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# Polynomials of Best Approximation which are Monotone

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## 1. INTRODUCTION

This paper deals with the problem of approximating functions which are monotone on a closed interval by polynomials which are also monotone there. In particular, we study the following problem.

Let $f \in C[a, b]$, let $1 \leqslant k_1 < k_2 < \cdots < k_p$ be integers, and let $\epsilon_i = \pm 1$, $i = 1,..., p$, be given signs. Assume $f^{(k_p)} \in C[a, b]$, and that $\epsilon_i f^{(k_i)}(x) \geqslant 0$ for all $x \in [a, b]$ and $i = 1,..., p$. For a given positive integer $n$ ($\geqslant k_p$) let $P_n$ be the polynomial of best approximation to $f$ from $\pi_n$, the set of algebraic polynomials of degree $n$ or less. In this paper we find conditions on $f$ insuring that $\epsilon_i P_n^{(k_i)}(x) \geqslant 0$ for all $x$ in $[a, b]$ and $i = 1,..., p$. This is obviously related to the degree of approximation by monotone polynomials as studied in [1–3], [5–7].

It is known that the problem is not trivial. That is, examples of such functions exist where the polynomials of best approximation are not monotone even for $n$ large. Such examples can be found in [2], [3], and [6], and in Section 3 of this paper.

## 2. THE MAIN RESULT

THEOREM. *Let $f$ have $2m - 1$ continuous derivatives on $[-1, +1]$ for some integer $m \geqslant 1$. Let $1 \leqslant k_1 < k_2 < \cdots < k_p < m$ be $p$ fixed integers and $\epsilon_i = \pm 1$, $i = 1,..., p$ fixed signs. For each positive integer $n$ let $P_n$ be the polynomial from $\pi_n$ of best approximation to $f$ on $[-1, +1]$. If $\epsilon_i f^{(k_i)}(x) > 0$ on $[-1, +1]$ for $i = 1,..., p$ and if $\sum_{k=1}^{\infty} (1/k)\, w(1/k) < +\infty$, then for $n$ sufficiently large we have $\epsilon_i P_n^{(k_i)}(x) \geqslant 0$ on $[-1, +1]$ for $i = 1,..., p$. (Here $w$ is the modulus of continuity of $f^{(2m-1)}$).*

The proof of this theorem requires two lemmas. The first lemma is well known and can be found in [4, p. 74]. We first make the following definition as in [4].

DEFINITION. $\Delta_n(x) = \max(n^{-1}(1 - x^2)^{1/2}, n^{-2})$ for $-1 \leqslant x \leqslant 1$ and $n = 1, 2, \ldots$ and $\Delta_0(x) = 1$.

LEMMA 1. *There are constants $M_q$, $q = 1, 2, \ldots$, so that if $w$ is any modulus of continuity for which $\sum_{k=1}^{\infty} (1/k)\, w(1/k) < +\infty$, and if for $f \in C[-1, +1]$ and polynomials $p_n \in \pi_n$*

$$|f(x) - p_n(x)| \leqslant \Delta_n(x)^q\, w(\Delta_n(x)),$$

*then $f$ has continuous derivatives $f', \ldots, f^{(q)}$, and*

$$|f^{(q)}(x) - p_n^{(q)}(x)| \leqslant M_q \sum_{k \geqslant [\Delta_n(x)^{-1}]} \frac{1}{k}\, w\left(\frac{1}{k}\right)$$

*for $-1 \leqslant x \leqslant 1$.*

We now state and prove the second lemma, which follows from Lemma 1.

LEMMA 2. *Let $f$ have $2m - 1$ continuous derivatives on $[-1, +1]$ and let $w$ be the modulus of continuity of $f^{(2m-1)}$. Assume $w$ satisfies*

$$\sum_{j=1}^{\infty} \frac{1}{j}\, w\left(\frac{1}{j}\right) < +\infty.$$

*For each $n$, let $P_n$ be the polynomial from $\pi_n$ of best approximation to $f$ on $[-1, +1]$. Then there is a constant $B_m$ for which*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leqslant B_m \sum_{j=n}^{+\infty} \frac{1}{j}\, w\left(\frac{1}{j}\right)$$

