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Complexity of matrix problems

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Abstract

In representation theory, the classification problem is called wild if it contains the problem of classifying pairs of matrices up to simultaneous similarity. We show in an explicit form that the last problem contains all classification matrix problems given by quivers or posets. Then we prove that this problem does not contain (but is contained in) the problem of classifying three-valent tensors. Hence, every wild classification problem given by a quiver or poset has the same complexity; moreover, a solution of one of them implies a solution of each of the remaining problems. The problem of classifying three-valent tensors is more complicated.

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1. Introduction

Classification problems of representation theory split into two types: *tame* (or classifiable) and *wild* (containing the problem of classifying pairs of matrices up

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to simultaneous similarity); wild problems are hopeless in a certain sense. These terms were introduced by Donovan and Freislich [6] in analogy with the partition of animals into tame and wild ones.

Gelfand and Ponomarev [13] proved that the problem of classifying pairs of commuting matrices up to simultaneous similarity contains the problem of classifying t-tuples of matrices up to simultaneous similarity for an arbitrary t. (Hence, the problem of classifying pairs of linear operators is as complicated as the problem of classifying 1,000,000-tuples of linear operators.) This implies that the problem of classifying pairs of matrices contains the problem of classifying representations of an arbitrary finite-dimensional algebra,[3] whence it contains matrix problems given by arbitrary quivers.

In Section 2, we give the proof of the last statement by methods of linear algebra; it was sketched in [28, Section 3.1]. The notions of a quiver and its representations were introduced by Gabriel [10] and allow to formulate problems of classifying systems of linear mappings (without relations).

In Section 3, we prove that the problem of classifying pairs of matrices up to simultaneous similarity contains matrix problems given by partially ordered sets. The notion of poset representations was introduced by Nazarova and Roiter [22] and allows to formulate problems of classifying block matrices $[A_1|A_2|\cdots|A_t]$ up to elementary row-transformations of the whole matrix, elementary column-transformations within each vertical strip, and additions of a column of A_i to a column of A_j for a certain set of pairs (i, j).

In Section 4, we prove that the problem of classifying three-valent tensors contains the problem of classifying pairs of matrices up to simultaneous similarity, but it is not contained in the last problem. Three-valent tensors are given by spatial matrices, so we first consider the problem of classifying $m \times n \times q$ spatial matrices $\mathbb{A} = [a_{ijk}]_{i=1}^{m}{}_{j=1}^{n}{}_{k=1}^{q}$ up to *equivalence transformations:*

$$[a_{ijk}] \mapsto [a'_{ijk}], \quad a'_{i'j'k'} = \sum_{ijk} a_{ijk} r_{ii'} s_{jj'} t_{kk'}, \tag{1}$$

where

$$R = [r_{ii'}], \quad S = [s_{jj'}], \quad T = [t_{kk'}] \tag{2}$$

are nonsingular $m \times m$, $n \times n$, and $q \times q$ matrices. We classify $m \times n \times 2$ spatial matrices up to equivalence and prove that the problem of classifying $m \times n \times 3$ spatial matrices up to equivalence contains (but is not contained in) the problem of classifying pairs of matrices up to simultaneous similarity.

Every matrix problem $\mathscr{A}$ is given by a set $\mathscr{A}_1$ of a-tuples of matrices and a set $\mathscr{A}_2$ of admissible transformations with them. We say that a matrix problem $\mathscr{A}$ *is*

[3] Each t-dimensional algebra is a factor algebra $\Lambda = k\langle x_1, \dots, x_t\rangle/J$ of the free algebra of noncommutative polynomials in $x_1, \dots, x_t$. Let $g_1, \dots, g_r$ be generators of J. Then each matrix representation of Λ is a t-tuple of $n \times n$ matrices $(A_1, \dots, A_t)$ satisfying $g_i(A_1, \dots, A_t) = 0, i = 1, \dots, r$; it is determined up to simultaneous similarity.

contained in a matrix problem $\mathscr{B}$ if there exists a b-tuple $\mathscr{T}(x) = \mathscr{T}(x_1, \dots, x_a)$ of matrices, whose entries are noncommutative polynomials in $x_1, \dots, x_a$, such that:

(i) $\mathscr{T}(A) = \mathscr{T}(A_1, \dots, A_a) \in \mathscr{B}_1$ if $A = (A_1, \dots, A_a) \in A_1$;
(ii) for every $A, A' \in \mathscr{A}_1$, A reduces to A' by transformations $\mathscr{A}_2$ if and only if $\mathscr{T}(A)$ reduces to $\mathscr{T}(A')$ by transformations $\mathscr{B}_2$.

In this article (except for Theorem 4.8), the entries of matrices from $\mathscr{T}(x)$ are 0, scalars, or x_i, and we replace them by zero matrices, scalar matrices, or A_i. Suppose $\mathscr{A}$ is contained in $\mathscr{B}$ and a set of *canonical b-tuples* for the problem $\mathscr{B}$ is known (this set has to posses the following property: each b-tuple $A \in \mathscr{B}_1$ reduces to a canonical $A_{\text{can}} \in \mathscr{B}_1$, and A reduces to B if and only if $A_{\text{can}} = B_{\text{can}}$). We reduce to the form $\mathscr{T}(A)$ those canonical b-tuples, for which this is possible. Then all a-tuples A from the obtained set of $\mathscr{T}(A)$ may be considered as canonical a-tuples for $\mathscr{A}$. Hence, a solution of the problem $\mathscr{B}$ implies a solution of $\mathscr{A}$.

In [28], the entries of matrices in the considered matrix problems satisfied systems of linear equations, for this reason the entries of matrices from $\mathscr{T}(x)$ were linear polynomials. In the theory of representations of quivers with relations, the entries of matrices from $\mathscr{T}(x)$ are noncommutative polynomials.

A quiver or poset is called $tame$ $(wild)$ if the problem of classifying its representations is tame (wild). We sum up results of this article in the following theorem:

Theorem 1.1. *All problems of classifying representations of wild quivers or posets have the same complexity*: *any of them contains every other* (*moreover*, *a solution of one implies solutions of the others*). *The problem of classifying three-valent tensors is more complicated since it contains each of them but is not contained in them.*

This theorem explains the existence of the "universal" algorithm [3] (see also [4] or [28]) for reducing the matrices of an arbitrary representation of a quiver or poset to canonical form. The algorithm [3] was used in [26] in order to receive a canonical form of 4×4 matrices up to simultaneous similarity. The algorithm was also used in [28] to prove that the set of canonical $m \times n$ matrices for a *tame* matrix problem forms a finite number of points and straight lines in the affine space of $m \times n$ matrices. This statement is a strengthened form of Drozd's Tame–Wild Theorem [8] and holds for a large class of matrix problems, which includes representations of quivers and posets. A full system of invariants for pairs of matrices up to simultaneous similarity was obtained by Friedland [9].

