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Semi-regular Graph Automorphisms and Generalized Quadrangles*

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We use topological techniques to verify the existence of non-trivial semi-regular automorphisms of certain graphs, and use this to obtain a characterization of certain families of Moufang quadrangles.

In Part I of this paper we consider certain types of graphs on which a group of automorphisms acts semi-regularly, and we prove a result (Theorem 1.3) which guarantees the existence of certain semi-regular automorphisms. This result uses the topological notion of a covering space. The graphs we consider can arise from various geometrical considerations, and in Part II we give an application to generalized quadrangles, obtaining a geometric characterization of certain families of Moufang quadrangles; the necessary concepts are defined as they arise. There have been several characterizations of finite generalized quadrangles in recent years (we refer to Thas [2], for references and a survey), but none have previously dealt with the infinite case. Our characterization is based on the notion of two quadrilaterals being in perspective from a line, as described in Section 2.

I. GRAPHS

Let X be a graph (undirected without loops or multiple edges) whose set of vertices is partitioned into subsets, and write V_x for the subset containing the vertex $x \in X$. Write $x \sim y$ if x and y are adjacent vertices of X, or if $x = y$.

CONDITION A. *If $x' \in V_x$ and $x \sim y$, then there is a unique vertex $y' \in V_y$ such that $x' \sim y'$.*

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Remark. Setting $y = x$ in condition A implies that the vertices of V_x are pairwise non-adjacent. Setting $x' = x$ implies that every vertex of V_x is adjacent to at most one vertex of V_y.

Notation. Let G denote the group of automorphisms of X which fix each set V_x. A group is *semi-regular* on a set S if only the identity fixes an element of S, and *regular* if it is semi-regular and transitive.

THEOREM 1.1. *If X is connected and satisfies condition* A, *then G is semi-regular on each set V_x.*

Proof. Let $g \in G$ fix $x \in X$, let $y \in X$ be any vertex of X, and let y^g be the image of y under g. If $y \sim x$, then $y^g \sim x^g = x$, and since $y^g \in V_y$ by definition of G, condition A implies that $y^g = y$. The result now follows by induction along a chain joining x to y.

A Covering of X

With the previous notation in which X satisfies condition A, let $\tilde{X}$ denote the graph

vertices: ordered pairs (x, x'), where $x' \in V_x$, for all $x \in X$.
edges: pairs $((x, x'), (y, y'))$ of vertices of $\tilde{X}$ such that $x \sim y$ and $x' \sim y'$.

The map $\pi : \tilde{X} \to X$ sending the vertex (x, x') to x and the edge $((x, x'), (y, y'))$ to (x, y) is a *covering projection*, by which is meant that for each vertex v of $\tilde{X}$, the subgraph of $\tilde{X}$ induced at v is mapped isomorphically onto the subgraph of X induced at $\pi(v)$. One calls $\tilde{X}$ a *covering graph* of X in this case.

THEOREM 1.2. *With the previous notation, for each $g \in G$, the subgraph X_g of $\tilde{X}$ having vertex set $\{(x, x^g) \mid x \in X\}$ is a connected component of $\tilde{X}$ such that $\pi \mid X_g$ is an isomorphism onto X, and any such connected component arises in this way. In particular the number of such components equals the order of G.*

Proof. Suppose (y, y') is adjacent to (x, x^g) in $\tilde{X}$. Then $y \sim x$ and $y' \sim x^g$, and hence by condition A, $y' = y^g$, so $(y, y') = (y, y^g) \in X_g$. Clearly X_g is connected since X is, and the map $\pi \mid X_g$ is an isomorphism onto X.

Conversely, if Y is a connected component of $\tilde{X}$ such that $\pi \mid Y$ is an isomorphism onto X, then for each vertex $x \in X$ let $(x, x') = (\pi \mid Y)^{-1}(x)$. Clearly x' is well-defined and if $x \sim y$, then $x' \sim y'$, so $x \mapsto x'$ defines an element of the group G.

DEFINITION. We say X satisfies the *triangle condition* if given any set of three mutually adjacent vertices p, q and r, then for each $p' \in V_p$ there is a unique $q' \in V_q$ and $r' \in V_r$ such that p', q' and r' are mutually adjacent.

