Existence and global attractivity of unique positive periodic solution for a model of hematopoiesis

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Abstract

In this paper, we consider the generalized model of hematopoiesis

\[ x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t-\tau_i(t))}. \]

By using a fixed point theorem, some criteria are established for the existence of the unique positive \( \omega \)-periodic solution \( \tilde{x} \) of the above equation. In particular, we not only give the conclusion of convergence of \( \{x_k\} \) to \( \tilde{x} \), where \( \{x_k\} \) is a successive sequence, but also show that \( \tilde{x} \) is a global attractor of all other positive solutions.

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1. Introduction

The nonlinear delay differential equation

\[ P'(t) = -\gamma P(t) + \frac{\beta_0 \theta^n}{\theta^n + P^n(t-\tau)}. \] (1.1)

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where \( n, \gamma, \beta_0, \theta, \tau \in (0, \infty) \), has been used by Mackey and Glass [15] to describe the dynamics of hematopoiesis (blood cell production). Here, \( P(t) \) denotes the density of mature cells in blood circulation, and \( \tau \) is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream. For more details of Eq. (1.1), see [15].

The change of variables \( P(t) = \theta x(t) \) transforms Eq. (1.1) to the delay differential equation:

\[
x'(t) = -\gamma x(t) + \frac{\beta}{1 + x^n(t - \tau)}.
\]

where \( \beta = \beta_0/\theta \). Subsequently, Eqs. (1.1) and (1.2) have been studied by many authors [2,4,9,11,12,16–18] from various angles.

In 1991, Györi and Ladas [9] obtained sufficient conditions for the global attractivity of unique positive equilibrium of Eq. (1.2). In addition, they also proposed Open problem 11.6.3 [9, p. 322] of extending this result to equations with several delays.

As we know, the variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In addition, we remark that in recent years, periodic population dynamics has become a very popular subject. In fact, several different periodic models have been studied in [1,3,5,6,9,10,12–14,19–26].

In view of this, the authors [10,21] recently considered the following modification of (1.2),

\[
x'(t) = -a(t)x(t) + \frac{b(t)}{1 + x^n(t - \tau(t))},
\]

where \( a, b \in C(R, (0, \infty)) \) and \( \tau \in C(R, R) \) are all \( \omega \)-periodic functions. However, they only studied the existence of positive periodic solutions to Eq. (1.3).

Motivated by Open problem 11.6.3 [9], in this paper, we shall study the existence and global attractivity of unique positive periodic solution of the following generalized model of hematopoiesis:

\[
x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))},
\]

where \( n > 0 \) is a constant.

In Eq. (1.4), we shall use the following hypothesis:

\((H_1)\) \( a, b_i, \tau_i \ (i = 1, 2, \ldots, m) \) are positive continuous \( \omega \)-periodic functions.

In what follows for a continuous \( \omega \)-periodic function \( h \), we shall denote by

\[
\bar{h} = \max_{t \in [0,\omega]} \{h(t)\}, \quad \underline{h} = \min_{t \in [0,\omega]} \{h(t)\}.
\]

Also, let

\[
\tau = \max_{1 \leq i \leq m} \{\bar{\tau}_i\}, \quad b = \int_{0}^{\omega} \sum_{i=1}^{m} b_i(s) \, ds
\]
and
\[ M = \frac{e^{\int_0^\omega a(u)\,du}}{e^{\int_0^\omega a(u)\,du} - 1}, \quad N = \frac{1}{e^{\int_0^\omega a(u)\,du} - 1}. \]

In view of the actual applications of Eq. (1.4), we shall only consider the solutions of Eq. (1.4) with the initial condition
\[ x(t) = \phi(t) \quad \text{for} \quad -\tau \leq t \leq 0, \quad \phi \in C\left([-\tau, 0], [0, \infty)\right), \quad \phi(0) > 0. \]

2. Existence of unique positive periodic solution

We shall use a fixed point theorem to obtain the existence of a unique positive \( \omega \)-periodic solution to Eq. (1.4). In order to prove our results, the following definitions and lemmas are needed.

Let \( E \) be a real Banach space. \( \theta \) denotes the zero element in \( E \), and \( P \) is a cone of \( E \). The semi-order induced by the cone \( P \) is denoted by “\( \leq \)”. That is, \( x \leq y \) if and only if \( y - x \in P \).

**Definition 2.1.** A cone \( P \) of \( E \) is said to be normal if there exists a positive constant \( \delta \) such that \( \|x + y\| \geq \delta \) for any \( x, y \in P, \|x\| = \|y\| = 1 \).

**Definition 2.2.** \( P \) is a cone of \( E \) and \( A: P \to P \) is an operator. \( A \) is called decreasing, if \( \theta \leq x \leq y \) implies \( Ax \geq Ay \).