For each matrix problem, one has an alternative: to solve it or to prove that it is wild and hence is hopeless in a certain sense. Examples of wild problems:

(a) The problem of classifying pairs of $m \times n$ and $n \times n$ matrices up to transformations

$$(A, B) \mapsto (R^{-1}AR, SBR),$$

where R and S are nonsingular matrices (that is, the replacement of the quiver with does not simplify the problem of classifying its representations; see the list (8)).

(b) The problem of classifying pairs of commuting nilpotent matrices (A, B) up to simultaneous similarity, see [13]; this problem was solved in [23] if $AB = BA = 0$.

(c) The problem of classifying quintuples of subspaces in a vector space. A classification of quadruples of subspaces (they may be given by representations of the quiver) was given in [14].

(d) The problem of classifying triples of quadratic forms; its wildness follows from the method of classifying pairs of quadratic forms used in [25, Theorem 4]. A classification of all *tame* systems of linear mappings, bilinear forms, and quadratic forms (without relations) was obtained in [24, Section 4].

(e) The problem of classifying of metric (or self-adjoint) operators in a space with symmetric bilinear form; the problem was solved by many authors if this form is nonsingular, see [25, Theorems 5 and 6].

(f) The problem of classifying normal operators in a space with indefinite scalar product, see [15] or [24, Theorem 5.5].

In the theory of unitary matrix problems, the role of pairs of matrices up to simultaneous similarity is played by the problem of classifying matrices up to unitary similarity; it contains the problem of classifying unitary representations of an arbitrary quiver (its points and arrows correspond to unitary spaces and linear operators), see [27, Section 2.3] and [19].

The partition into tame and wild problems was first exhibited for representations of Abelian groups (see [16]): Bashev [1] and Heller and Reiner [17] classified all representations of the Klein group (i.e., pairs of commuting matrices (A, B) satisfying $A^2 = B^2 = 0$ up to simultaneous similarity) over an algebraically closed field of characteristic 2. In contrast to this, Krugljak [18] showed that if one could solve the corresponding problem for groups of type (p, p) with $p > 2$, then one could classify the representations of *any* group over an algebraically closed field of characteristic p; Heller and Reiner [17] showed this for groups of type $(2, 2, 2)$.

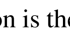
2. Representations of quivers

Classification problems for systems of linear mappings may be formulated in terms of a quiver and its representations introduced by Gabriel [10] (see also [11]). A *quiver* is a directed graph. Its *representation* $\mathscr{A}$ over a field $\mathbb{F}$ is given by assigning to each vertex v a vector space V_v over $\mathbb{F}$ and to each arrow $\alpha : u \to v$ a linear mapping $\mathscr{A}_\alpha : V_u \to V_v$ of the corresponding vector spaces. Two representations $\mathscr{A}$

and $\mathscr{A}'$ are *isomorphic* if there exists a system of linear bijections $\mathscr{S}_v : V_v \to V'_v$ transforming $\mathscr{A}$ to $\mathscr{A}'$; that is, for which the diagram

$$\begin{tikzcd} V_u \arrow[r, "\mathscr{A}_\alpha"] \arrow[d, "\mathscr{S}_u"'] & V_v \arrow[d, "\mathscr{S}_v"] \\ V'_u \arrow[r, "\mathscr{A}'_\alpha"] & V'_v \end{tikzcd} \tag{3}$$

is commutative ($\mathscr{S}_v\mathscr{A}_\alpha = \mathscr{A}'_\alpha\mathscr{S}_u$) for every arrow $\alpha : u \to v$. The *direct sum* of $\mathscr{A}$ and $\mathscr{A}'$ is the representation $\mathscr{A} \oplus \mathscr{A}'$ formed by $V_v \oplus V'_v$ and $\mathscr{A}_\lambda \oplus \mathscr{A}'_\lambda$.

For example, the problems of classifying representations of the quivers , , and  are the problems of classifying linear operators (whose solution is the Jordan or Frobenius normal form), pairs of linear mappings from one space to another (the matrix pencil problem, solved by Kronecker), and pairs of linear operators in a vector space (i.e., pairs of matrices up to simultaneous similarity).

Furthermore, a representation of the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\varepsilon"] & \\ 1 \arrow[loop left, "\alpha"] \arrow[ur, "\beta"] \arrow[rr, shift left, "\gamma"] \arrow[rr, shift right, "\delta"'] & & 3 \arrow[loop right, "\zeta"] \end{tikzcd} \tag{4}$$

over a field $\mathbb{F}$ is a set of linear mappings

$$\begin{tikzcd} & V_2 \arrow[dr, "\mathcal{A}_\varepsilon"] & \\ V_1 \arrow[loop left, "\mathcal{A}_\alpha"] \arrow[ur, "\mathcal{A}_\beta"] \arrow[rr, shift left, "\mathcal{A}_\gamma"] \arrow[rr, shift right, "\mathcal{A}_\delta"'] & & V_3 \arrow[loop right, "\mathcal{A}_\zeta"] \end{tikzcd} \tag{5}$$

Let n_1, n_2, n_3 be the dimensions of V_1, V_2, V_3; selecting bases in these spaces, we can give the representation (5) by the sequence

$$\begin{aligned} A &= (A_\alpha, A_\beta, A_\gamma, A_\delta, A_\varepsilon, A_\zeta) \\ &\in \mathbb{F}^{n_1\times n_1} \times \mathbb{F}^{n_2\times n_1} \times \mathbb{F}^{n_3\times n_1} \times \mathbb{F}^{n_3\times n_1} \times \mathbb{F}^{n_3\times n_2} \times \mathbb{F}^{n_3\times n_3} \end{aligned} \tag{6}$$

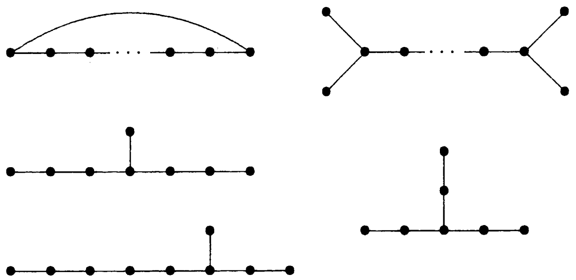
of matrices of linear mappings $\mathscr{A}_\alpha, \mathscr{A}_\beta, \mathscr{A}_\gamma, \mathscr{A}_\delta, \mathscr{A}_\varepsilon, \mathscr{A}_\zeta$. If a sequence of matrices $A' = (A'_\alpha, A'_\beta, \dots, A'_\zeta)$ gives an isomorphic representation, then

$$A' = \left(S_1A_\alpha S_1^{-1}, S_2A_\beta S_1^{-1}, S_3A_\gamma S_1^{-1}, S_3A_\delta S_1^{-1}, S_3A_\varepsilon S_2^{-1}, S_3A_\zeta S_3^{-1}\right), \tag{7}$$

where S_1, S_2, S_3 are the matrices of linear bijections $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ (see (3)). Note that the change of bases in V_1, V_2, V_3 by matrices $S_1^{-1}, S_2^{-1}, S_3^{-1}$ also transforms A to A'; that is, A and A' give the same representation (5) but in different bases.