*for all $x \in [-1, +1]$ and $1 \leqslant k < m$.*

*Proof of Lemma* 2. We first note that if

$$E_n(f) = \inf_{p \in \pi_n} \max_{-1 \leqslant x \leqslant 1} |f(x) - p(x)|,$$

then

$$E_n(f) \leqslant \frac{K}{n^{2m-1}}\, w\left(\frac{1}{n}\right), \tag{1}$$

where $K$ is some absolute constant depending only on $m$. This follows from the well known Jackson theorems. It is easy to see that

$$\frac{1}{n^2} \leqslant \Delta_n(x) \leqslant \frac{1}{n}. \tag{2}$$

We also see that for each $x \in [-1, +1]$ and each $k = 1,\ldots, m-1$ that

$$\begin{aligned}\frac{1}{n^{2m-1}} w\left(\frac{1}{n}\right) &\leqslant \frac{1}{n^{2k+1}} w\left(\frac{1}{n}\right)\\ &\leqslant \Delta_n(x)^k \cdot \frac{1}{n} w\left(\frac{1}{n}\right)\\ &\leqslant \Delta_n(x)^k \cdot w\left(\frac{1}{n^2}\right)\\ &\leqslant \Delta_n(x)^k\, w(\Delta_n(x)).\end{aligned}$$

The second and last of these inequalities follow from (2), and the third follows from the well known inequality for moduli of continuity $w(n\delta) \leqslant nw(\delta)$ if $n$ is a positive integer (see [4]).

Thus using (1) and this we have

$$E_n(f) \leqslant K\,\Delta_n(x)^k\, w(\Delta_n(x)) \tag{3}$$

for $-1 \leqslant x \leqslant 1$ and $k = 1, 2,\ldots, m-1$.

That is,

$$|f(x) - P_n(x)| \leqslant K\,\Delta_n(x)^k\, w(\Delta_n(x)) \tag{4}$$

if we define

$$\bar{w}(\delta) = Kw(\delta),$$

then

$$|f(x) - P_n(x)| \leqslant \Delta_n(x)^k\, \bar{w}(\Delta_n(x)).$$

Thus by Lemma 1 we have

$$\begin{aligned}|f^{(k)}(x) - P_n^{(k)}(x)| &\leqslant M_p \sum_{j\geqslant[\Delta_n(x)^{-1}]} \frac{1}{j}\, \bar{w}\left(\frac{1}{j}\right)\\ &\leqslant M_p K \sum_{j=n}^{\infty} \frac{1}{j}\, w\left(\frac{1}{j}\right)\end{aligned}$$

for $-1 \leqslant x \leqslant 1$ and $k = 1,\ldots, m-1$. This completes the proof of Lemma 2.

The proof of the theorem now follows easily from Lemma 2 by observing that $\sum_{j=n}^{\infty} (1/j)\, w(1/j) \to 0$ as $n \to \infty$.

## 3. Examples

We will not give examples of functions satisfying the theorem in this section, since these are usually easy to recognize. Instead we will give examples of functions that do not completely satisfy the hypotheses of the theorem.

For some of these the theorem will still hold and for some it will fail. The basic tools used to examine these functions will be the theorem and the well known remainder formula in polynomial interpolation.

EXAMPLE 1. Let $f(x) = \sin x$ on $[\pi/4, \pi/2]$ and $p = 1$ with $k_1 = 1$. Note that $f'(\pi/2) = 0$ and that $f''(x) = -\sin x < -(2)^{1/2}/2$ on $[\pi/4, \pi/2]$. Clearly, the theorem applies to $f$ if we only consider $f''$. That is, $P_n''(x) < 0$ for all $x$ in $[\pi/4, \pi/2]$ for $n$ sufficiently large. Thus for $n$ sufficiently large $P_n'$ is strictly decreasing on $[\pi/4, \pi/2]$. But $P_n'$ must interpolate $f'$ at at least $n$ distinct points $x_0 < x_1 < \cdots < x_{n-1}$ in $(\pi/4, \pi/2)$ since $P_n$ is the polynomial of best approximation to $f$. Thus for each $x$ in $[\pi/4, \pi/2]$ is a number $\zeta_x$ in $(\pi/4, \pi/2)$ for which

