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Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems

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ABSTRACT

The concepts of collective sensitivity and compact-type collective sensitivity are introduced as stronger conditions than the traditional sensitivity for dynamical systems and Hausdorff locally compact second countable (HLCSC) dynamical systems, respectively. It is proved that sensitivity of the induced hyperspace system defined on the space of non-empty compact subsets or non-empty finite subsets (Vietoris topology) is equivalent to the collective sensitivity of the original system; sensitivity of the induced hyperspace system defined on the space of all non-empty closed subsets (hit-or-miss topology) is equivalent to the compact-type collective sensitivity of the original HLCSC system. Moreover, relations between these two concepts and other dynamics concepts that describe chaos are investigated.

1. Introduction

1.1. The scope

Given a dynamical system (E, d, f), let $(2^E, \tau, 2^f)$ be its induced hyperspace topological dynamical system where 2^E is the space of all non-empty closed subsets of E, τ is an appropriate hyperspace topology on 2^E, and 2^f is the hyperspace map defined by $2^f : 2^E \to 2^E$, $2^f(F) = f(F)$, $F \in 2^E$. Here, f is assumed to be compatible with 2^E (i.e., $f(F) \in 2^E$ for every $F \in 2^E$). In particular, when τ is metrizable by a metric ρ, we then write $(2^E, \rho, 2^f)$ and call it a hyperspace dynamical system. If the topology τ and metric ρ are clear, we simply write $(2^E, 2^f)$.

The induced system $(2^E, 2^f)$ may inherit some dynamical properties of the original system (E, d, f). However, the dynamical properties of $(2^E, 2^f)$ are much more complex than that of (E, d, f) due to the structural complexity of the hyperspace 2^E, as explored by the recent studies (see Section 1.2).

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If $\mathcal{H}$ is a dense subspace of 2^E and f is compatible with $\mathcal{H}$, the subsystem $(\mathcal{H}, \tau, 2^f)$ or $(\mathcal{H}, \rho, 2^f)$ may be considered. Interesting subsystems include those defined on the space $\mathcal{C}$ of all non-empty compact subsets and the space $\mathcal{F}_\infty$ of all non-empty finite subsets of E.

In particular, we are interested in the relation between (E, d, f) and the hyperspace systems $(2^E, \rho, 2^f)$, $(\mathcal{C}, d_H, 2^f)$, and $(\mathcal{F}_\infty, d_H, 2^f)$ regarding sensitivity, where ρ is the metric of the hit-or-miss topology and d_H is the Hausdorff metric which is consistent with the Vietoris topology when restricted to $\mathcal{C}$ and $\mathcal{F}_\infty$ [12,18,31,1].

1.2. Recent developments on hyperspace dynamics and existing problems

The study of hyperspaces could be traced back to the early 20th century, e.g., the work of Hausdorff, Vietoris, Hahn and Kuratowski (see Engelking [12]), and has been active since then (see e.g., books by Beer [5] and Illanes [18]). The study of dynamical systems has become a central part of mathematics and its applications since the middle of the 20th century when scientists from all related disciplines realized the power and beauty of the geometric and qualitative techniques developed during this period for nonlinear systems (see e.g., Robinson [32]).

In contrast, a systematic study of hyperspace dynamical systems was given by Bauer and Sigmund in 1975 [4] (maps on spaces of probability measures and on hyperspaces). During the last quarter of the 20th century, the investigation of hyperspace dynamics was fairly inactive. The study of hyperspace dynamics has recently become active again:

(1) Using Hausdorff metric topology: Román-Flores on transitivity [35], 2003; Fedeli on transitivity, dense periodic points and collective chaos [13], 2005; Román-Flores and Chalco-Cano on Robinson's chaos [36], 2005; Peris on mixing, weak mixing, transitivity and Auslander–Yorke chaos [31], 2005; Liao, Wang, and Zhang on transitivity, mixing and chaos [21], 2006; Gu and Guo on mixing [17], 2006; Liao, Ma and Wang on Devaney's chaos [22], 2007; Peña and López on entropy [30], 2006; Liao, Ma and Wang on Devaney's chaos [22], 2007; Kwietniak and Oprocha on entropy, mixing and weak mixing [19], 2007; Ma, Hou and Liao on entropy, Li–Yorke chaos and distributional chaos [24], 2007 (Regarding the topological entropy of compact systems, Kwietniak and Oprocha proved that $ent(f) > 0$ implies $ent(2^f) = \infty$; Ma, Hou and Liao gave a system where $ent(f) = 0$ but $ent(2^f) > 0$.)