Therefore, the problem of classifying representations of the quiver (4) reduces to the problem of classifying matrix sequences (6) up to transformations (7) with nonsingular matrices S_1, S_2, S_3.

The list of tame quivers and a classification of their representations were obtained independently by Donovan and Freislich [7] and Nazarova [20] (see also [11, Section 11]). They proved that a connected quiver is tame if and only if it is a subquiver of (or coincides with) one of the quivers



(8)

with an arbitrary orientation of edges.

As follows from the next theorem, the problem of classifying quiver representations has the same complexity for all wild quivers.

Theorem 2.1. *The problem of classifying pairs of matrices up to simultaneous similarity contains the problem of classifying representations of an arbitrary quiver.*

Proof. We will prove the theorem for representations of the quiver (4) since the proof for the other quivers is analogous. For each sequence (6), we construct the pair of matrices

$$(M, N) = \left(\begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 2I_{n_2} & 0 & 0 \\ 0 & 0 & 3I_{n_3} & 0 \\ 0 & 0 & 0 & 4I_{n_3} \end{bmatrix}, \begin{bmatrix} A_\alpha & 0 & 0 & 0 \\ A_\beta & 0 & 0 & 0 \\ A_\gamma & 0 & 0 & 0 \\ A_\delta & A_\varepsilon & I_{n_3} & A_\zeta \end{bmatrix} \right). \quad (9)$$

Let (M, N') be analogously constructed from

$$A' = \left(A'_\alpha, A'_\beta, A'_\gamma, A'_\delta, A'_\varepsilon, A'_\zeta\right),$$

and let the pairs (M, N) and (M, N') be simultaneously similar:

$$S^{-1}MS = M, \quad S^{-1}NS = N'. \quad (10)$$

The equality $MS = SM$ implies

$$S = S_1 \oplus S_2 \oplus S_3 \oplus S_4.$$

Equating in $NS = SN'$ the blocks with indices $(4, 3)$ gives $S_3 = S_4$. By the second equality in (10), the pairs (M, N) and (M, N') are simultaneously similar if and only if A' is obtained from A by transformations (7). $\square$

3. Representations of posets

Many matrix problems may be formulated in terms of representations of partially ordered sets (posets) introduced by Nazarova and Roiter [22]; see also [11, Section 1.3].

Let $(T, \preceq)$ be a finite poset. Since every partial ordering on a finite set is supplemented to a total ordering, we suppose that $T = \{1, 2, \ldots, t\}$ and

$$i \prec j \implies i < j. \tag{11}$$

A *representation* of $(T, \preceq)$ over a field $\mathbb{F}$ is an arbitrary matrix

$$A = [A_1|A_2|\cdots|A_t] \tag{12}$$

over $\mathbb{F}$ divided into t vertical strips. Two representations are *isomorphic* if one reduces to the other by the following transformations:

(a) elementary row-transformations of the whole matrix;
(b) elementary column-transformations within each vertical strip;
(c) additions of a column of A_i to a column of A_j if $i \prec j$.

(By a sequence of transformations (b) and (c), we may add an arbitrary linear combination of columns of A_i to a column of A_j if $i \prec j$; indeed, we may multiply the columns of A_i by scalars, add them to a column of A_j, and then divide them by the same scalars.) The *direct sum* of representations A and A' is the representation

$$A \oplus A' = \left[\begin{array}{cc|cc|c|cc} A_1 & 0 & A_2 & 0 & \cdots & A_t & 0 \\ 0 & A'_1 & 0 & A'_2 & \cdots & 0 & A'_t \end{array}\right].$$

These notions arose in the theory of representations of finite-dimensional algebras. The assumption that $\preceq$ is a partial ordering is not a limitation. Suppose that $\trianglelefteq$ is an arbitrary reflective binary relation on $T = \{1, 2, \ldots, t\}$. We may define representations of $(T, \trianglelefteq)$ and their isomorphisms as above replacing $i \prec j$ in (c) by $i \vartriangle j$. Let (12) be a representation of $(T, \trianglelefteq)$. If $i \vartriangle j$ and $j \vartriangle i$, then we may join strips i and j to a single strip with arbitrary column-transformations within it. If $i \vartriangle j$ and $j \vartriangle l$, then we may add a column a of A_i to a column c of A_l through a column b of A_j:

$$\begin{aligned}(a, b, c) &\mapsto (a, a + b, c) \mapsto (a, a + b, a + b + c)\\ &\mapsto (a, b, a + b + c) \mapsto (a, b, a + c).\end{aligned}$$

And so we may assume that $i \vartriangle l$, leaving the set of admissible transformations unchanged. Then $(T, \trianglelefteq)$ becomes a poset.

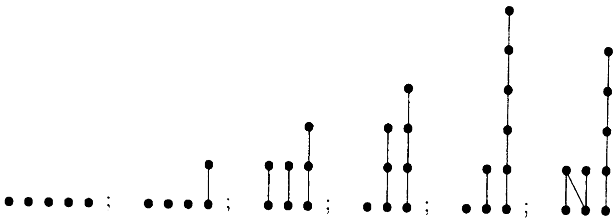
For instance, every representation of $(\{1, 2, 3\}, \leqslant)$ reduces to the form

$$\left[\begin{array}{cc|cc|cc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The following theorem is a well-known corollary of the Krull–Schmidt theorem [2, Section 1, Theorem 3.6] for additive categories (the categories of representations of quivers and posets are additive).

Theorem 3.1. *Every representation of a quiver or poset decomposes into a direct sum of indecomposable representations uniquely, up to isomorphism of summands.*

Nazarova [21] proved that a poset is wild if and only if it contains a poset from the following list:



(the points represent the elements of a poset, and $a \prec b$ if and only if the point a is under the point b and they are linked by a line).

As follows from the next theorem and from the definition of wildness, the problem of classifying representations of a poset has the same complexity for all wild posets.

Theorem 3.2. *The problem of classifying pairs of matrices up to simultaneous similarity contains the problem of classifying representations of an arbitrary poset.*

Proof. *Step* 1. Let us prove that the problem of classifying pairs of matrices up to simultaneous similarity contains the problem of classifying block matrices

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}, \quad A_l = \begin{bmatrix} A_{l11} & \cdots & A_{l1t} \\ \cdots & \cdots & \cdots \\ A_{lt1} & \cdots & A_{ltt} \end{bmatrix}, \tag{13}$$

up to transformations:

(i) arbitrary elementary transformations within each of rt horizontal strips and each of t vertical strips;
(ii) additions of columns of strip i to columns of strip j if $i < j$;
(iii) within each A_l, additions of rows of strip i to rows of strip j if $i < j$ ($i, j \in \{1, \dots, t\}$).

