathbf{A}_j^{kl}]_{k,l=1}^M$. Then it follows that $\mathbf{A}_j^{kl} = 0$ for $k > l - j$. Since $\dim \ker \mathbf{A}^j = D_j = \sum_{i=1}^j d_j$, it follows that the nonzero column cross sections of $\mathbf{A}_j$ are linearly independent; in particular, the column cross sections of $\mathbf{A}_j^{1,j+1}$ are linearly independent. Thus, it follows that we can find in $\mathscr{A}$ elements $T_h^j = [T^{jkl}]_{k,l=1}^M$ such that $T_h^{jkl} = 0$ for $k > l - j$ and

$$T_j^{j1,j+1} = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0],$$

where 1 is in hth position. The element T_h^j, $j \in \underline{M-1}$, $h \in \underline{d_{j+1}}$, together with the identity matrix I, are clearly linearly independent, and there are

$$1 + \sum_{j=1}^{M-1} d_{j+1} = N$$

of them. Therefore $\dim \mathscr{A} \geq N$, and so with (7) we have that $\dim \mathscr{A} = N$. ■

COROLLARY 2. *The algebra* $\mathscr{A}$ *is a maximal commutative subalgebra of* $M_N(\mathbb{C})$.

Proof. Suppose that $B \in M_N(\mathbb{C})$ is such that $AB = BA$ for all $A \in \mathscr{A}$. Write $B = [B_{ij}]_{i,j=1}^M$ and $B_{11} = [b_{11}]$. Let matrices T_h^j, $j \in \underline{M-1}$, $h \in \underline{d_{j+1}}$, be defined as in the proof of Theorem 2. Because $T_h^j B = BT_h^j$ for all j and h, we first obtain that B is upper triangular, and furthermore, we see that

$$B_{jj} = b_{11} I_{d_j}. \tag{8}$$

Now, let $A_{n+1} = B - b_{11}I$ and $\mathbf{A}' = \{A_s;\ s \in \underline{n+1}\}$. Then $\mathbf{A}'$ is a commutative array, and it is simple. Thus Theorem 2 implies that the algebra $\mathscr{A}'$ generated by $\mathbf{A}'$ and I has dimension equal to N. Since $\mathscr{A} \subset \mathscr{A}'$ and $\dim \mathscr{A} = N$, it follows that $\mathscr{A} = \mathscr{A}'$. Then $B = A_{n+1} + b_{11}I$ is in $\mathscr{A}$, and hence $\mathscr{A}$ is maximal. ■

COROLLARY 3. *If a set* $\mathbf{A}$ *of* $N \times N$ *commutative matrices is such that the eigenspace at each joint eigenvalue is one-dimensional, then the dimension of the algebra generated by* $\mathbf{A}$ *(and the identity matrix) is* N.

Proof. Since $\mathbb{C}^N$ is the direct sum of all joint spectral subspaces of matrices of $\mathbf{A}$, the result follows if we show it for each joint spectral subspace. For each joint eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of $\mathbf{A}$ let $\mathscr{A}_{\boldsymbol{\lambda}}$ be the algebra generated by the restrictions of elements of $\mathbf{A}$ and the identity to the joint spectral subspace $V_{\boldsymbol{\lambda}}$ of $\mathbf{A}$ at $\boldsymbol{\lambda}$. The algebra $\mathscr{A}$ generated by $\mathbf{A}$ and I is a direct sum of the algrebras $\mathscr{A}_{\boldsymbol{\lambda}}$ as $\boldsymbol{\lambda}$ ranges over all the joint eigenvalues of $\mathbf{A}$. But then it follows by Theorem 2 that $\dim \mathscr{A}_{\boldsymbol{\lambda}} = \dim V_{\boldsymbol{\lambda}}$, and thus $\dim \mathscr{A} = \sum_{\boldsymbol{\lambda}} \dim \mathscr{A}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda}} \dim V_{\boldsymbol{\lambda}} = N$. ■

4. CANONICAL BASIS AND STRUCTURE CONSTANTS FOR THE ALGEBRA $\mathscr{A}$ IN THE SIMPLE CASE

Here we still assume that $\mathbf{A}$ is simple. Then Corollary 1 and Theorem 2 imply that for $g \in \underline{M-1}$ and $h \in \underline{d_{g+1}}$ there exist matrices $T_h^g = [T_h^{gkl}]_{k,l=1}^M \in \mathscr{A}$ such that

$$T_h^{gkl} = 0 \tag{9}$$

if either $k < l - g$ or $k = 1$ and $l \neq g + 1$, and

$$T_h^{g1,g+1} = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0], \tag{10}$$

where 1 is in the hth position. Moreover, the matrices T_h^g are uniquely determined by the conditions (9) and (10), and

$$\mathscr{T} = \{I\} \cup \left\{T_h^g;\, g \in \underline{M-1}\,,\, h \in \underline{d_{g+1}}\right\}$$

is a (canonical) basis for $\mathscr{A}$. We write $T_h^{gkl} = [t_{hij}^{gkl}]_{i=1,j=1}^{d_k\ \ d_l}$. Then we have that

$$T_i^k T_j^l = \sum_{g=k+l}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{klg} T_h^g = \sum_{g=k+l}^{M-1} \sum_{h=1}^{d_{g+1}} t_{jih}^{lkg} T_h^g. \tag{11}$$