**Lemma 2.1.** [7,8] Suppose that

(i) \( P \) is a normal cone of a real Banach space \( E \) and \( A: P \to P \) is decreasing and completely continuous;
(ii) \( A\theta > \theta, A^2\theta \geq \varepsilon_0 A\theta \), where \( \varepsilon_0 > 0 \);
(iii) For any \( \theta < x \leq A\theta \) and \( 0 < \lambda < 1 \), there exists \( \eta = \eta(x, \lambda) > 0 \) such that
\[ A(\lambda x) \leq \left[ \frac{1}{\lambda(1 + \eta)} \right]^{-1} Ax. \]  
(2.1)

Then \( A \) has exactly one positive fixed point \( \tilde{x} > \theta \). Moreover, constructing successive sequence \( x_k = Ax_{k-1} \) (\( k = 1, 2, 3, \ldots \)) for any initial \( x_0 \in P \), it follows that
\[ \|x_k - \tilde{x}\| \to 0 \quad (k \to \infty). \]

In order to apply Lemma 2.1, let \( X \) be the real Banach space \( \{x: x \in C(R, R), x(t + \omega) = x(t), \ t \in R\} \) with \( \|x\| = \sup_{t \in [0, \omega]} |x(t)|, x \in X \).

Let \( P \) be a cone of \( X \) defined by
\[ P = \{x \in X: x(t) \geq 0, \ t \in [0, \omega]\}. \]

It is easy to see that \( P \) is a normal cone of \( X \) and its interior is \( P^0 = \{x \in X: x(t) > 0, \ t \in [0, \omega]\} \).

Define the operator \( A \) by
\[ (Ax)(t) = \int_t^{t+\omega} G(t, s) \sum_{i=1}^m \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \, ds, \]  
(2.2)
where
\[ G(t, s) = \frac{e^{\int_{t}^{s} a(u) \, du}}{e^{\int_{0}^{\omega} a(u) \, du}} - 1, \quad s \in [t, t + \omega]. \]

In addition, we have
\[ G(t + \omega, s + \omega) = G(t, s), \quad s \in [t, t + \omega], \quad (2.3) \]
and
\[ 0 < N = \min_{t \in [0, \omega], s \in [t, t + \omega]} \{ G(t, s) \} \leq \max_{t \in [0, \omega], s \in [t, t + \omega]} \{ G(t, s) \} = M. \]

It is easy to see that Eq. (1.4) has unique \( \omega \)-periodic positive solution if and only if the operator \( A \) has exactly one fixed point in \( P^o \).

**Lemma 2.2.** Assume that \((H_1)\) holds. Then \( A : P \to P \) is completely continuous.

**Proof.** First, we prove \( A : P \to P \). In view of \((H_1)\), (2.2), (2.3) and the definition of \( P \), for any \( x \in P, t \in \mathbb{R} \), we have

\[
(Ax)(t + \omega) = \int_{t+\omega}^{t+2\omega} G(t + \omega, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \, ds
\]

\[
= \int_{t}^{t+\omega} G(t + \omega, u + \omega) \sum_{i=1}^{m} \frac{b_i(u + \omega)}{1 + x^n(u + \omega - \tau_i(u + \omega))} \, du
\]

\[
= \int_{t}^{t+\omega} G(t, u) \sum_{i=1}^{m} \frac{b_i(u)}{1 + x^n(u - \tau_i(u))} \, du
\]

\[
= (Ax)(t).
\]

In addition, it is clear that \( Ax \in C(\mathbb{R}, \mathbb{R}) \) and \((Ax)(t) \geq 0\) for any \( x \in P, t \in \mathbb{R} \). Hence, \( Ax \in P \) for any \( x \in P \). Thus \( A : P \to P \).

Next, we show that \( A : P \to P \) is completely continuous. Obviously, \( A : P \to P \) is continuous. For any \( x \in P, t \in [0, \omega] \), we have

\[
|(Ax)(t)| = \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \, ds \leq Mb
\]
and

\[
|(Ax)'(t)| = \left| G(t, t + \omega) \sum_{i=1}^{m} \frac{b_i(t + \omega)}{1 + x^n(t + \omega - \tau_i(t + \omega))} - G(t, t) \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))} \right|
\]

\[
- a(t) \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \, ds
\]
\[
\begin{align*}
\quad = & \left| -a(t)(Ax)(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))} \right| \\
\quad \leq & \bar{a}M + \tilde{b}.
\end{align*}
\]

Hence, \(\{Ax: x \in P\}\) is a family of uniformly bounded and equicontinuous functions on \([0, \omega]\). By using the Arzela–Ascoli Theorem, \(A: P \to P\) is compact. Therefore, \(A: P \to P\) is completely continuous. The proof of Lemma 2.2 is complete. \(\Box\)

**Lemma 2.3.** Assume that \((H_1)\) holds. Then every solution to Eq. (1.4) is positive and bounded on \([0, \infty)\).

**Proof.** From (1.4), for any \(\phi \in C(\mathbb{R}, [0, \infty))\), \(\phi(0) > 0\) and \(t \geq 0\), we have

\[
x(t) = \phi(0)e^{-\int_0^t a(u)du} + \int_0^t e^{-\int_u^t a(s)ds} \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} ds.
\]