Remark. Setting $q = r$ we see that condition A is a consequence of the triangle condition.

DEFINITION. If X satisfies the triangle condition, let $\Delta(X)$ denote the 2-dimensional simplicial complex obtained from X by the addition of 2-simplexes (p, q, r) for each triple of mutually adjacent vertices.

THEOREM 1.3. *If X satisfies the triangle condition and $\Delta(X)$ is simply-connected in the topological sense, then the group G is regular on each set V_x.*

Proof. We let $\tilde{X}$ be the covering graph constructed previously, and notice that if (p, q, r) is a 2-simplex of $\Delta(X)$, then for $p' \in V_p$, the vertices (p, p'), (q, q') and (r, r') of $\tilde{X}$ are mutually adjacent, by the triangle condition, where $q' \in V_q$ and $r' \in V_r$ are adjacent to p'. If we therefore define a 2-dimensional simplical complex $\Delta(\tilde{X})$ by adding 2-simplexes for each triangle of $\tilde{X}$, it is clear that the projection $\pi: \tilde{X} \to X$ which sends (p, p') to $p \in X$ induces a covering projection, also denoted by π, from $\Delta(\tilde{X})$ to $\Delta(X)$. The simple-connectivity of $\Delta(X)$ implies that each connected cover of $\Delta(X)$ maps isomorphically onto $\Delta(X)$ inder π (see, for example, [6, pp. 72, 80, 81]). Therefore by Theorem 1.3 each connected component of $\Delta(\tilde{X})$ defines an element of G, and hence G is regular on each set V_x.

II. AN APPLICATION TO GENERALIZED QUADRANGLES

2. *Introduction and Theorems*

Recall that a *generalized quadrangle* **Q** is a geometry of points and lines such that two points lie on at most one line, and given a point p not on a line L, there is exactly one point of L collinear with p. We shall assume at least three points per line and three lines per point. A non-degenerate circuit of four points and four lines of **Q** will be called a *quadrilateral.*

In the theory of projective planes (equivalently generalized triangles) one uses the notion of two triangles being in perspective from a point. For a generalized quadrangle, one cannot have two quadrilaterals in perspective from a point as this would imply the existence of triangles. One can, however, define perspectivity of quadrilaterals from a line.

Let us first define a quadrilateral Σ to be *opposite* a line L if the lines of Σ do not meet L, and we call the four lines passing through points of Σ and meeting L, *lines of perspectivity* of Σ from L, and L the *axis* of perspectivity.

Now we say that two quadrilaterals Σ and Σ' are in *perspective* from L if either $\Sigma = \Sigma'$ is opposite L, or $\Sigma \neq \Sigma'$ are both opposite L and the lines of perspectivity of Σ from L are the same as the lines of perspectivity of Σ' from L.

THEOREM 2.1. *The generalized quadrangle* $\mathbf{Q}$ *is Moufang, of type* B_2 *(see below), if and only if given a quadrilateral* Σ *opposite a line* L, *and a point* p, *not on* L, *lying on a line of perspectivity of* Σ *from* L, *then there is a quadrilateral* Σ' *containing* p *and in perspective with* Σ *from* L.

Remark. By a Moufang quadrangle of type B_2 we understand one in which orthogonal line root groups commute (see Section 4 for a definition of this terminology). The Moufang quadrangles have been classified by Tits [5], and those of type B_2 arise from nondegenerate pseudo-quadratic forms of Witt index 2, or from mixed groups of type B_2, as described in Chap. 8 and Sect. 10.3.2 of [4]. A partial classification in the case where the characteristic is not 2 (i.e., no involutions in the root groups) is given in Faulkner [1, Chap. 5].

Digression. Apart from the Moufang quadrangles of type B_2, and their duals, of type C_2, all other Moufang quadrangles are of type BC_2 and occur from a hermitian form or from an exceptional group of ${}^2\mathrm{E}_6^{16'}$, E_7^{31} or E_8^{66} as described in [3].