We first consider the case $r = 1$, $t = 3$, and all A_{lij} of size 1×1. Then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{14}$$

Basing on A, we construct the pair of matrices

$$(M, N) = \left(\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \begin{bmatrix} 0 & N_1 \\ 0 & 0 \end{bmatrix} \right),$$

where

$$M_1 = \left[\begin{array}{c|cc|ccc} 1 & & & & & \\ \hline & 1 & & & & \\ & 1 & 1 & & & \\ \hline & & & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{array}\right], \quad M_2 = \left[\begin{array}{c|cc|ccc} 2 & & & & & \\ \hline & 2 & & & & \\ & 1 & 2 & & & \\ \hline & & & 2 & & \\ & & & 1 & 2 & \\ & & & & 1 & 2 \end{array}\right]$$

(we omit zeros), and

$$N_1 = \left[\begin{array}{c|cc|ccc} a_{11} & a_{12} & 0 & a_{13} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{23} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & a_{33} & 0 & 0 \end{array}\right]. \tag{15}$$

Let (M, N') be analogously constructed based on

$$A' = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{16}$$

and let (M, N) be simultaneously similar to (M, N'):

$$(S^{-1}MS, S^{-1}NS) = (M, N'). \tag{17}$$

Then $MS = SM$. Since M is a Jordan matrix with two distinct eigenvalues, $S = S_1 \oplus S_2$, where

$$S_1 = \left[\begin{array}{c|cc|ccc} x_{11} & x_{12} & 0 & x_{13} & 0 & 0 \\ \hline 0 & x_{22} & 0 & x_{23} & 0 & 0 \\ x_{21} & x'_{22} & x_{22} & x'_{23} & x_{23} & 0 \\ \hline 0 & 0 & 0 & x_{33} & 0 & 0 \\ 0 & x_{32} & 0 & x'_{33} & x_{33} & 0 \\ x_{31} & x'_{32} & x_{32} & x''_{33} & x'_{33} & x_{33} \end{array}\right]$$

and

$$S_2 = \left[\begin{array}{c|cc|ccc} y_{11} & y_{12} & 0 & y_{13} & 0 & 0 \\ \hline 0 & y_{22} & 0 & y_{23} & 0 & 0 \\ y_{21} & y'_{22} & y_{22} & y'_{23} & y_{23} & 0 \\ \hline 0 & 0 & 0 & x_{33} & 0 & 0 \\ 0 & y_{32} & 0 & y'_{33} & y_{33} & 0 \\ y_{31} & y'_{32} & y_{32} & y''_{33} & y'_{33} & y_{33} \end{array}\right]$$

(the form of all matrices commuting with a given Jordan matrix is in [10, Section VIII, §2]. By (17), $NS = SN'$ and hence $N_1S_2 = S_1N_1'$, where N_1' is of the form (15) with b_{ij} instead of a_{ij}. Equating nonzero entries in the matrices $N_1S_2 = S_1N_1'$, we obtain $AY = XA'$ (see (14) and (16)), where

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}. \tag{18}$$

Therefore, (M, N) is simultaneously similar to (M, N') if and only if there exist nonsingular matrices X and Y of the form (18) such that $A' = X^{-1}AY$; that is, if and only if A reduces to A' by transformations (i)–(iii).

In the general case, A has the form (13). Basing on A, we construct (M, N) as follows:

$$M = M_1 \oplus \cdots \oplus M_{r+1}, \quad M_l = lI \oplus J_2(lI) \oplus \cdots \oplus J_t(lI),$$

where

$$J_i(lI) = \begin{bmatrix} lI & & & \\ I & lI & & \\ & \ddots & \ddots & \\ & & I & lI \end{bmatrix}$$

is obtained from the $i \times i$ Jordan block $J_i(l)$. The matrix N consists of the blocks A_{lij} (see (13)) and zeros: each block A_{lij} is located at the place of those block of M that is the intersection of the last horizontal strip of $J_i(lI)$ and the first vertical strip of $J_j((r+1)I)$.

Let (M, N') be analogously constructed based on A' of the form (13), and let (M, N) be simultaneously similar to (M, N'). By (17), the transforming matrix S commutes with M, and hence

$$S = S_1 \oplus \cdots \oplus S_{r+1}, \quad S_l = [S_{lij}]_{i,j=1}^{t},$$

where each S_{lij} is of the form

$$\begin{bmatrix} X_1 & & & \\ X_2 & X_1 & & \\ \vdots & \ddots & \ddots & \\ X_p & \cdots & X_2 & X_1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} X_1 & & & & \\ X_2 & X_1 & & & \\ \vdots & \ddots & \ddots & & \\ X_p & \cdots & X_2 & X_1 & \end{bmatrix}$$

(see [10, Section VIII, §2]). The form of S implies that (M, N) is simultaneously similar to (M, N') if and only if A reduces to A' by transformations (i)–(iii).

Step 2. We prove that the problem of classifying matrices (13) up to transformations (i)–(iii) contains the problem of classifying representations of each poset

$$\mathscr{P} = (T, \preceq), \quad T = \{1, \ldots, t\},$$

satisfying (11).

Namely, we show that there exists a block matrix $A^{(r)} = [A_l]_{l=1}^{r}$ of the form (13) such that the set of admissible transformations (i)–(iii) that preserve $A_2, \dots, A_r$ produces on its first strip

$$\left[A_{111}|A_{112}|\cdots|A_{11t}\right] \tag{19}$$

the matrix problem given by $\mathscr{P}$; we say in this situation that $A^{(r)}$ *simulates* the poset $\mathscr{P}$. Of course, this property does not depend on the entries of A_1.

We will suppose that $\preceq$ is not a total ordering; that is, it does not coincide with $\leqslant$ (otherwise, $\mathscr{P}$ is simulated by $A^{(1)} = [A_1]$). We construct $A_2, \dots, A_r$ sequentially. Assume that $A^{(m)} = [A_l]_{l=1}^{m},\ m \geqslant 1$, has been constructed, and it simulates a poset $(T, \trianglelefteq)$ with

$$G(\preceq) \subsetneqq G(\trianglelefteq) \subseteq G(\leqslant),$$

where $G(\preceq)$, $G(\trianglelefteq)$, and $G(\leqslant)$ are the sets of pairs $(i, j) \in T \times T$ such that $i \preceq j,\ i \trianglelefteq j$, and $i \leqslant j$.

Let us construct A_{m+1} so that $A^{(m+1)} = [A_l]_{l=1}^{m+1}$ simulates a poset $(T, \sqsubseteq)$, for which

$$G(\preceq) \subseteq G(\sqsubseteq) \subsetneqq G(\trianglelefteq). \tag{20}$$

If we take

$$A_{m+1} = \begin{bmatrix} 0 & \cdots & 0 & I \\ 0 & \cdots & I & 0 \\ \cdots & \cdots & \cdots & \cdots \\ I & \cdots & 0 & 0 \end{bmatrix}, \tag{21}$$

then the matrix $A^{(m+1)}$ simulates the same poset $(T, \trianglelefteq)$ as $A^{(m)}$ since every admissible (with respect to $\trianglelefteq$) transformation (ii) with columns of $A^{(m+1)}$ spoils (21), but it is restored by transformations (iii).