(2) Using Vietoris topology: Banks on mixing, weak mixing, transitivity, and dense periodic points [1], 2005; Zhang, Zeng and Liu on Devaney's chaos [52] (also W^e topology), 2006; Liu, Wang and Wei on entropy [23], 2007.

(3) Using hit-or-miss topology: Wang and Wei on transitivity, weak mixing and mixing [45], 2007; Wang, Wei, Campbell and Bourquin on a framework of using hit-or-miss topology [47], 2008, where a comparison of the above three topologies was also given.

In 2005, Román-Flores proved that, for $(\mathcal{C}, d_H, 2^f)$, 2^f being sensitive implies f is sensitive [36], and pointed out that the reverse implication does not hold.

Theorem I. *(Banks [1], 2005; for $\mathcal{H} = \mathcal{C}$, see Peris [31], 2005) Let E be a metric space, 2^E be equipped with the Vietoris topology, and $\mathcal{H}$ be a dense subspace of 2^E. If $f : E \to E$ is continuous and compatible with $\mathcal{H}$, then $2^f : \mathcal{H} \to \mathcal{H}$ is continuous and (E, f) being weakly mixing $\Leftrightarrow$ $(\mathcal{H}, 2^f)$ being weakly mixing $\Leftrightarrow$ $(\mathcal{H}, 2^f)$ being transitive. Moreover, $(\mathcal{H}, 2^f)$ is mixing if and only if (E, f) is mixing.*

However, in Theorem I, the Vietoris topology on 2^E is non-metrizable when E is not a compact metric space (Michael [26]). In particular, when E is the n-dimensional Euclidean space R^n, 2^E is non-metrizable. Thus, the Vietoris topology does not admit the definition of metric related dynamics concepts for the induced hyperspace system $(2^E, 2^f)$, such as sensitivity and metric based entropy (e.g., under Bowen's definition [9,10,32]) (or limiting the scope of such concepts to some special subhyperspaces, e.g., $\mathcal{C}$ and $\mathcal{F}_\infty$).

According to the popularly accepted definition given by Devaney in 1980's [11], a dynamical system (E, d, f) is chaotic if it meets three conditions: (i) transitivity, (ii) the set of periodic points is dense in E, and (iii) sensitive dependence on initial conditions (sensitivity). Regarding this definition, the following result is known:

Theorem II. *(Banks [2], 1992; Silverman [37], 1992; Glasner and Weiss [16], 1993) The first two measures* (i) *and* (ii) *together imply the third measure* (iii)*.*

This result, however, is unavailable if the space of the system is not metrizable. Again, to ensure this result available for the induced hyperspace system $(2^E, 2^f)$, the space 2^E needs to be metrizable.

1.3. The motivation and the approach

Sensitivity is widely understood as being the central idea in chaos (Román-Flores [35], 2003; see also Viana [40]). The hit-or-miss topology on 2^E is metrizable when E is Hausdorff, locally compact and second countable (HLCSC), and concrete metrics of this hyperspace topology are available (see Wang and Wei [46,51]). Thus, the hit-or-miss topology will be employed for investigating the sensitivity of the induced hyperspace dynamical systems in this paper.

HLCSC (equivalently, locally compact separable metrizable) spaces appear as natural domains for many applications, and consequently, dynamics on such spaces becomes extremely important. As such, our emphasis is the hyperspace dynamical system $(2^E, \rho, 2^f)$ induced by any dynamical system (E, d, f) where E is HLCSC (and ρ is a compatible metric of the hit-or-miss topology). The hit-or-miss topology plays a key role for the study of hyperspace dynamical systems, as well as for random sets (see e.g., [28]), due to its metrizability: when E is HLCSC, the hit-or-miss topology on 2^E is again HLCSC (Matheron [25] and Beer [5,6]), thus metrizable by the Urysohn's metrization theorem (Engelking [12]). Other remarkable properties were summarized in Wang and Wei [46,51].