Hence, \(x(t)\) is defined on \([-\tau, \infty)\) and positive on \([0, \infty)\). Now, we prove that every solution to Eq. (1.4) is bounded. Otherwise, there exists an unbounded solution \(x(t)\) to Eq. (1.4). We then have

\[
x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))} \leq -a(t)x(t) + \sum_{i=1}^{m} b_i(t) \leq -a(t)x(t) + \sum_{i=1}^{m} \tilde{b}_i.
\]

In addition, there exist \(0 < t_1 < t_2\) such that

\[
x(t_1) < x(t_2) \quad \text{and} \quad -a(t_2) + \sum_{i=1}^{m} \tilde{b}_i < 0,
\]

which, combined with (2.4), imply that \(x'(t_2) < 0\). Let \(x(t_3) = \max_{t_1 \leq t \leq t_2} \{x(t)\}\). It is easy to see that \(t_3 \neq t_1\) and \(t_3 \neq t_2\). So \(x'(t_3) = 0\). From (2.4) we have

\[
x'(t_3) \leq -a(t_3) + \sum_{i=1}^{m} \tilde{b}_i \leq -a(t_2) + \sum_{i=1}^{m} \tilde{b}_i < 0,
\]

which is a contradiction. Consequently, \(x(t)\) is bounded. The proof of Lemma 2.3 is complete. \(\Box\)

**Theorem 2.1.** Assume that \((H_1)\) holds and one of the following two conditions is satisfied:

\(\text{(H2)} \quad n \leq 1;\)

\(\text{(H3)} \quad n > 1 \quad \text{and} \quad (n - 1)(Mb)^n \leq 1.\)

Then Eq. (1.4) has a unique \(\omega\)-periodic positive solution \(\bar{x}(t)\). Moreover,

\[
\|x_k - \bar{x}\| \to 0 \quad (k \to \infty),
\]

where \(x_k = Ax_{k-1}\) (\(k = 1, 2, \ldots\)) for any initial \(x_0 \in P\).
Proof. First, it is clear that \( P \) is normal and that \( A \) is decreasing. As a result of Lemma 2.2, the condition (i) of Lemma 2.1 is satisfied.

Next, we show that the condition (ii) of Lemma 2.1 is satisfied. For \( t \in R \), we have

\[
Mb \geq (A\theta)(t) = \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} b_i(s) \, ds \geq Nb > 0.
\]

Thus, \( A\theta > \theta \) and

\[
(A^2\theta)(t) = \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + (A\theta)^n(s - \tau_i(s))} \, ds \\
\geq \frac{1}{1 + (Mb)^n} \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} b_i(s) \, ds \\
= \frac{1}{1 + (Mb)^n} (A\theta)(t).
\]

So \( A^2\theta \geq \varepsilon_0 A\theta \), where \( \varepsilon_0 = \frac{1}{1 + (Mb)^n} > 0 \).

Finally, we prove that the condition (iii) of Lemma 2.1 is also satisfied. For any \( \theta < x \leq A\theta \) and \( 0 < \lambda < 1 \), we have

\[
0 < \|x\| \leq \|A\theta\| \leq Mb
\]

and

\[
A(\lambda x)(t) = \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + \lambda^n x^n(s - \tau_i(s))} \, ds \\
= \frac{1}{\lambda} \int_{t}^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \frac{\lambda [1 + x^n(s - \tau_i(s))]}{1 + \lambda^n x^n(s - \tau_i(s))} \, ds.
\]

Further, for \( \theta < x \leq A\theta \),

\[
\frac{\lambda [1 + x^n(s - \tau_i(s))]}{1 + \lambda^n x^n(s - \tau_i(s))} \leq \lambda^{1-n} \left[ 1 + \frac{\lambda^n - 1}{1 + \lambda^n x^n(s - \tau_i(s))} \right] \\
\leq \lambda^{1-n} \left[ 1 + \frac{\lambda^n - 1}{1 + \lambda^n (Mb)^n} \right] = \lambda^{1-n} \frac{1 + (Mb)^n}{1 + \lambda^n (Mb)^n}.
\]

Set \( f(\lambda) = \frac{\lambda [1 + (Mb)^n]}{1 + \lambda^n (Mb)^n} \). Then, we have

\[
f'(\lambda) = \frac{[1 + (Mb)^n][1 + \lambda^n (Mb)^n] - [1 + (Mb)^n] \lambda n (Mb)^n \lambda^{n-1}}{[1 + \lambda^n (Mb)^n]^2} \\
= \frac{[1 + (Mb)^n][1 + (1 - n)\lambda^n (Mb)^n]}{[1 + \lambda^n (Mb)^n]^2}.
\]

which, together with the fact that (H2) or (H3) is satisfied, implies that \( f'(\lambda) > 0 \) (\( 0 < \lambda < 1 \)). Therefore, for any \( 0 < \lambda < 1 \), we have