Since T_h^g are linearly independent, the relation (11) implies that $t_{ijh}^{klg} = t_{jih}^{lkg}$. Note also that (11) implies that constants t_{ijh}^{klg} are the structure constants for multiplication in $\mathscr{A}$ expressed in the basis $\mathscr{T}$.

Since $\mathscr{T}$ is a basis for $\mathscr{A}$, it follows that $A_i = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} a_{1hi}^{1g} T_h^g$. Then we obtain that

$$\mathbf{a}_{ij}^{kl} = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{klg} \mathbf{a}_{1h}^{1g}.$$

Thus we have proved the first of the following two theorems. The second then follows easily.

THEOREM 3. *If t_{ijh}^{klg} are the structure constants for the multiplication in $\mathscr{A}$ expressed in the basis $\mathscr{T}$, then $\mathbf{a}_{ij}^{kl} = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{klg} \mathbf{a}_{ij}^{1g}$.*

THEOREM 4. *A simple commutative array* $\mathbf{A}$ *in the reduced form* (3) *is uniquely determined by the arrays* $\mathbf{A}^{1l}$, $1 = 2, 3, \ldots, M$, *and structure constants for* $\mathscr{A}$, *the algebra generated by* $\mathbf{A}$.

Note that if we write $X_j = [t_{klj}^{112}]_{k,l=1}^{d_2}$, $j \in d_3$, then X_j are symmetric and such that $C_j^{23} = R_1^{12} X_j$, where matrices R_1^{12} and C_j^{23} are defined in (5) and (6). Thus it follows that the entries of the symmetric matrices X_j in [11, Theorem 2] are precisely the structure constants for multiplication in $\mathscr{A}$. Similar construction can be obtained also for the column cross sections of arrays

$$\begin{bmatrix} \mathbf{A}^{k2} \\ \mathbf{A}^{k3} \\ \vdots \\ \mathbf{A}^{k,k+1} \end{bmatrix} \qquad \text{for} \quad k \geq 3.$$

Because t_{ijh}^{klg} are the structure constants for multiplication in a commutative algebra $\mathscr{A}$, they satisfy higher-order symmetries. These symmetries arise because the products of three or more matrices in $\mathscr{T}$ do not depend on the order of multiplication. We include the precise statement, since it is needed in the application to multiparameter spectral theory. First we introduce some further notation.

For $m = 2, 3, \ldots, M$ and $2 \leq q \leq m$ we denote by $\Phi_{m,q}$ the set of multiindices $\{(k_1, k_2, \ldots, k_q);\ k_i \geq 1,\ \sum_{i=1}^{q} k_i \leq m\}$. For $\mathbf{k} = (k_1, k_2, \ldots, k_q) \in \Phi_{m,q}$ we define a set

$$\chi_{\mathbf{k}} = \underline{d_{k_1}} \times \underline{d_{k_2}} \times \cdots \times \underline{d_{k_q}}.$$

The set of all permutations of the set $\underline{q}$ is denoted by π_q. For a permutation $\sigma \in \pi_q$ and multiindices $\mathbf{k} \in \Phi_{m,q}$ and $\mathbf{i} = (i_1, i_2, \ldots, i_q)$ we write $\mathbf{k}_\sigma = (k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(q)})$ and $\mathbf{i}_\sigma = (i_{\sigma(1)} i_{\sigma(2)}, \ldots, i_{\sigma(q)})$. Then we define recursively numbers $s_{\mathbf{i}h}^{\mathbf{k}g}$: for $\mathbf{k} \in \Phi_{m,2}$ and $\mathbf{i} \in \chi_{\mathbf{k}}$ we write $s_{\mathbf{i}h}^{\mathbf{k}g} = t_{i_1 i_2 h}^{k_1 k_2 g}$, and for $q > 2$ and $\mathbf{k} \in \Phi_{m,q}$ and $\mathbf{i} \in \chi_{\mathbf{k}}$ we write

$$s_{\mathbf{i}h}^{\mathbf{k}g} = \sum_{l=k_1+k_2}^{m-k_3-\cdots-k_q} \sum_{j=1}^{d_l} t_{i_1 i_2 j}^{k_1 k_2 l}\, s_{(j, i_1, i_2, \ldots, i_q)h}^{(l, k_3, k_4, \ldots, k_q)g}.$$