Parallel to the hit-or-miss topology, we will also investigate the hyperspace subsystems defined on the space $\mathcal{C}$ of all non-empty compact subsets and on the space $\mathcal{F}_\infty$ of all non-empty finite subsets of E, equipped with the Vietoris topology.

1.4. The settings of the induced hyperspace dynamical systems

Let (E, d, f) be a dynamical system where f is continuous. $\mathcal{C}$ and $\mathcal{F}_\infty$ are dense subsets of 2^E under the Vietoris topology. A continuous mapping f is always compatible with $\mathcal{C}$ and with $\mathcal{F}_\infty$. Hence, $2^f : \mathcal{C} \to \mathcal{C}$ and $2^f : \mathcal{F}_\infty \to \mathcal{F}_\infty$ are well-defined continuous maps (see e.g., Banks [1] and Peris [31]). As the Vietoris topology and d_H are consistent on $\mathcal{C}$, $(\mathcal{F}_\infty, d_H, 2^f)$ and $(\mathcal{C}, d_H, 2^f)$ are well-defined hyperspace dynamical systems. The former is a subsystem of the latter, and the latter can be considered as a subsystem of the topological dynamical system $(2^E, \tau_v, 2^f)$ when f is a closed mapping (thus compatible with 2^E).

Let X and Y be two topological spaces. A continuous mapping $f : X \to Y$ is perfect if f is a closed mapping and all fibers $f^{-1}(y)$ $(y \in Y)$ are compact subsets of X [12].

Let (E, d, f) be a dynamical system where E is (non-compact) HLCSC, d be a compact-type metric (Definition 3.1), and f be a perfect mapping. Then 2^f is compatible with 2^E and $2^f : 2^E \to 2^E$ is continuous where 2^E is equipped with the hit-or-miss topology (Wang, Wei, Campbell and Bourquin [47]). The necessity of the requirement of f being a perfect mapping is to ensure the compatibility and continuity of 2^f with 2^E under the hit-or-miss topology, and was investigated in the above paper. Hence, $(2^E, \rho, 2^f)$ is a well-defined hyperspace dynamical system. $\mathcal{C}$ and $\mathcal{F}_\infty$ are dense subsets of 2^E under the hit-or-miss topology too. $(\mathcal{F}_\infty, \rho, 2^f)$ and $(\mathcal{C}, \rho, 2^f)$ are well-defined hyperspace dynamical systems. The former is a subsystem of the latter, and the latter is a subsystem of the hyperspace dynamical system $(2^E, \rho, 2^f)$ when f is perfect.

1.5. The outline of contents

Section 2 introduces the concept of collective sensitivity for any dynamical system to characterize the sensitivity of induced hyperspace dynamical systems defined on $\mathcal{C}$ and $\mathcal{F}_\infty$, equipped with the Vietoris topology. Section 3 introduces the concept of compact-type collective sensitivity for HLCSC dynamical systems to characterize the sensitivity of induced hyperspace dynamical systems defined on 2^E, equipped with the hit-or-miss topology. In Section 4, the relation between collective sensitivity and weak mixing for any dynamical systems, and the relation among compact-type collective sensitivity, c-transitivity, weak mixing, and the existence of a dense set of periodic points for HLCSC systems are explored. In Section 5, we investigate the metric independence regarding sensitivity for HLCSC dynamical systems, in particular for induced hyperspace dynamical systems equipped with the hit-or-miss topology. In Section 6, we provide two examples to explore dynamical properties related to the concepts introduced and results established in Sections 2–5. Related dynamical terms and hyperspace topologies are defined in Appendix A.

2. Sensitivity of induced (sub)hyperspace dynamical systems equipped with the Vietoris topology

We begin with two trivial but useful facts.