Thus we patch up (21) as follows. Choose $(a, b) \in G(\trianglelefteq) \backslash G(\preceq)$. In the set of horizontal strips of (21) intersecting at I with vertical strips $a, a+1, \dots, b$ we make the transposition that gathers at the top the strips intersecting at I with vertical strips $a, a_2, \dots, a_l$, where

$$\mathscr{A} = \{a, a_2, \dots, a_l\} = \{i \mid a \preceq i < b\}. \tag{22}$$

For instance, if $t = 8,\ (a, b) = (3, 7)$, and $\mathscr{A} = \{3, 5, 6\}$, then we obtain

$$A_{m+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each addition of a column of strip i to a column of strip j, $i < j$, spoils A_{m+1}, but it is restored by transformations (iii) for all (i, j) except when $(i, j) \in \mathscr{A} \times \mathscr{B}$, where $\mathscr{B} = \{a, a+1, \dots, b\} \backslash \mathscr{A}$. Hence, the obtained block matrix $A^{(m+1)}$ simulates the poset $(T, \sqsubseteq)$ with

$$G(\sqsubseteq) = G(\trianglelefteq) \backslash \mathscr{A} \times \mathscr{B}.$$

Then $(a, b) \in G(\trianglelefteq) \backslash G(\sqsubseteq)$. By (22), $i \not\preceq j$ for all $(i, j) \in \mathscr{A} \times \mathscr{B}$. Therefore, the relation $\sqsubseteq$ satisfies (20).

We construct $A_2, A_3, \dots$ until obtain a block matrix $A^{(r)} = [A_l]_{l=1}^{r}$ that simulates the poset $\mathscr{P} = (T, \preceq)$. $\square$

4. Spatial matrices and tensors

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let V^* be its dual space. A *tensor of type* $\binom{q}{q}$ (in other words, a tensor of covariant order p and contravariant order q) on V is a multilinear map

$$A : \underbrace{V \times \cdots \times V}_{p \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{q \text{ copies}} \to \mathbb{F}, \tag{23}$$

see [5, Chapter 11]. A tensor of type $\binom{0}{2}$ is a bilinear form on V. A tensor of type $\binom{1}{1}$ determines a linear map $V \to V$ as follows: $v \mapsto A(v, ?) \in V^{**} = V$. A finite-dimensional algebra is given by a tensor of type $\binom{1}{2}$.

Relatively to a basis $e_1, \dots, e_m$ of V, the tensor (23) is given by the $m \times \cdots \times m$ spatial matrix

$$\begin{aligned} &\mathbb{A} = \left[a_{i_1 \cdots i_{p+q}}\right]_{i_1, \dots, i_{p+q}=1}^{m}, \\ &a_{i_1 \cdots i_{p+q}} = A\left(e_{i_1}, \dots, e_{i_p}, e_{i_{p+1}}^*, \dots, e_{i_{p+q}}^*\right), \end{aligned} \tag{24}$$

where $e_1^*, \dots, e_m^*$ is the dual basis of V^*. Then

$$A\left(\sum_j \beta_{1j} e_j, \dots, \sum_j \beta_{p+q,j} e_j^*\right) = \sum_{j_1, \dots, j_{p+q}} \beta_{1,j_1} \cdots \beta_{p+q, j_{p+q}} a_{i_1 \cdots i_{p+q}}.$$

Let $\mathbb{B} = [b_{i_1 \cdots i_{p+q}}]$ be the spatial matrix of this tensor relatively to another basis $f_1, \dots, f_m$ of V, and let $C = [c_{ij}]$ be the transition matrix. Then

$$b_{j_1 \cdots j_{p+q}} = \sum_{i_1, \dots, i_{p+q}=1}^{m} a_{i_1 \cdots i_{p+q}} c_{i_1 j_1} \cdots c_{i_p j_p} d_{i_{p+1} j_{p+1}} \cdots d_{i_{p+q} j_{p+q}}, \tag{25}$$

where $[d_{ij}] = C^{\vee} = (C^{\mathrm{T}})^{-1}$.

Therefore, the problem of classifying tensors of type $\binom{q}{p}$ over a field $\mathbb{F}$ is the problem of classifying $m \times \cdots \times m$ spatial matrices $[a_{i_1 \cdots i_{p+q}}]$ over $\mathbb{F}$ up to transformations $[a_{i_1 \cdots i_{p+q}}] \mapsto [b_{j_1 \cdots j_{p+q}}]$ of the form (25), where $[c_{ij}] = C$ is an arbitrary nonsingular $m \times m$ matrix and $[d_{ij}] = C^{\vee}$.

In this section, we study the problem of classifying three-valent tensors ($p + q = 3$). For every $p \in \{0, 1, 2, 3\}$, we prove that the problem of classifying tensors of type $\binom{3-p}{p}$ contains the problem of classifying pairs of matrices up to simultaneous similarity, but is not contained in it.

We start with an investigation of spatial matrices up to equivalence since each tensor of type $\binom{3-p}{p}$ is an $m \times m \times m$ spatial matrix $\mathbb{A}$, and admissible transformations with it are equivalence transformations (1) given by matrices (2) of the form

$$(R, S, T) = (\underbrace{C, \ldots, C}_{p \text{ copies}}, \underbrace{C^{\vee}, \ldots, C^{\vee}}_{3-p \text{ copies}}). \tag{26}$$

Lemma 4.1. *For every $m \times n \times q$, the following three classification problems are equivalent*:

(i) *The problem of classifying $m \times n \times q$ spatial matrices up to equivalence.*

(ii) *The problem of classifying q-tuples of $m \times n$ matrices $(A_1, \ldots, A_q)$ up to*
   (a) *simultaneous elementary transformations with $A_1, \ldots, A_q$, and*
   (b) *the replacement of $(A_1, \ldots, A_q)$ with*

$$(A_1, \ldots, A_q)T = (A_1 t_{11} + \cdots + A_q t_{q1}, \ldots, A_1 t_{1q} + \cdots + A_q t_{qq}), \tag{27}$$

   where $T = [t_{ij}]$ is a nonsingular $q \times q$ matrix.

(iii) *The problem of classifying spaces of $m \times n$ matrices of dimension at most q up to multiplication by a nonsingular matrix from the left and by a nonsingular matrix from the right.*

Proof. An $m \times n \times q$ spatial matrix $\mathbb{A} = [a_{ijk}]_{i=1}^{m}{}_{j=1}^{n}{}_{k=1}^{q}$ may be given by the q-tuple $m \times n$ matrices

$$\mathscr{A} = (A_1, \ldots, A_q), \quad A_k = [a_{ijk}]_{ij}. \tag{28}$$

If $\mathbb{A}$ is determined up to equivalence, then $\mathscr{A}$ is determined up to transformations (a) and (b); furthermore, the vector space of $m \times n$ matrices generated by $A_1, \ldots, A_q$ is determined up to simultaneous multiplications of its matrices by a nonsingular $m \times m$ matrix from the left and a nonsingular $n \times n$ matrix from the right. $\square$

Remark 4.2. The matrix T from (27) is a product of elementary matrices. Hence, every transformation (b) is a sequence of elementary transformations: the transposition of A_i and A_j, the multiplication of A_i by a nonzero scalar, and the replacement of A_i by $A_i + bA_j$, $i \neq j$.