\[
0 = f(0) < f(\lambda) < f(1) = 1.
\]
Set $f(\lambda) = \frac{1}{1 + \eta}$, where $\eta = \eta(\lambda) > 0$, $0 < \lambda < 1$. From (2.5) and (2.6), we have

$$A(\lambda x) \leq \left[\lambda (1 + \eta)\right]^{-1} Ax.$$ 

Hence, (2.1) holds. By Lemma 2.1, we see that $A$ has exactly one positive fixed point $\tilde{x} > \theta$. Moreover, $\|x_k - \tilde{x}\| \to 0$ ($k \to \infty$), where $x_k = Ax_{k-1}$ ($k = 1, 2, \ldots$) for any initial $x_0 \in P$. In addition, for $t \in R$, we obtain

$$\tilde{x}(t) = (A\tilde{x})(t) = \int_t^{t+\omega} G(t, s) \sum_{i=1}^{m} \frac{b_i(s)}{1 + \tilde{x}^n(s - \tau_i(s))} \, ds \geq \frac{Nb}{1 + \|\tilde{x}\|^n} > 0.$$ 

Thus, $\tilde{x} \in P^0$. The proof of Theorem 2.1 is complete. □

**Remark 2.1.** Theorem 2.1 gives the sufficient conditions for the existence of unique positive periodic solution to Eq. (1.4). In particular, it contains the conclusion of convergence of $x_k$ to $\tilde{x}$.

**Remark 2.2.** Theorem 2.1 generalizes and improves Corollary 3.2 [10] and Corollary 3.2 [21].

**Remark 2.3.** From the proof of Theorem 2.1, we see that the unique periodic positive solution $\tilde{x}$ to Eq. (1.4) satisfies

$$\frac{Nb}{1 + (Mb)^n} \leq \tilde{x}(t) \leq Mb, \quad t \geq 0.\quad (2.7)$$

### 3. Global attractivity of unique positive periodic solution

In the previous section, we have showed the existence of unique positive periodic solution to Eq. (1.4). In this section, we study its attractivity. For this purpose, we need some preparation.

Consider the function

$$f(x) = -\bar{a}x + \sum_{i=1}^{m} \frac{b_i}{1 + x^n},$$

where $x \in [0, \infty)$. Then, $f(x) = 0$ has a unique solution $x_1$ such that

$$f(x) \begin{cases} > 0, & \text{if } 0 < x < x_1, \\ < 0, & \text{if } x > x_1. \end{cases} \quad (3.1)$$

**Lemma 3.1.** Let $x(t)$ be a noneventually monotonic positive solution to Eq. (1.4). Then, there exists a $T \in R$ such that

$$x(t) \geq x_1 e^{-\int_{-T}^{0} a(\alpha) \, d\alpha} := X_1, \quad \text{for } t \geq T.$$

**Proof.** Since $x(t)$ is not eventually monotonic, there exists a $T_1 \in R$ such that $x(t)$ attains relative minimum on $(T_1, \infty)$. Let $x(t)$ attain relative minimum at $t_1 > T_1$. If $x(t_1) < x_1$, then

$$0 = x'(t_1) = -a(t_1)x(t_1) + \sum_{i=1}^{m} \frac{b_i(t_1)}{1 + x^n(t_1 - \tau_i(t_1))}$$

$$> -\bar{a}x_1 + \sum_{i=1}^{m} \frac{b_i}{1 + x^n(t_1 - \tau_i(t_1))}.$$
This, combined with (3.1), implies that there exists some 1 \leq i_1 \leq m such that \( x(t_1 - \tau_{i_1}(t_1)) > x_1 \). From (1.4), we have

\[
\left(xe^{\int_0^t a(u)du}\right)' = e^{\int_0^t a(u)du} \sum_{i=1}^m \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}.
\]  

(3.2)

Now, integrating (3.2) from \( t_1 - \tau_{i_1}(t_1) \) to \( t_1 \), we obtain

\[
x(t_1) = x(t_1 - \tau_{i_1}(t_1)) e^{-\int_{t_1 - \tau_{i_1}(t_1)}^{t_1} a(u)du} + \int_{t_1 - \tau_{i_1}(t_1)}^{t_1} e^{\int_0^s a(u)du} \sum_{i=1}^m \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} ds.
\]

Further,

\[
x(t_1) = x(t_1 - \tau_{i_1}(t_1)) e^{-\int_{t_1 - \tau_{i_1}(t_1)}^{t_1} a(u)du} + \int_{t_1 - \tau_{i_1}(t_1)}^{t_1} e^{-\int_{t_1 - \tau_{i_1}(t_1)}^{s} a(u)du} \sum_{i=1}^m \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} ds
\]

> \( x_1 e^{-\int_0^{t_1 - \tau_{i_1}(t_1)} a(u)du} = X_1 \).