Fact 1. *Let (X, d, f) be a subsystem of a dynamical system (E, d, f) where X is dense in E. Then (E, d, f) is sensitive if and only if (X, d, f) is.*

Fact 2. *Let (E, d, f) be a dynamical system. Then $(\mathcal{C}, d_H, 2^f)$ is sensitive if and only if $(\mathcal{F}_\infty, d_H, 2^f)$ is.*

If E is compact metrizable, $\mathcal{C}$ is the same as 2^E and the Vietoris topology on 2^E is metrizable (by d_H). Hence, for compact dynamical systems (E, d, f), $(2^E, d_H, 2^f)$ is sensitive if and only if $(\mathcal{F}_\infty, d_H, 2^f)$ is.

Definition 2.1. Let (E, d, f) be a dynamical system and $\delta > 0$ a constant. (E, d, f) is said to be collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \ldots, x_n$ of E and an arbitrary $\epsilon > 0$, there exist the same number of distinct points $y_1, y_2, \ldots, y_n$ of E and $k \in N$ satisfying the following two conditions:

(i) $d(x_i, y_i) < \epsilon$ for all $1 \leqslant i \leqslant n$;
(ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that $d(f^k(x_i), f^k(y_{i_0})) \geqslant \delta$ for all $1 \leqslant i \leqslant n$ or $d(f^k(x_{i_0}), f^k(y_i)) \geqslant \delta$ for all $1 \leqslant i \leqslant n$.

Collective sensitivity is a stronger condition than sensitivity.

Theorem 2.2. *Let (E, d, f) be a dynamical system. Then $(\mathcal{F}_\infty, d_H, 2^f)$ is sensitive if and only if (E, d, f) is collectively sensitive.*

Proof. *Necessity*. Suppose that $(\mathcal{F}_\infty, d_H, 2^f)$ be sensitive with a sensitivity constant δ. For any distinct points $x_1, x_2, \dots, x_n$ of E and an arbitrary $\epsilon > 0$, without loss of generality, assume $\epsilon < \frac{1}{2}\min\{d(x_i, x_j) \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$. By the assumption, there exist $B \in \mathcal{F}_\infty$ and $k \in N$ satisfying $d_H(A, B) < \epsilon$ and $d_H((2^f)^k(A), (2^f)^k(B)) \geqslant \delta$ where $A = \{x_1, x_2, \dots, x_n\}$. From $d_H(A, B) < \epsilon$ and the assumption on ϵ, for any $y \in B$ there is only one $1 \leqslant i \leqslant n$ such that $d(y, x_i) < \epsilon$. Put $B_i = \{y \mid y \in B$ and $d(y, x_i) < \epsilon\}$ for $1 \leqslant i \leqslant n$. Because of the selection of ϵ, we have $B_i \neq \emptyset$. $d_H((2^f)^k(A), (2^f)^k(B)) \geqslant \delta$ implies (1) $S((2^f)^k(A), \delta) \not\supseteq (2^f)^k(B)$; or (2) $S((2^f)^k(B), \delta) \not\supseteq (2^f)^k(A)$.

If (1) holds, then there exists $\bar{y} \in B_{i_0}$ satisfying $d(f^k(\bar{y}), f^k(x_i)) \geqslant \delta$ for all $1 \leqslant i \leqslant n$. For each i, we choose $y_i \in B_i$. In particular, choose $y_{i_0} = \bar{y}$. Consequently, we have (i) $d(x_i, y_i) < \epsilon$ for all $1 \leqslant i \leqslant n$; and (ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ satisfying $d(f^k(x_i), f^k(y_{i_0})) \geqslant \delta$ for all $1 \leqslant i \leqslant n$.

If (2) holds, then there exists an i_0 with $1 \leqslant i_0 \leqslant n$ satisfying $d(f^k(x_{i_0}), f^k(y)) \geqslant \delta$ for all $y \in B$. For each $1 \leqslant i \leqslant n$, choose $y_i \in B_i$. We have (i) $d(x_i, y_i) < \epsilon$ for all $1 \leqslant i \leqslant n$; and (ii) $d(f^k(x_{i_0}), f^k(y_i)) \geqslant \delta$ for all $1 \leqslant i \leqslant n$.

Hence, (E, d, f) is collectively sensitive with a collective sensitivity constant δ.