Remark 4.3. It follows from the equivalence of classification problems (i) and (ii) that a q-tuple of $m \times n$ matrices $(A_1, \dots, A_q)$ reduces to a q-tuple $(B_1, \dots, B_q)$ by transformations (a) and (b) if and only if there exists a nonsingular $q \times q$ matrix T such that $(A_1, \dots, A_q)T$ (see (27)) is simultaneously equivalent to $(B_1, \dots, B_q)$.

4.1. Classification of $m \times n \times 2$ spatial matrices

For every natural number r, we define two $(r-1) \times r$ matrices

$$F_r = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad G_r = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}.$$

Theorem 4.4. *Over an algebraically closed field, every pair of $m \times n$ matrices reduces by transformations* (a) *and* (b) *to a direct sum of the form*

$$\bigoplus_i (F_{r_i}, G_{r_i}) \oplus \bigoplus_j \left(F_{s_j}^{\mathrm{T}}, G_{s_j}^{\mathrm{T}}\right) \oplus \bigoplus_{k=1}^{q} (I_{l_k}, J_{l_k}(\lambda_k)). \tag{29}$$

This sum is determined uniquely, up to permutation of summands and up to linear-fractional transformations of the sequence of eigenvalues:

$$(\lambda_1, \dots, \lambda_q) \mapsto \left(\frac{a + b\lambda_1}{c + d\lambda_1}, \dots, \frac{a + b\lambda_q}{c + d\lambda_q}\right), \tag{30}$$

where $c + d\lambda_1 \neq 0, \dots, c + d\lambda_q \neq 0$, and $ad - bc \neq 0$.

Proof. Let $\mathscr{A} = (A_1, A_2)$ be a pair of $m \times n$ matrices. Using transformations (a) from Lemma 4.1, we reduce it to the form

$$\bigoplus_i (F_{r_i}, G_{r_i}) \oplus \bigoplus_j \left(F_{s_j}^{\mathrm{T}}, G_{s_j}^{\mathrm{T}}\right) \oplus \bigoplus_{k=1}^{q_1} (I_{l_k}, J_{l_k}(\lambda_k)) \oplus \bigoplus_{k=q_1+1}^{q} (J_{l_k}(0), I_{l_k}) \tag{31}$$

(the classification of pencils of matrices, see [12, Section XII]). This sum is determined uniquely up to permutation of summands.

We will say that a pair of matrices is *pencil-decomposable* if it reduces by transformations (a) to a direct sum of pairs. Each transformation (b) with

$$T = \begin{bmatrix} c & a \\ d & b \end{bmatrix}, \quad ad - bc \neq 0$$

replaces each summand (P, Q) of (31) with

$$(P', Q') = (cP + dQ, aP + bQ). \tag{32}$$

This pair is pencil-indecomposable (otherwise, T^{-1} transforms its direct decomposition to the direct decomposition of (P, Q), but each summand of (31) is pencil-indecomposable). All indecomposable pairs of $(r-1) \times r$ matrices reduce to (F_r, G_r) by transformations (a). Hence, if $(P, Q) = (F_r, G_r)$, then (P', Q') reduces to (F_r, G_r) too. This proves that every transformation (b) with the pair $\mathscr{A} = (A_1, A_2)$ does not change the summand $\bigoplus_i (F_{r_i}, G_{r_i})$ in the decomposition (31). The same holds for the summand $\bigoplus_j (F_{s_j}^{\mathrm{T}}, G_{s_j}^{\mathrm{T}})$ too.

If $q_1 < q$, then we reduce the pair (31) to the pair (29) (with other $\lambda_1, \ldots, \lambda_{q_1}$) as follows. We convert all summands $(I_{l_k}, J_{l_k}(\lambda_k))$ and $(J_{l_k}(0), I_{l_k})$ to pencil-indecomposable pairs with nonsingular first matrices by transformation (32) with $c = b = 1$, $a = 0$, and a nonzero d such that $d\lambda_1 \neq -1, \ldots, d\lambda_{q_1} \neq -1$. Then we reduce these summands to the form $(I, J(\lambda))$ by transformations (a).

Each transformation (32) converts all summands $(I_{l_k}, J_{l_k}(\lambda_k))$ of (29) to the pairs of matrices $(cI_{l_k} + dJ_{l_k}(\lambda_k), aI_{l_k} + bJ_{l_k}(\lambda_k))$, which are simultaneously equivalent to

$$\big(I_{l_k}, (aI_{l_k} + bJ_{l_k}(\lambda_k)) \cdot (cI_{l_k} + dJ_{l_k}(\lambda_k))^{-1}\big). \tag{33}$$

The matrices $aI_{l_k} + bJ_{l_k}(\lambda_k)$ and $cI_{l_k} + dJ_{l_k}(\lambda_k)$ are triangular; their diagonal entries are $a + b\lambda_k$ and $c + d\lambda_k$. Hence, the pair of matrices (33) is simultaneously equivalent to

$$\left(I_{l_k}, J_{l_k}\left(\frac{a + b\lambda_k}{c + d\lambda_k}\right)\right),$$

this gives the transformation (30). $\square$

4.2. Wildness of tensors and $m \times n \times 3$ spatial matrices

Theorem 4.5. *The problem of classifying $m \times n \times 3$ spatial matrices up to equivalence is wild.*

Proof. For every pair (X, Y) of $r \times r$ matrices, we construct the triple of matrices

$$(A_1, A_2, A_3(X, Y)) = (B_1, B_2, B_3) \oplus (I_r, I_r, I_r) \oplus (C_1, C_2, C_3(X, Y)),$$

where

$$(B_1, B_2, B_3) = \left(\begin{bmatrix} I_{4r} & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & I_{2r} & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & I_{2r} \end{bmatrix}\right)$$

and

$$(C_1, C_2, C_3(X, Y)) = \left(I_{4r}, \begin{bmatrix} 0 & & & \\ I_r & 0 & & \\ 0 & I_r & 0 & \\ 0 & 0 & I_r & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ X & 0 & 0 & \\ 0 & Y & 0 & 0 \end{bmatrix}\right).$$

We will prove that $(A_1, A_2, A_3(X, Y))$ reduces to $(A_1, A_2, A_3(X', Y'))$ by transformations (a) and (b) from Lemma 4.1 if and only if the pairs of matrices (X, Y) and (X', Y') are simultaneously similar.

We write $(M_1, M_2, M_3) \sim (N_1, N_2, N_3)$ if these triples of matrices are simultaneously equivalent.