So, if \( x(t) \) attains relative minimum at \( t_1 > T_1 \), then \( x(t_1) \geq X_1 \). Hence, we can pick a \( T \geq T_1 \) such that \( x(t) \geq X_1 \), for \( t \geq T \). The proof of Lemma 3.1 is complete.

\[\square\]

**Lemma 3.2.** Let \( \bar{x}(t) \) be a positive \( \omega \)-periodic solution to Eq. (1.4). Then

\[
\bar{x}(t) \geq X_1, \quad \text{for } t \in R.
\]  

(3.3)

**Proof.** If \( \bar{x}(t) \) is not a constant function, then it is not monotonic. Thus, (3.3) follows from Lemma 3.1. Now, if \( \bar{x}(t) = l \) is a constant function, then

\[-a(t)l + \sum_{i=1}^m \frac{b_i(t)}{1 + l^n} = 0.\]

It follows that

\[-\bar{a}l + \sum_{i=1}^m \frac{b_i}{1 + l^n} \leq 0.\]

This, combined with (3.1), implies that \( x_1 \leq l \). Note that \( X_1 \leq x_1 \). So (3.3) holds. The proof of Lemma 3.2 is complete. \[\square\]

**Lemma 3.3.** Let \( x(t) \) be a positive solution to Eq. (1.4) such that \( x(t) - \bar{x}(t) \) is oscillatory, where \( \bar{x}(t) \) is a positive \( \omega \)-periodic solution to Eq. (1.4). Then, there exists a \( T \in R \) such that

\[
x(t) \geq X_1, \quad \text{for } t > T.
\]  

(3.4)

**Proof.** If \( x(t) \) is not eventually monotonic, then (3.4) follows from Lemma 3.1. Now, assume that \( x(t) \) is eventually monotonic. Since \( x(t) - \bar{x}(t) \) is oscillatory and \( \bar{x}(t) \) is periodic, it follows that there exists a \( T \in R \) such that
min_{s \in [0, \omega]} \{ \tilde{x}(s) \} \leq x(t), \quad \text{for } t > T.

This, combined with (3.3), implies (3.4) immediately. The proof of Lemma 3.3 is complete. \hfill \square

The following result provides some sufficient conditions for the global attractivity of unique \( \omega \)-periodic solution to Eq. (1.4).

**Theorem 3.1.** Assume that (H\(_1\)) holds and one of the following two conditions is satisfied:

(H\(_4\)) \( n \leq 1 \) and \( \frac{n \tilde{x}^{(n-1)}(t)}{1 + \tilde{x}^n(t - \tau_i(t))} Mb \leq 1; \)

(H\(_5\)) \( n > 1 \) and \( (n - 1)(n - 1)(Mb)^n \leq 1. \)

Then, Eq. (1.4) has a unique positive \( \omega \)-periodic solution \( \tilde{x}(t) \). Moreover, every solution \( x(t) \) to Eq. (1.4) satisfies

\[ \lim_{t \to \infty} [x(t) - \tilde{x}(t)] = 0, \]

i.e., \( \tilde{x}(t) \) is a global attractor of all other positive solutions to Eq. (1.4).

**Proof.** By Theorem 2.1, it is clear that Eq. (1.4) has a unique positive \( \omega \)-periodic solution \( \tilde{x}(t) \). Let \( y(t) = x(t) - \tilde{x}(t) \). Then Eq. (1.4) reduces to

\[ y'(t) = -a(t)y(t) + \sum_{i=1}^{m} b_i(t) \frac{1}{1 + \tilde{x}^n(t - \tau_i(t))} \left[ \frac{1 + \tilde{x}^n(t - \tau_i(t))}{1 + \tilde{x}^n(t - \tau_i(t))} - 1 \right]. \tag{3.5} \]

Now, we prove \( \lim_{t \to \infty} y(t) = 0 \). First, suppose that \( y(t) \) is eventually positive solution to (3.5). Thus, in view of (3.5), \( y'(t) < 0 \) for all sufficiently large \( t \). So \( \lim_{t \to \infty} y(t) = l \geq 0 \). We claim \( l = 0 \). Otherwise, \( l > 0 \) and from (3.5) we see that there exists \( T > 0 \) such that

\[ y'(t) < -la(t), \quad t \geq T. \]

Integrating the above inequality from \( T \) to \( \infty \), we obtain

\[ l - y(T) < -l \int_{T}^{\infty} a(t) \, dt = -\infty, \]

which is a contradiction. For the case that \( y(t) \) is eventually negative, the proof is similar and will be omitted.

Next, assume that \( y(t) \) is oscillatory. From Lemma 2.3, we know that \( y(t) \) is bounded. Set

\[ c = \limsup_{t \to \infty} y(t) \quad \text{and} \quad d = \liminf_{t \to \infty} y(t). \tag{3.6} \]

Then \( c \geq 0 \) and \( d \leq 0 \). For arbitrary small positive constant \( \varepsilon \), \( d - \varepsilon < 0 \) and \( c + \varepsilon > 0 \), in view of (3.6), there exists \( T_{\varepsilon} > 0 \) such that

\[ d - \varepsilon < y(t) < c + \varepsilon \quad \text{for all } t \geq T_{\varepsilon} - \tau. \tag{3.7} \]