COROLLARY 2.2. *If* $\mathbf{Q}$ *is a finite generalized quadrangle, then* $\mathbf{Q}$ *satisfies the hypothesis of Theorem* 2.1 *on perspective quadrilaterals if and only if* $\mathbf{Q}$ *is associated with one of the groups* $O_5(q)$ *or* $O_6^-(q)$.

Remark. The quadrangles associated with $O_5(q)$ and $O_6^-(q)$ are the classical quadrangles with parameters (q, q) and (q, q^2), respectively.

For proofs of Theorem 2.1 and Corollary 2.2, see Section 5.

3. *Preliminary Definitions and Lemmas*

DEFINITIONS 3.1. If X and Y are two lines of a generalized quadrangle which do not intersect, we say Y is *opposite* X, and write $X * Y$ for the set of lines meeting both X and Y. Notice that each point of X (or Y) is on a unique line of $X * Y$ and vice versa. ($X * Y$ is $\mathrm{tr}(X, Y)$ in the notation of S. E. Payne; see [2]).

LEMMA 3.2. *If the condition of Theorem* 2.1 *on perspective qudrilaterals is satisfied, then given any line* M *opposite* L *we have* $A * B = A * C$ *for any three lines* A, B, C, $\in L * M$.

Proof. If $N \in A * B$, it suffices to show that N meets C (hence

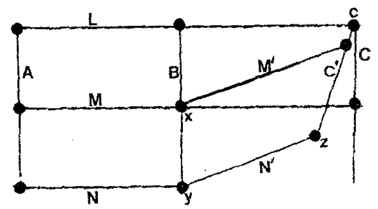

FIGURE 1

$A * B \subset A * C$, and the reverse inclusion follows similarly). Write $x = B \cap M$, $y = B \cap N$, and $c = C \cap L$ (see Fig. 1).

Let N' be any line through y that is different from B and N, and let C' be the unique line through c meeting N' in z. It suffices to show that $C' \neq C$, and hence N is the unique line through y meeting C.

Let Σ_1 be a quadrilateral opposite L, containing $A \cap N$, y and z (hence also N and N'), and let Σ_2 be the quadrilateral containing x and in perspective with Σ_1 from L. Clearly C' is a line of perspective of Σ_1 and Σ_2 from L, and so the line $M' \neq M$ of Σ_2 which passes through x, meets C'. Since M meets C, it follows that $C' \neq C$, and this completes the proof.

Notation 3.3. Given opposite lines L and M such that $A * B = A * C$ for all $A, B, C \in L * M$, we write $\langle L, M \rangle = A * B$ for any two lines $A, B \in L * M$. This is sp(X, Y) in the notation of S. E. Payne; see [2].

4. *Root Groups and Moufang Quadrangles*

If $\mathbf{Q}$ is a generalized quadrangle, a *root* ρ is a chain of length 4 in the flag-graph of $\mathbf{Q}$. Given $\rho = (x_0, x_1, x_2, x_3, x_4)$ we call ρ a *point* (resp. *line*) root if the *middle element* x_2 is a point (resp. a line). Writing $\partial\rho = \{x_0, x_4\}$ one defines the *root group* U_ρ to be the subgroup of Aut $\mathbf{Q}$ fixing all points and lines incident with lines and points of $\rho \backslash \partial\rho$. This definition and the following lemma are due to J. Tits, in a more general context.

LEMMA 4.1 (J. Tits). *The root group U_ρ acts semi-regularly on quadrilaterals containing ρ.*

Proof. Suppose without loss of generality that ρ is a point root and z its middle point. If $g \in U_\rho$ fixes a quadrilateral containing ρ then it fixes a point w opposite z. Now each line through w meets a unique line through z, fixed by g, and also opposite a line of ρ through z, fixed pointwise by g. Therefore g fixes pointwise all lines through w, and hence all lines through z. It is now straightforward to check that g fixes all lines of $\mathbf{Q}$ and is hence the identity.

DEFINITION. $\mathbf{Q}$ is said to be *Moufang* if all root groups U_ρ are transitive (hence regular) on apartments containing ρ.