Suppose that $(A_1, A_2, A_3(X, Y))$ reduces to $(A_1, A_2, A_3(X', Y'))$ by transformations (a) and (b). By Remark 4.3, there exists a nonsingular 3×3 matrix $T = [t_{ij}]$ such that $(A_1, A_2, A_3(X, Y))T \sim (A_1, A_2, A_3(X', Y'))$. Hence,

$$\operatorname{rank}(A_1 t_{1j} + A_2 t_{2j} + A_3(X, Y)t_{3j}) = \begin{cases} \operatorname{rank} A_j & \text{if } j = 1 \text{ or } j = 2, \\ \operatorname{rank} A_3(X', Y') & \text{if } j = 3. \end{cases}$$

This implies $t_{ij} = 0$ if $i \neq j$ since

$$\begin{aligned} \operatorname{rank}(A_1 + A_2\alpha + A_3(X, Y)\beta) > 9r = \operatorname{rank} A_1 &> \operatorname{rank}(A_2 + A_3(X, Y)\gamma) \\ &> 6r = \operatorname{rank} A_2 > \operatorname{rank} A_3(X', Y') \end{aligned}$$

for all α, β, γ such that $(\alpha, \beta) \neq (0, 0)$ and $\gamma \neq 0$.

Therefore, t_{11}, t_{22}, and t_{33} are nonzero, and

$$\begin{aligned} (A_1 t_{11}, A_2 t_{22}, A_3(X, Y)t_{33}) = &(B_1 t_{11}, B_2 t_{22}, B_3 t_{33}) \oplus (I_r t_{11}, I_r t_{22}, I_r t_{33}) \\ &\oplus (C_1 t_{11}, C_2 t_{22}, C_3(X, Y)t_{33}) \end{aligned}$$

is simultaneously equivalent to

$$(A_1, A_2, A_3(X', Y')) = (B_1, B_2, B_3) \oplus (I_r, I_r, I_r) \oplus (C_1, C_2, C_3(X', Y')).$$

They can be considered as isomorphic representations of the quiver $1 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} 2$. By Theorem 3.1,

$$(I_r t_{11}, I_r t_{22}, I_r t_{33}) \sim (I_r, I_r, I_r), \tag{34}$$

$$(C_1 t_{11}, C_2 t_{22}, C_3(X, Y)t_{33}) \sim (C_1, C_2, C_3(X', Y')) \tag{35}$$

since $(B_1 t_{11}, B_2 t_{22}, B_3 t_{33}) \sim (B_1, B_2, B_3)$, the triples (34) are direct sums of triples of 1×1 matrices, and each of the triples (35) cannot be simultaneously equivalent to a direct sum containing a triple of 1×1 matrices. By (34), $t_{11} = t_{22} = t_{33}$. Then $(C_1 t_{11}, C_2 t_{22}, C_3(X, Y)t_{33}) \sim (C_1, C_2, C_3(X, Y))$, and by (35) $(C_1, C_2, C_3(X, Y)) \sim (C_1, C_2, C_3(X', Y'))$. Since $C_1 = I$, $(C_2, C_3(X, Y))$ is simultaneously similar to $(C_2, C_3(X', Y'))$.

Therefore, there is a nonsingular matrix R such that

$$C_2 R = R C_2, \quad C_3(X, Y)R = R C_3(X', Y').$$

By the first equality,

$$R = \begin{bmatrix} R_1 & & & \\ R_2 & R_1 & & \\ R_3 & R_2 & R_1 & \\ R_4 & R_3 & R_2 & R_1 \end{bmatrix}.$$

By the second equality, $XR_1 = R_1 X'$ and $YR_1 = R_1 Y'$. $\square$

In the remaining part of Section 4.2, we prove the following theorem.

Theorem 4.6. *For each* $p \in \{0, 1, 2, 3\}$, *the problem of classifying tensors of type* $\binom{3-p}{p}$ *is wild since it contains the problem of classifying spatial matrices up to equivalence.*

An $m \times n \times q$ spatial matrix $\mathbb{A} = [a_{ijk}]$ may be given by any of the following sequences of matrices:

$$\mathscr{A}^{(1)} = \left(A_1^{(1)}, \ldots, A_m^{(1)}\right), \quad A_i^{(1)} = [a_{ijk}]_{jk}, \tag{36}$$

$$\mathscr{A}^{(2)} = \left(A_1^{(2)}, \ldots, A_n^{(2)}\right), \quad A_j^{(2)} = [a_{ijk}]_{ik}, \tag{37}$$

$$\mathscr{A}^{(3)} = \left(A_1^{(3)}, \ldots, A_q^{(3)}\right), \quad A_k^{(3)} = [a_{ijk}]_{ij}. \tag{38}$$

The last sequence coincides with (28). They play the same role in the theory of spatial matrices as the sequences of rows and columns in the theory of matrices. If $\mathbb{A}$ is determined up to equivalence, we may produce arbitrary elementary transformations within each of the sequences (36)–(38) by analogy with transformations (b) from Lemma 4.1 for (28). Moreover, two spatial matrices are equivalent if and only if one reduces to the other by elementary transformations within (36)–(38).

It follows that the triple

$$\operatorname{rank} \mathbb{A} = (r_1, r_2, r_3), \quad r_i = \operatorname{rank} \mathscr{A}^{(i)} \tag{39}$$

(r_i is the rank of the system of matrices $\mathscr{A}^{(i)}$ in the vector space of matrices of the corresponding size), is invariant with respect to equivalence transformations with $\mathbb{A}$. We will say that $\mathbb{A}$ is *regular* if $\operatorname{rank} \mathbb{A} = (m, n, q)$, where $m \times n \times q$ is the size of $\mathbb{A}$.

Let us make the first r_1 matrices in $\mathscr{A}^{(1)}$ linearly independent and the others zero by elementary transformations with $\mathbb{A}$. Then we reduce the "new" $\mathscr{A}^{(2)}$ and $\mathscr{A}^{(3)}$ in the same way. The obtained spatial matrix is equivalent to $\mathbb{A}$ and has the form $\mathbb{A}' \oplus \mathbb{O}$, where $\mathbb{A}'$ is a regular $r_1 \times r_2 \times r_3$ spatial matrix and $\mathbb{O}$ is the zero $(m - r_1) \times (n - r_2) \times (q - r_3)$ spatial matrix. We will call $\mathbb{A}'$ a *regular part* of $\mathbb{A}$.

Lemma 4.7. *Two spatial matrices of the same size are equivalent if and only if their regular parts are equivalent.*

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be $m \times n \times q$ spatial matrices. Without loss of generality, we will assume that

$$\mathbb{A} = \mathbb{A}' \oplus \mathbb{O}, \quad \mathbb{B} = \mathbb{B}' \oplus \mathbb{O}, \tag{40}$$

where $\mathbb{A}'$ and $\mathbb{B}'$ are their regular parts.