Sufficiency. Now we assume that (E, d, f) is collectively sensitive with a collective sensitivity constant δ. For any $A \in \mathcal{F}_\infty$ and $\epsilon > 0$, put $A = \{x_1, x_2, \dots, x_n\}$. By the assumption, there exist n distinct points $y_1, y_2, \dots, y_n$ of E and $k \in N$ satisfying (i) and (ii) in Definition 2.1. Let $B = \{y_1, y_2, \dots, y_n\}$. (i) implies $d_H(A, B) < \epsilon$; and (ii) implies $S((2^f)^k(A), \delta) \not\supseteq (2^f)^k(B)$ or $S((2^f)^k(B), \delta) \not\supseteq (2^f)^k(A)$, i.e., $d_H((2^f)^k(A), (2^f)^k(B)) \geqslant \delta$. This proves that $(\mathcal{F}_\infty, d_H, 2^f)$ is sensitive with a sensitivity constant δ. □

Fact 2 and Theorem 2.2 together imply Theorem 2.3 below.

Theorem 2.3. *Let (E, d, f) be a dynamical system. Then the following conditions are equivalent*:

(i) $(\mathcal{C}, d_H, 2^f)$ *is sensitive*;
(ii) $(\mathcal{F}_\infty, d_H, 2^f)$ *is sensitive*;
(iii) (E, d, f) *is collectively sensitive.*

In particular, for compact dynamical systems, we have

Corollary 2.4. *Let (E, d, f) be a compact dynamical system. Then the following conditions are equivalent*:

(i) $(2^E, d_H, 2^f)$ *is sensitive*;
(ii) $(\mathcal{F}_\infty, d_H, 2^f)$ *is sensitive*;
(iii) (E, d, f) *is collectively sensitive.*

3. Sensitivity of induced hyperspace dynamical systems equipped with the hit-or-miss topology

Throughout this section, E represents a non-compact HLCSC space (if E is also compact, the Vietoris topology and hit-or-miss topology coincide on 2^E, which is already covered in Section 2), d is a compact-type metric of E (Definition 3.1), and ρ is a metric of the hit-or-miss topology on $\mathcal{F} = 2^E \cup \{\emptyset\}$ (Appendix A.2). $f : E \to E$ is a perfect mapping. $\mathcal{C}$ and $\mathcal{F}_\infty$ are subspaces of $(\mathcal{F}, \tau_f)$ where τ_f is the hit-or-miss topology.

Definition 3.1. A metric d of E is of compact-type if it can be extended to a metric $\bar{d}$ of the Alexandroff compactification ωE. In other words, d is the restriction on E from some metric $\bar{d}$ of ωE.

With the hit-or-miss topology, $\mathcal{C}$ is a dense subset of $\mathcal{F}$ and $\mathcal{F}_\infty$ a dense subset of $\mathcal{C}$. Hence, by Fact 1 of Section 2, we have the following theorem:

Theorem 3.2. *The following conditions are equivalent*:

(i) $(2^E, \rho, 2^f)$ *is sensitive*;
(ii) $(\mathcal{C}, \rho, 2^f)$ *is sensitive*;
(iii) $(\mathcal{F}_\infty, \rho, 2^f)$ *is sensitive.*

Definition 3.3. Let $\bar{d}$ be a metric of ωE and let d be the restriction of $\bar{d}$ on E. (E, d, f) is said to be compact-type collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \dots, x_n$ of E and an arbitrary $\epsilon > 0$, there exist the same number of distinct points $y_1, y_2, \dots, y_n$ of E and $k \in N$ satisfying the following two conditions:

(i) $d(x_i, y_i) < \epsilon$ for all $1 \leqslant i \leqslant n$;
(ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that $d(f^k(x_i), f^k(y_{i_0})) \geqslant \delta$ for all $1 \leqslant i \leqslant n$ and $\bar{d}(f^k(y_{i_0}), \omega) \geqslant \delta$, or $d(f^k(x_{i_0}), f^k(y_i)) \geqslant \delta$ for all $1 \leqslant i \leqslant n$ and $\bar{d}(f^k(x_{i_0}), \omega) \geqslant \delta$.

It follows from Definitions 3.3 and 2.1 that a compact-type collective sensitivity constant is also a collective sensitivity constant.