From (3.5), we obtain

\[ \frac{d}{dt} [y(t)e^{\int_{0}^{t} a(u) \, du}] = e^{\int_{0}^{t} a(u) \, du} \left[ \sum_{i=1}^{m} b_i(t) \frac{1}{1 + \tilde{x}(t - \tau_i(t))} \left( \frac{1 + \tilde{x}^n(t - \tau_i(t))}{1 + \tilde{x}^n(t - \tau_i(t))} - 1 \right) \right]. \tag{3.8} \]
In addition,
\[
\ln \left[ \frac{1 + \tilde{x}^n(t - \tau_i(t))}{1 + x^n(t - \tau_i(t))} \right] = \ln \left[ 1 + \tilde{x}^n(t - \tau_i(t)) \right] - \ln \left[ 1 + x^n(t - \tau_i(t)) \right]
\]
\[
= - \frac{n\xi^{n-1}(t - \tau_i(t))}{1 + \xi^n(t - \tau_i(t))} y(t - \tau_i(t)),
\]
where \( \xi(t - \tau_i(t)) \) lies between \( x(t - \tau_i(t)) \) and \( \tilde{x}(t - \tau_i(t)) \), \( i = 1, 2, \ldots, m \). From Lemmas 3.2 and 3.3, there exists \( T_0 > 0 \) such that
\[
\xi(t - \tau_i(t)) \geq X_1, \quad t \geq T_0, \quad i = 1, 2, \ldots, m.
\]
We assume first that (H$_4$) holds. It is easy to see that \( \frac{n^\alpha(n-1)}{1 + \alpha x^n} \) is decreasing on \( (0, +\infty) \). Let \( T_1 = \max\{T_s, T_0\} \). From (3.7), (3.9) and (3.10), we obtain
\[
\ln \left[ \frac{1 + \tilde{x}^n(t - \tau_i(t))}{1 + x^n(t - \tau_i(t))} \right] \leq - \frac{n\xi^{n-1}(t - \tau_i(t))}{1 + \xi^n(t - \tau_i(t))} (d - \varepsilon) < - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon), \quad t \geq T_1.
\]
From (3.8) and (3.11), we have
\[
\frac{d}{dr}[y(t)e^{\int_0^r a(u)du}]
\]
\[
= e^{\int_0^r a(u)du} \sum_{i=1}^m b_i(t) \left( \frac{1}{1 + \tilde{x}^n(t - \tau_i(t))} \left[ \exp \left( \ln \left[ \frac{1 + \tilde{x}^n(t - \tau_i(t))}{1 + x^n(t - \tau_i(t))} \right] - 1 \right) \right) \right)
\]
\[
\leq e^{\int_0^r a(u)du} \sum_{i=1}^m b_i(t) \left( \frac{1}{1 + \tilde{x}^n(t - \tau_i(t))} \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right) \right)
\]
\[
= \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right] e^{\int_0^r a(u)du} \left[ \tilde{x}'(t) + a(t)\tilde{x}(t) \right]
\]
\[
= \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right] \frac{d}{dr}[\tilde{x}(t)e^{\int_0^r a(u)du}].
\]
Integrating both sides of (3.12) from \( T_1 \) to \( t (> T_1) \), we obtain
\[
y(t)e^{\int_0^r a(u)du} \leq y(T_1)e^{\int_0^{T_1} a(u)du} + \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right] \left[ \tilde{x}(t)e^{\int_0^r a(u)du} - \tilde{x}(T_1)e^{\int_0^{T_1} a(u)du} \right],
\]
which, combined with (2.7), yields
\[
y(t) \leq y(T_1)e^{-\int_{T_1}^r a(u)du} + \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right] \left[ Mb - \tilde{x}(T_1)e^{-\int_{T_1}^r a(u)du} \right],
\]
\( t \geq T_1 \).
In view of (3.6) and (3.13), we find
\[
c \leq Mb \left[ \exp \left( - \frac{nX_1^{n-1}}{1 + X_1^n} (d - \varepsilon) - 1 \right) \right].
As $\varepsilon$ is arbitrary small, we conclude that
\[
  c \leq M_b \left[ \exp \left( -\frac{n \chi_{1}^{(n-1)}}{1 + \chi_{1}^{n}} d \right) - 1 \right].
\]  
(3.14)

By using an argument similar to that given above, we obtain
\[
d \geq M_b \left[ \exp \left( -\frac{n \chi_{1}^{(n-1)}}{1 + \chi_{1}^{n}} c \right) - 1 \right].
\]  
(3.15)

From a result in [24, p. 360], $M_b \frac{n \chi_{1}^{(n-1)}}{1 + \chi_{1}^{n}} \leq 1$ implies that (3.14) and (3.15) have unique solution $c = d = 0$. Therefore
\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ x(t) - \tilde{x}(t) \right] = 0.
\]