$$P_n'(x) = f'(x) - \frac{f^{(n+1)}(\zeta_x)}{n!}(x - x_0) \cdots (x - x_{n-1}).$$

$$= \cos x - \frac{f^{(n+1)}(\zeta_x)}{n!}(x - x_0) \cdots (x - x_{n-1})$$

Now

$$f^{(4k-1)}(x) = -\cos x \leqslant 0$$

and

$$f^{(4k-2)}(x) = -\sin x < 0, \qquad k = 1, 2, \ldots .$$

Also, $(\pi/2 - x_0) \cdots (\pi/2 - x_{n-1}) > 0$. So, $P_n'(\pi/2) > f'(\pi/2) \geqslant 0$ if $n = 4k - 2$ or $n = 4k - 3$ for $k = 1, 2, \ldots$ .

But then $P_n'(x) \geqslant 0$ on $[\pi/4, \pi/2]$ since $P_n'$ is decreasing if $n$ is large and $n = 4k - 2$ or $n = 4k - 3$. It is easy to see that for all other $nP_n'(\pi/2) < 0$.

EXAMPLE 2. Let $f(x) = xe - e^x$ on $[0, 1]$, and $k_1 = 1$ and $p = 1$. Then $f'(x) = e - e^x$ and $f^{(k)}(x) = -e^x$, $k = 2, 3, \ldots$ . Note that $f'(1) = 0$.

Using the same arguments as in Example 1 we see that $P_n'(x) > 0$ for $n$ sufficiently large.

EXAMPLE 3. Let $f(x) = e^x - x^2e/2$ on $[0, 1]$, $p = 1$, $k_1 = 1$. For $n \geqslant 2$ the polynomial $P_n$ from $\pi_n$ of best approximation to $f$ is not increasing on $[0, 1]$ even though $f$ is. In fact we have $P_n'(1) < 0$ for $n \geqslant 2$. To see this, we use the remainder theorem mentioned in Example 1 at $x = 1$.

To be sure, the results obtained here are not complete, but they are a beginning where this author and others had no theorem of this type a few years ago. At that time each individual example was a result in itself. Some of the examples obtained in [3] and [6] can be obtained using the theorem or techniques described herein.

In fact, the techniques used in these examples could have just as easily been used to state theorems describing what happened. It is felt however that the examples are more valuable since the theorems would appear quite "doctored up" to get the signs of the various derivatives of $f$ to correspond in the right way. The examples demonstrate techniques just as well.

## 4. Remarks

It is still not known whether the theorem holds without requiring $2m - 1$ derivatives of the function. Perhaps a deeper study using divided differences and the properties of the deviation points in Chebyshev approximation will answer this.

This paper answers at least in part and in another sense more completely a question posed by G. G. Lorentz in [3]:

If $f$ satisfies

$$\epsilon_i f^{(k_i)}(x) > 0 \qquad \text{in} \quad [a, b]$$

for $i = 1,..., p$ and if

$$E_n^*(f) = \inf_{p \in H_n} \max_{a \leqslant x \leqslant b} |f(x) - p(x)|,$$

where $H_n = \{p \in \pi_n : \epsilon_i p^{(k_i)}(x) \geqslant 0 \text{ for } a \leqslant x \leqslant b \text{ and } i = 1,..., p\}$ then is

$$\limsup_{n \to \infty} \frac{E_n^*(f)}{E_n(f)} < +\infty ?$$

Here $E_n(f)$ is as defined in the proof of Lemma 2. Our theorem shows under somewhat stronger conditions on $f$ that for $n$ sufficiently large $E_n^*(f)/E_n(f) = 1$.

If one assumes that $f^{(2m)} \in Lip_M \alpha$ for some $M > 0$ and $0 < \alpha \leqslant 1$, then one can prove a stronger version of the theorem assuming only $1 \leqslant k_1 < \cdots < k_p \leqslant m$. The same proof goes through if we use $E_n(x) \leqslant K \cdot 1/n^{2m+\alpha}$ and $1/n^\alpha \leqslant \Delta_n(x)^{\alpha/2}$.

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