Necessity. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are equivalent, and their equivalence is given by matrices R, S, and T (see (2)). Then $\mathbb{A}'$ and $\mathbb{B}'$ have the same size $r_1 \times r_2 \times r_3$, where $(r_1, r_2, r_3) = \operatorname{rank} \mathbb{A}$. Following (28), we will give $\mathbb{A}$ and $\mathbb{B}$ by the sequences $\mathscr{A} = (A_1, \ldots, A_q)$ and $\mathscr{B} = \left(B_1, \ldots, B_q\right)$. Put

$$(C_1, \ldots, C_q) = (A_1, \ldots, A_q)T, \tag{41}$$

then

$$(R^{\mathrm{T}}C_1S, \ldots, R^{\mathrm{T}}C_qS) = \big(B_1, \ldots, B_q\big) \tag{42}$$

by analogy with transformations (a) and (b) from Lemma 4.1.

Let us partition R, S, and T into blocks $R = [R_{ij}]_{i,j=1}^2$, $S = [S_{ij}]_{i,j=1}^2$, and $T = [T_{ij}]_{i,j=1}^2$ in accordance with the decompositions (40), where R_{11}, S_{11}, and T_{11} have sizes $r_1 \times r_1$, $r_2 \times r_2$, and $r_3 \times r_3$. If $T_{21} \neq 0$, then $C_l \neq 0$ for a certain $l > r_3$ since $A_1, \ldots, A_{r_3}$ are linearly independent (see (41) and (40)). By (42), $B_l \neq 0$; a contradiction.

Hence, $T_{21} = 0$; analogously $R_{21} = 0$ and $S_{21} = 0$. It follows that R_{11}, S_{11}, and T_{11} are nonsingular and produce an equivalence of $\mathbb{A}'$ to $\mathbb{B}'$.

Sufficiency. Suppose that $\mathbb{A}'$ and $\mathbb{B}'$ are equivalent and their equivalence is given by matrices R, S, and T. Then the matrices $R \oplus I_{m-r_1}$, $S \oplus I_{n-r_2}$, and $T \oplus I_{q-r_3}$ produce an equivalence of $\mathbb{A}$ to $\mathbb{B}$. $\square$

Proof of Theorem 4.6. For an $m \times n \times q$ spatial matrix $\mathbb{A}$, we construct the spatial block matrix

$$\mathbb{H}(\mathbb{A}) = [\mathbb{H}_{ijk}]_{i,j,k=1}^3, \quad \mathbb{H}_{ijk} = \begin{cases} \mathbb{A} & \text{if } (i,j,k) = (1,2,3), \\ \mathbb{O} & \text{otherwise}, \end{cases} \tag{43}$$

where the diagonal blocks $\mathbb{H}_{111}$, $\mathbb{H}_{222}$, and $\mathbb{H}_{333}$ have sizes $m \times m \times m$, $n \times n \times n$ and $q \times q \times q$.

Let $\mathbb{B}$ be another $m \times n \times q$ spatial matrix. Then $\mathbb{A}$ is equivalent to $\mathbb{B}$ if and only if $\mathbb{H}(\mathbb{A})$ and $\mathbb{H}(\mathbb{B})$ determine the same tensor of type $\binom{3-p}{p}$. Indeed, if matrices Q_1, Q_2, Q_3 give an equivalence of $\mathbb{A}$ to $\mathbb{B}$, then $\mathbb{H}(\mathbb{A})$ reduces to $\mathbb{H}(\mathbb{B})$ by equivalence transformations (1) satisfying (26), where C is

$$Q_1^\vee \oplus Q_2^\vee \oplus Q_3^\vee, \quad Q_1 \oplus Q_2^\vee \oplus Q_3^\vee, \quad Q_1 \oplus Q_2 \oplus Q_3^\vee, \quad \text{or} \quad Q_1 \oplus Q_2 \oplus Q_3$$

if, respectively, p is 0, 1, 2, or 3. Conversely, if $\mathbb{H}(\mathbb{A})$ is reduced to $\mathbb{H}(\mathbb{B})$ by transformations (1), then they are equivalent. Since their regular parts are regular parts of $\mathbb{A}$ and $\mathbb{B}$ too, $\mathbb{A}$ and $\mathbb{B}$ are equivalent by Lemma 4.7.

We have proved that the problem of classifying tensors of type $\binom{3-p}{p}$ contains the problem of classifying spatial matrices up to equivalence. By Theorems 4.5, the first problem is wild. $\square$

4.3. Spatial matrices and tensors are "very wild"

Theorem 4.8. *The problem of classifying pairs of matrices up to simultaneous similarity does not contain both*

(i) *the problem of classifying $m \times n \times 2$ spatial matrices up to equivalence, and*
(ii) *the problem of classifying tensors of type $\binom{3-p}{p}$ for each $p \in \{0, 1, 2, 3\}$.*

Proof. (i) To the contrary, suppose there exists a pair $\mathscr{T}(x_1, x_2)$ of matrices, whose entries are noncommutative polynomials in x_1, x_2, such that a pair $A = (A_1, A_2)$ of $m \times n$ matrices reduces to $A' = (A'_1, A'_2)$ by transformations (a) and (b) from Lemma 4.1 if and only if $\mathscr{T}(A)$ is simultaneously similar to $\mathscr{T}(A')$.

Put

$$A = \left(\begin{bmatrix}1 & & \\ & 0 & \\ & & 1\end{bmatrix}, \begin{bmatrix}0 & & \\ & 1 & \\ & & 0\end{bmatrix}\right), \quad A' = \left(\begin{bmatrix}1 & & \\ & 0 & \\ & & 1\end{bmatrix}, \begin{bmatrix}0 & & \\ & 1 & \\ & & 1\end{bmatrix}\right).$$

Since the pair of 1×1 matrices ([1], [0]) reduces to ([1], [1]) by transformations (b), the pair $\mathscr{T}([1], [0])$ is simultaneously similar to $\mathscr{T}([1], [1])$. Therefore, $\mathscr{T}(A) = \mathscr{T}([1], [0]) \oplus \mathscr{T}([0], [1]) \oplus \mathscr{T}([1], [0])$ is simultaneously similar to $\mathscr{T}(A') = \mathscr{T}([1], [0]) \oplus \mathscr{T}([0], [1]) \oplus \mathscr{T}([1], [1])$, and hence A reduces to A' by transformations (a) and (b). By definition of transformations (a) and (b), there exist $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma \neq 0$ and

$$A'' = \left(\begin{bmatrix}\alpha & & \\ & \beta & \\ & & \alpha\end{bmatrix}, \begin{bmatrix}\gamma & & \\ & \delta & \\ & & \gamma\end{bmatrix}\right)$$

is simultaneously equivalent to A'. Equating the ranks of matrices in A'' and A' gives $\beta = \delta = 0$, contrary to $\alpha\delta - \beta\gamma \neq 0$.

(ii) Suppose the problem of classifying pairs of matrices up to simultaneous similarity contains the problem of classifying tensors of type $\binom{3-p}{p}$. Then, by Theorem 4.6, the first problem contains the problem of classifying spatial matrices up to equivalence, contrary to (i). □

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