Let us call two roots *orthogonal* if they lie in the same quadriliteral, and their middle elements are collinear points or intersecting lines (this corresponds to the two roots being at 90° in the root system). We will also call the corresponding root groups orthogonal.

LEMMA 4.2. *Suppose* **Q** *is Moufang, and let* ρ *be a line root with middle line* L. *Then* $[U_\rho, U_\sigma]=1$ *for all roots* σ *orthogonal to* ρ *if and only if* U_ρ *fixes all lines meeting* L.

Proof. Let $\rho=(X,x,L,y,Y)$ and suppose $[U_\rho, U_\sigma]=1$ for all roots σ orthogonal to ρ. Now let M be any line meeting L. If M passes through x or y, then U_ρ fixes M, by definition, so suppose M meets L in another point z. Now let σ be a root containing X and L, and orthogonal to ρ, and let $g \in U_\sigma$ send y to z. If $h \in U_\rho$, then $h=g^{-1}hg$ and so $M^h=M^{g^{-1}hg}$. But $M^{g^{-1}}$ passes through y and is therefore fixed by h, hence $M^h=M$, as required.

Conversely suppose U_ρ fixes all lines meeting L, and let σ be orthogonal to ρ; without loss of generality σ contains X and L. If $g \in U_\sigma$ and $h \in U_\rho$, then since h fixes every line through y^g, it follows that $[g,h]$ fixes every line through y, and therefore, since $[g,h]$ obviously fixes all lines through x and all points of L, we have $[g,h] \in U_\rho$. But $[g,h]$ also fixes all points of X, and hence by Lemma 4.1, $[g,h]=1$.

DEFINITION. An automorphism fixing all lines meeting a line L is called an *axial automorphism* with *axis* L, and the group of all such is denoted U_L. Clearly $U_L \leqslant U_\rho$ for all roots ρ with middle line L, and hence, by Lemma 4.1, is semi-regular on all points of a line meeting L, not on L. If U_L is transitive on these sets of points, then we call it a *full* group of axial automorphisms, and in this case it is clear, by Lemma 4.3, that $U_L=U_\rho$ for all roots ρ having middle line L.

LEMMA 4.3. *If* **Q** *satisfies the conditions of Theorem* 2.1 *on quadrilaterals in perspective from some line* L, *then* U_L *is a full group of axial automorphisms.*

Proof. (1) Consider the graph X comprising the three types of vertices

(i) points not on L,

(ii) lines opposite L,

(iii) quadrilaterals opposite L,

where adjacency is given by containment.

If x is a vertex of type (i) (resp. (ii), resp. (iii)), we let V_x be the set of points on the line through x meeting L, and not on L (resp. $V_x=\langle L,x\rangle\backslash\{L\}$ (see 3.3), resp. $V_x=\{$quadrilaterals in perspective with x from $L\}$). It is easily checked that this partition of the vertices of X into sets V_x satisfies the triangle condition (see Part 1), and we now show that $\Delta(X)$ is simply-connected, in order to apply Theorem 1.3.

(2) *$\Delta(X)$ is simply-connected.* Let γ be a simplicial path in $\Delta(X)$ (i.e., in the graph X) having as a vertex some quadrilateral Σ. If x and y are the vertices of γ adjacent to Σ, then the path x, Σ, y is homotopic to a path of points and lines in Σ. It follows that every path γ in $\Delta(X)$ is homotopic to a chain of points not on L and lines opposite L.

Now let γ be a circuit (i.e., closed path) of such points and lines, and let $n(\gamma)$ denote the number of points in γ. If $n(\gamma)=4$, then γ forms a quadrilateral Σ opposite L, and is therefore null-homotopic in $\Delta(X)$ since it may be contracted to the vertex Σ of $\Delta(X)$. If $n(\gamma)>4$, we will show that γ is homotopic to γ' such that $n(\gamma')<n(\gamma)$, and so γ is null-homotopic, by induction. It clearly suffices to consider circuits γ which have no repetitions. Let the chain $p_1, L_1, p_2, L_2, p_3, L_3, p_4, L_4$ of points and lines be part of γ and let q_3 be the unique point of L_3 collinear with p_1.