A minor adaptation of the proof of Theorem 2.2 gives the following result:

Theorem 3.4. *Let (E, d, f) be a dynamical system. Then $(\mathcal{F}_\infty, \rho, 2^f)$ is sensitive if and only if (E, d, f) is compact-type collectively sensitive.*

From Theorems 3.2 and 3.4, we have established the following result:

Theorem 3.5. *Let (E, d, f) be a dynamical system. Then the following conditions are equivalent*:

(i) $(2^E, \rho, 2^f)$ *is sensitive*;
(ii) $(\mathcal{C}, \rho, 2^f)$ *is sensitive*;
(iii) $(\mathcal{F}_\infty, \rho, 2^f)$ *is sensitive*;
(iv) (E, d, f) *is compact-type collectively sensitive.*

4. Other properties of collective sensitivity and compact-type collective sensitivity

4.1. Relation between collective sensitivity and weak mixing

First, recall the fact that transitivity and dense periodic points imply sensitivity (Section 1.2, Theorem II). In particular, for $f : I \to I$ where I is an interval (not necessarily finite) of the one-dimensional Euclidean space with the usual metric, transitivity alone implies dense periodic points and sensitivity, see e.g. Block and Coppel [8], Vellekoop and Berglund [39]. [Redundancy exists in Devaney's definition of chaos; on the other hand, there are systems which are sensitive with dense periodic points, but not transitive (Vellekoop and Berglund [39], or Example 6.2).]

Theorem II shows a relation between sensitivity and transitivity (assuming dense periodic points). It is also known that weak mixing alone implies sensitivity. In contrast, Theorem 4.1 below establishes a stronger result by revealing the relation between collective sensitivity and weak mixing.

Theorem 4.1. *Let (E, d, f) be a dynamical system. If (E, d, f) is weakly mixing, then (E, d, f) is collectively sensitive.*[2]

Proof. Let s_1 and s_2 be two points in E with $d(s_1, s_2) \geqslant 10\delta$. Then $G_i = \{x \in E \mid d(x, s_i) < \delta\}$, $i = 1, 2$, are two disjoint open balls. We will show that (E, d, f) is collectively sensitive with constant δ. Given a finite set of distinct points $x_1, x_2, \ldots, x_n$ of E and $\epsilon > 0$, let O_i be the open ball centered at x_i with radius ϵ, $i = 1, 2, \ldots, n$. By a result of Furstenberg (Proposition II.3 in [15]), any m-product of f is transitive. This means, for $m = 2n$, that there exists $k \in N$ so that $f^k(O_i) \cap G_j \neq \emptyset$ for $i = 1, 2, \ldots, n$, $j = 1, 2$. Then there are $z_i, z_i' \in O_i$ such that $f^k(z_i) \in G_1$ and $f^k(z_i') \in G_2$. So $d(f^k(z_i), f^k(z_i')) > 8\delta$, $i = 1, 2, \ldots, n$. In particular, either $d(f^k(x_1), f^k(z_1)) > 4\delta$ or $d(f^k(x_1), f^k(z_1')) > 4\delta$. An adequate selection of $y_i = z_i$ or $y_i = z_i'$ implies $d(f^k(x_1), f^k(y_i) > \delta$ for $i = 1, 2, \ldots, n$, i.e., (E, d, f) is collectively sensitive with constant δ. □

Chaos in the sense of Devaney implies weak mixing. In Proposition 2.14 of Bés and Peris [7], it was shown that Devaney chaotic operators on Fréchet spaces (metric and complete locally convex spaces) are weakly mixing, therefore collectively sensitive by Theorem 4.1. By another of Banks' results [3], any totally transitive map which is Devaney chaotic is weakly mixing, therefore collectively sensitive.[3]

4.2. Relation among compact-type collective sensitivity, c-transitivity, weak mixing and dense periodic points

Theorem 4.3 of this section reveals a relation among compact-type collective sensitivity, c-transitivity and the existence of a dense set of periodic points. In particular, Corollary 4.4 is comparable to Theorem 4.1 and Theorem II (Section 1.2).