COROLLARY 4. *For* $\mathbf{k} \in \Phi_{m,q}$ *and* $\mathbf{i} \in \chi_{\mathbf{k}}$ *the constants* $s_{\mathbf{i}h}^{\mathbf{k}g}$ *are symmetric in* $\mathbf{k}$ *and* $\mathbf{i}$, *i.e.*

$$s_{\mathbf{i}h}^{\mathbf{k}g} = s_{\mathbf{i}_\sigma h}^{\mathbf{k}_\sigma g} \tag{12}$$

for any permutation $\sigma \in \pi_q$.

We remark that the relations (12) are the matching conditions (in the simple case) mentioned at the end of Section 4 in [11].

A canonical form for a simple commutative array would be obtained if we replaced the basis $\mathscr{B}$ by another filtered basis $\mathscr{B}'$ so that the matrix

$$R = \begin{bmatrix} R_1^{12} & R_1^{13} & \cdots & R_1^{1M} \end{bmatrix}$$

is in a canonical form. This reduces to finding a canonical form for R by multiplying by permutation matrices on the left (if $\mathbf{A}$ is considered as a set only, i.e., the matrices A_i are not considered in any particular order) and by invertible block upper triangular matrices on the right. The first immediate reduction we can achieve is that the nonzero columns in R are linearly independent.

Then in a particular case $d_1 = n$ we can assume that $R_1^{1l} = 0$ for $l \geq 3$. If we replace the vectors z_j^2 by vectors $\hat{z}_j^2 = \sum_{k=1}^{d_2} a_{1jk}^{12} z_k^2$ in the basis $\mathscr{B}$, then $R_2^{12} = I$, and $A_h = T_h^1$ for $h \in \underline{n}$ is a canonical form for $\mathbf{A}$. In the general simple case a block version of the row reduced echelon form (see [9, §2.5] for the standard version and [3, §1] for some generalized versions) applied to R yields toward a canonical form for **A**. However, this requires an extensive case-by-case analysis, and we do not proceed with it. Rather we consider some examples.

5. EXAMPLES

EXAMPLE 1. Suppose $n = 2$. Then sets of matrices that span the algebra $\mathscr{A}$, generated by a pair of matrices $\mathbf{A} = \{A_1, A_2\}$ and the identity matrix I, are described in [2, 13] (see also [7, 20]). In general the sets of matrices given there are not a basis; their elements may be linearly dependent. For example, if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

then neither $\{I, A_1; A_2, A_1A_2\}$ nor $\{I, A_1; A_2; A_1^2\}$ are linearly independent, since $A_1A_2 = A_1^2 = 0$.

However, if **A** is a simple then $\dim \mathscr{A} = N$ by Theorem 3, and so the sets given in [2, 13] are a basis. For instance, if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

then $\mathscr{A} = \mathrm{Sp}\{I, A_1, A_1^2, A_2\}$. Furthermore, if $\{e_i; i \in \underline{4}\}$ is the standard basis for $\mathbb{C}^4$, then in the basis $\mathscr{B} = \{e_1; e_2, e_4; e_3\}$ the reduced form for the array **A** is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \end{bmatrix}.$$

Next we find that $T_h^1 = A_h$, $h = 1, 2$, and $T_1^2 = A_1^2$, and so $\mathscr{T} = \{I, A_1, A_1^2, A_2\}$ is a basis for $\mathscr{A}$.

EXAMPLE 2. We consider a commutative array

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}, \tag{15}$$

which is already in the reduced form (3). The columns of the first row of the array (15) are not linearly independent. To make them so, we substitute the vector $e_4 - \frac{1}{2}e_3$ for the vector e_4 in the basis $\mathscr{B}$. (Here we assume that $\mathscr{B} = \{e_i, i \in \underline{4}\}$ is the standard basis of $\mathbb{C}^4$.) Note that the new basis is still filtered. The array $\mathbf{A}$ in the new basis is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

To find a canonical form for $\mathbf{A}$ we finally replace vectors z_1^2 and z_2^2 by $z_1^2 + z_2^2$ and $2z_2^2$, respectively. The new basis is still filtered, and we find that

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}. \tag{16}$$

So we have that $T_h^1 = A_h$, $h = 1, 2$; $T_1^2 = A_1^2$; and (16) is a canonical form for $\mathbf{A}$. ■

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