Case 1. If the line p_1q_3 does not meet L, then let γ' be the path obtained from γ by replacing $p_1,..., L_3$ with p_1, p_1q_3, q_3, L_3 and let γ'' be the circuit $(p_1, L_1, p_2, L_2, p_3, L_3, q_3, q_3p_1)$. With the usual definition of addition for chains we have $\gamma=\gamma'+\gamma''$, and γ'' is null-homotopic, being a quadrilateral opposite L. Therefore γ is homotopic to γ', where clearly $n(\gamma')<n(\gamma)$.

Case 2. Now suppose that p_1q_3 meets L, and $q_3 \neq p_4$. Let q_1 denote the unique point of L_1 collinear with p_4, and notice that p_1q_3, p_4q_1, $L_2 \in L_1 * L_3$. If p_4q_1 meets L, then we have $L \in (p_1q_3) * (p_4q_1)$, and hence by Lemma 3.2, L_2 meets L, a contradiction. Therefore p_4q_1 does not meet L, and we may use the same procedure as in the preceding paragraph to replace γ be a shorter circuit γ' containing p_4q_1.

Case 3. If p_1q_3 meets L and $q_3=p_4$, then by Lemma 3.2, no two lines of $L_2 * L_4$ can meet L, because $L_3 \in L_2 * L_4$ is opposite L. We may therefore pick some line $L_3' \neq L_3$ opposite L meeting L_2 and L_4 in p_3' and p_4', respectively. Clearly the circuit $\delta=(p_3', L_3', p_4', L_4, p_4, L_3, p_3, L_2)$ is a quadrilateral opposite L, and is thus null-homotopic. Hence γ is homotopic to $\gamma'=\gamma+\delta$, which is the same as γ except that p_3, L_3, p_4 is replaced by p_3', L_3', p_4'. Moreover the line p_1q_3 through p_1 meeting L does not meet L_3', and so by Case 1 it follows that γ' is homotopic to a circuit of shorter length.

We have now shown that every circuit γ of $\Delta(X)$ is null-homotopic, and so it follows that $\Delta(X)$ is simply connected.

(3) *Completion of the proof.* We now apply Theorem 1.3 to see that the automorphism group of X which preserves each set V_x is transitive on each V_x. By restriction to the vertices of types (i) and (ii), and by extension to the

other points and lines of $\mathbf{Q}$ by the identity, we have the fact that U_L is a full group of axial automorphisms, and this completes the proof.

5. *Concluding Lemmas*

In order to use Lemma 4.3 to prove the Moufang condition we first need a lemma.

LEMMA 5.1. *If G is a permutation group (not necessarily faithful) on a set S, generated by subgroups A, B and C which act regularly on S, and commute with one another, then $G/K \cong A \cong B \cong C$ as permutation groups, where K is the kernel of the action of G on S.*

Proof. Fix an element $s \in S$, and define maps $\varphi: A \to B$ and $\psi: A \to C$ by $s \cdot \varphi(a) = s \cdot a$ and $s \cdot \psi(a) = s \cdot a$. Clearly φ and ψ are well-defined and bijective since A, B and C are regular. Now since A, B and C commute, we have, for $a, a' \in A$, $s \cdot aa' = s \cdot \varphi(a)\, a' = s \cdot a'\varphi(a) = s \cdot \psi(a')\, \varphi(a) = s \cdot \varphi(a)\, \psi(a') = s \cdot a\psi(a') = s \cdot \psi(a')\, a = s \cdot a'a$, hence $aa' = a'a$ and A is abelian. Also $s \cdot \varphi(aa') = s \cdot \varphi(a'a) = s \cdot a'a = s \cdot \varphi(a')\, a = s \cdot a\varphi(a') = s \cdot \varphi(a)\, \varphi(a')$, so φ is a homomorphism, and hence an isomorphism. Similarly ψ is an isomorphism, so A, B and C are isomorphic as abelian regular permutation groups on X. Moreover G is clearly abelian, being generated by commuting abelian subgroups A, B and C, and therefore G/K is regular, and the result follows.