Assume now that (H5) holds. Since $\frac{n \theta(n - 1)}{1 + \theta n}$ is increasing on $(0, (n - 1) \frac{1}{n})$ and decreasing on $((n - 1) \frac{1}{n}, \infty)$, we have
\[
0 < \frac{n \theta(n - 1)}{1 + \theta n} \leq (n - 1)\left(1 - \frac{1}{n}\right), \quad \text{for all } \theta > 0.
\]  
(3.16)

From (3.8), (3.9) and (3.16), we obtain
\[
\frac{d}{dt} \left[ y(t) e^{\int_{0}^{T} \alpha(u) \, du} \right] = e^{\int_{0}^{T} \alpha(u) \, du} \left\{ \sum_{i=1}^{m} b_i(t) \frac{1}{1 + \tilde{x}^{n}(t - \xi(t))} \left[ \exp \left( \ln \left[ \frac{1 + \tilde{x}^{n}(t - \xi(t))}{1 + \chi^{n}(t - \xi(t))} \right] \right) - 1 \right] \right\}
\[
\leq e^{\int_{0}^{T} \alpha(u) \, du} \left\{ \sum_{i=1}^{m} b_i(t) \frac{1}{1 + \tilde{x}^{n}(t - \xi(t))} \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] \right\}
\[
= \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] e^{\int_{0}^{T} \alpha(u) \, du} \left[ \tilde{x}^{'}(t) + a(t)\tilde{x}(t) \right]
\[
= \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] \frac{d}{dt} \left[ \tilde{x}(t) e^{\int_{0}^{T} \alpha(u) \, du} \right].
\]  
(3.17)

Integrating both sides of (3.17) from $T_{\varepsilon}$ to $t \left( > T_{\varepsilon} \right)$, we obtain
\[
y(t) e^{\int_{0}^{T_{\varepsilon}} \alpha(u) \, du} \leq y(T_{\varepsilon}) e^{\int_{0}^{T_{\varepsilon}} \alpha(u) \, du} + \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] \left[ \tilde{x}(t) e^{\int_{0}^{T} \alpha(u) \, du} - \tilde{x}(T_{\varepsilon}) e^{\int_{0}^{T_{\varepsilon}} \alpha(u) \, du} \right].
\]

This, combined with (2.7), implies that
\[
y(t) \leq y(T_{\varepsilon}) e^{-\int_{T_{\varepsilon}}^{T} \alpha(u) \, du} + \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] \left[ M_b - \tilde{x}(T_{\varepsilon}) e^{-\int_{T_{\varepsilon}}^{T} \alpha(u) \, du} \right],
\]  
(3.18)

In view of (3.6) and (3.18), we find
\[
c \leq M_b \left[ \exp \left( -(n - 1)\left(1 - \frac{1}{n}\right)(d - \varepsilon) \right) - 1 \right] - 1\].
\]  
(3.19)
By using an argument similar to that given above, we obtain
\[ d \geq Mb \left[ \exp\left(-\frac{(n - 1)^{1 - \frac{1}{n}}}{c}\right) - 1 \right]. \quad (3.20) \]

From (H5), we have
\[ Mb(n - 1)^{1 - \frac{1}{n}} \leq 1, \]
which implies that (3.19) and (3.20) have unique solution \( c = d = 0 \). Therefore
\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ x(t) - \tilde{x}(t) \right] = 0. \]

The proof of Theorem 3.1 is complete. \( \square \)

**Remark 3.1.** Theorem 3.1 is an answer to Open problem 11.6.3 due to Győri and Ladas [9].

**Remark 3.2.** From Remark 2.3 and Theorem 3.1, we see that every solution \( x(t) \) to Eq. (1.4) satisfies
\[ \frac{Nb}{1 + (Mb)^n} \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq Mb. \]

Thus, we have the following result.

**Corollary 3.1.** Suppose the same assumptions as those in Theorem 3.1. Then Eq. (1.4) is permanent (see [12, p. 273]).

In addition, by applying Theorem 3.1, we obtain the following corollary.

**Corollary 3.2.** Let \( K \) be the unique equilibrium to Eq. (1.2). Assume that \( \gamma, \beta, \tau \in (0, \infty) \), and one of the following conditions is satisfied:

(H6) \( n \leq 1 \) and \( n \frac{\beta}{\gamma} < Ke^{-\gamma \tau} + (Ke^{-\gamma \tau})^{1-n}; \)

(H7) \( n > 1 \) and \( (n - 1)^{(n-1)} (\frac{\beta}{\gamma})^n < 1. \)

Then, every positive solution \( x(t) \) to Eq. (1.2) satisfies
\[ \lim_{t \to \infty} x(t) = K, \]

i.e., \( K \) is a global attractor of all other positive solutions to Eq. (1.2).