Digression. If in the above theorem one has only two commuting subgroups A and B, then a similar result does not hold. For example, let A and B be the right and left regular representations respectively, of some non-abelian group.

LEMMA 5.2. *If $\mathbf{Q}$ admits all axial automorphisms (i.e., U_L is full for all lines L) and has at least four lines per point, then $\mathbf{Q}$ is Moufang.*

Proof. Let $\rho = (x, xz, z, zy, y)$ be a point root. Let Σ and Σ' be quadrilaterals containing ρ and thus determining unique points w and w', respectively, opposite z. Write $L = zx$, $M = zy$, $W = yw$, $W' = yw'$, and let

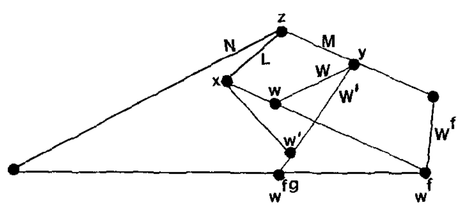

FIGURE 2

$1 \neq f \in U_L$. Now let N be the unique line of $\langle W', W'^f \rangle$ through z (see Fig. 2), and let $g \in U_N$ be the unique element of U_N sending W'^f to W', so $W'^{fg} = W'$. If we let $h \in U_M$ be the unique element of U_M sending w^{fg} to w', then writing $k = fgh$, we have $w^k = w'$. We will show that $k \in U_\rho$.

Clearly k fixes all lines through z, so it suffices to show that k fixes all points of L and M. Since **Q** has at least four lines per point, there is a third line K through z, apart from L and N, not meeting M. Now the existence of the group U_K implies, by Lemma 5.1, that U_L and U_N have isomorphic regular actions on $M \backslash \{z\}$. It follows that fg fixes M pointwise, since it fixes y, and similarly gh fixes L pointwise. Therefore k fixes L and M pointwise, so $k \in U_\rho$. Thus U_ρ is transitive on quadrilaterals containing ρ, and the result follows.

LEMMA 5.3. *If* **Q** *has three lines per point and satisfies the conditions of Theorem* 2.1, *then* **Q** *has parameters* (2, 2) *and is therefore Moufang.*

Proof. Supose we have at least four points per line, and let M be a line opposite L. If P, Q, R, S are four distinct lines of $L * M$, let P' be the third line through $L \cap P$ and p a point of P' not on L. Each set of lines $P' * Q$, $P' * R$, $P' * S$ contains a line through p distinct from P', and these lines must be distinct, because if, for example, $P' * Q$ and $P' * R$ have a line in common through p, then by Lemma 3.2, $P' * Q = P' * R = Q * R$, contradicting the fact that $Q, R \in L * M$, but $P' \notin L * M$. This shows that **Q** has exactly three points per line (we assume at least three, cf. Section 2), and so it follows that **Q** is the unique (Moufang) quadrangle of order (2, 2), associated with $O_5(2)$.

Proof of Theorem 2.1. If **Q** satisfies the condition on perspective quadrilaterals in Theorem 2.1, then by Lemmas 4.3, 5.2 and 5.3 we know that **Q** is Moufang, and by Lemmas 4.3 and 4.2 that $[U_\rho, U_\sigma] = 1$ for orthogonal line root groups U_ρ and U_σ, so **Q** is of type B_2.

Conversely if **Q** is a Moufang quadrangle of type B_2, then for any line L, U_L is a full axial automorphism group by Lemma 4.2. Now, given a quadrilateral Σ opposite L, a line of perspectivity of Σ from L meeting Σ in x, and a point p on this line but not on L, then let $g \in U_L$ send x to p. Clearly then Σ^g is a quadrilateral containing p, and in perspective with Σ from L, as required.

Proof of Corollary 2.2. This follows immediately from Theorem 2.1, since the mixed groups of type B_2 are only defined over non-perfect (hence infinite) fields, and a pseudo-quadratic form on a division ring is a quadratic form if the division ring is a field. The quadrangles associated with $O_4^+(q)$ do not occur in our case since we assume at least three lines per point.

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