**Proof.** Clearly, \( a(t) \equiv \gamma, b_1(t) \equiv \beta, \tau_1(t) \equiv \tau \). Therefore, \( a, b, \tau \) are positive continuous \( \omega \)-periodic functions, where \( \omega \) is a arbitrary positive constant. Further,
\[ b = \beta \omega, \quad M = \frac{e^{\gamma \omega}}{e^{\gamma \omega} - 1}, \quad x_1 = K, \quad X_1 = Ke^{-\gamma \tau}. \]

Let
\[ g(\omega) = Mb = \frac{\beta \omega e^{\gamma \omega}}{e^{\gamma \omega} - 1}, \quad h(\omega) = e^{\gamma \omega} - \gamma \omega - 1, \quad \omega \in (0, \infty). \]

Hence, for \( \omega \in (0, \infty) \),
\[ g'(\omega) = \beta \frac{e^{\gamma \omega} [e^{\gamma \omega} - \gamma \omega - 1]}{(e^{\gamma \omega} - 1)^2} = \frac{\beta e^{\gamma \omega} h(\omega)}{(e^{\gamma \omega} - 1)^2}, \quad h'(\omega) = \gamma (e^{\gamma \omega} - 1) > 0. \]
Further,

\[ h(\omega) > h(0) = 0, \quad \omega \in (0, \infty), \]

which implies \( g'(\omega) > 0, \omega \in (0, \infty) \). On the other hand,

\[
\lim_{\omega \to 0} g(\omega) = \lim_{\omega \to 0} \frac{\beta \omega e^{\gamma \omega}}{e^{\gamma \omega} - 1} = \lim_{\omega \to 0} \frac{\beta(1 + \gamma \omega)e^{\gamma \omega}}{\gamma e^{\gamma \omega}} = \frac{\beta}{\gamma}.
\]

Therefore,

\[ Mb = g(\omega) > \frac{\beta}{\gamma}, \quad \omega \in (0, \infty). \tag{3.21} \]

Assume (H6) holds. From (H6) and (3.21), we know that, there exists \( \omega > 0 \) such that

\[ nMb < Ke^{\gamma \tau} + (Ke^{\gamma \tau})^{1-n} = X_1 + X_1^{1-n}. \]

Hence, \( \frac{nX_1^{n-1}Mb}{1 + X_1^n} < 1. \) So (H4) holds.

Assume (H7) holds. From (H7) and (3.21), we know that, there exists \( \omega > 0 \) such that

\[ (n-1)^{n-1}(Mb)^n < 1. \]

Hence, (H5) holds. \( \square \)

Using Theorem 3.1, we know that \( K \) is a global attractor of all other positive solutions to Eq. (1.2).

**Remark 3.3.** Theorem 11.2.1 [9] and Corollary 8.2 [12] required that \( \gamma(K + \beta \tau)\tau < K, [\gamma + \frac{1}{4}\beta n^{-1}(n + 1)\frac{n+1}{n} \tau] < 1 \), respectively. This amounts to saying that the delay \( \tau \) must be reasonably small. But in our paper, for \( n > 1 \), the delay \( \tau \) has no effect on the global attractivity of unique equilibrium \( K \) to Eq. (1.2).

In order to verify applicability of Theorems 2.1 and 3.1, we shall take a special case of Eq. (1.4) for an example.

**Example 3.1.** Consider the following equation:

\[
x'(t) = -\frac{2 + \sin t}{4}x(t) + \frac{2 + \cos t}{64} + \frac{1}{1 + x^n(t - 2 - \sin t)} + \frac{2 + \sin t}{64} + \frac{1}{1 + x^n(t - 3 - \cos t)}.
\]

Clearly, \( a(t) = \frac{2 + \sin t}{4}, b_1(t) = \frac{2 + \cos t}{64}, b_2(t) = \frac{2 + \sin t}{64}, \tau_1(t) = 2 + \sin t \) and \( \tau_2(t) = 3 + \cos t \) are all continuous positive \( 2\pi \)-periodic functions. Furthermore, we have

\[
M = \frac{\int_0^{2\pi} \frac{2 + \sin t}{4} dt}{\int_0^{2\pi} \frac{2 + \sin t}{4} dt - 1} = \frac{e^{\pi} - 1}{e^{\pi} - 1} = 1 + \frac{1}{e^{\pi} - 1} < 1 + \frac{1}{2^3 - 1} = \frac{8}{7},
\]

and

\[
b = \int_0^{2\pi} \left( \frac{2 + \cos t}{64} + \frac{2 + \sin t}{64} \right) dt = \frac{\pi}{8}.
\]
If \( n = 1 \), then
\[
\frac{nX_1^{(n-1)}Mb}{1 + X_1^n} = \frac{Mb}{1 + X_1} \leq Mb < \frac{\pi}{7} < 1.
\]

If \( n = 2 \), then
\[
(n-1)^{n-1}(Mb)^n = (Mb)^2 < \left(\frac{\pi}{7}\right)^2 < 1.
\]

If \( n = 3 \), then
\[
(n-1)^{n-1}(Mb)^n = 4(Mb)^3 < 4 \times \left(\frac{\pi}{7}\right)^3 < 1.
\]

Using Theorem 3.1, we know that, for \( n = 1, 2, 3 \), Eq. (3.22) has a unique positive \( 2\pi \)-periodic solution. In addition, this positive \( 2\pi \)-periodic solution is a global attractor of all other positive solutions.

References