The concept of c-transitivity was introduced in a previous paper (Wang and Wei [45]):

[2] The referee pointed out this stronger result. The proof adopted here belongs to the referee. The original theorem had an additional assumption (dense set of periodic points), and the proof was based on Theorems I and II of Section 1.2.
[3] These connections of collective sensitivity to other concepts are kindly provided by the referee, too.

Definition 4.2. Let (E, f) be a topological dynamical system. (E, f) (or simply f) is said to be topologically co-compact transitive, or c-transitive, if for any two sets of open subsets $U_1, V_1^1, V_2^1, \dots, V_s^1$ and $U_2, V_1^2, V_2^2, \dots, V_t^2$ of E satisfying

(i) U_1 and U_2 are co-compact open subsets of E, and
(ii) $U_1 \cap V_j^1 \neq \emptyset$ $(1 \leqslant j \leqslant s)$ and $U_2 \cap V_j^2 \neq \emptyset$ $(1 \leqslant j \leqslant t)$,

there exists $m \in N$ such that

$$f^m(U_1) \cap \left(V_j^2 \cap U_2\right) \neq \emptyset \quad (1 \leqslant j \leqslant t) \quad \text{and} \quad f^m\left(V_j^1 \cap U_1\right) \cap U_2 \neq \emptyset \quad (1 \leqslant j \leqslant s). \tag{1}$$

c-transitivity is an invariant of topological conjugation, and is weaker than weak mixing. But for compact dynamical systems, c-transitivity and weak mixing are equivalent. The relation between c-transitivity and transitivity is complex. First, c-transitivity “nearly” implies transitivity. Second, if (E, f) is c-transitive, then for any co-compact subset U of E, $\bigcup_{n=1}^{\infty} f^n(U)$ is a dense subset of E, which is a similar property of transitive mappings. Third, for compact dynamical systems, c-transitivity implies and can be stronger than transitivity (Wang and Wei [45]).

Theorem 4.3. *Let E be a HLCSC space, d be a compact-type metric of E, and $f : E \to E$ be a perfect mapping. If (E, d, f) is c-transitive and has a dense set of periodic points, then (E, d, f) is compact-type collectively sensitive.*

Proof. As (E, d, f) is c-transitive, $(2^E, \rho, 2^f)$ is transitive [45]. (E, d, f) has a dense set of periodic points, so is $(2^E, \rho, 2^f)$. Hence, $(2^E, \rho, 2^f)$ is sensitive by Theorem II (Section 1.2). It then follows from Theorem 3.5 that (E, d, f) is compact-type collectively sensitive. □

The following corollary of Theorem 4.3 is comparable to Theorem 4.1 and Theorem II (Section 1.2).

Corollary 4.4. *Let E be a HLCSC space, d be a compact-type metric of E, and $f : E \to E$ be a perfect mapping. If (E, d, f) is weakly mixing and has a dense set of periodic points, then (E, d, f) is compact-type collectively sensitive.*

5. Sensitivity: A metric independent property among compact-type metrics

Let E be HLCSC. Every metric $\bar{d}$ of ωE determines a metric ρ of the hit-or-miss topology on $\mathcal{F} = 2^E \cup \{\emptyset\}$. As $\mathcal{F}$ is compact, the restriction of ρ on 2^E is of compact-type (Definition 3.1). In this section, we will prove that sensitivity of induced hyperspace dynamical systems is metric independent for the compact-type metrics, i.e., independent of the choice of the metric ρ of the hit-or-miss topology on 2^E (equivalently, independent of the choice of the metric $\bar{d}$ of ωE that defines ρ). Because 2^E is only locally compact when E is non-compact HLCSC, this result is compared to the well-known fact that sensitivity is metric independent for compact dynamical systems.

Our proof will be given for any HLCSC dynamical system, thus also valid for $(2^E, \rho, 2^f)$. Moreover, it follows from Theorems 3.5 that the subsystems $(\mathcal{F}_\infty, \rho, 2^f)$ and $(\mathcal{C}, \rho, 2^f)$ are also metric independent for the compact-type metrics, though these subsystems may not be HLCSC.

Theorem 5.1. *Let E be non-compact HLCSC, d_1 and d_2 be any two compact-type metrics of E, and $f : E \to E$ be a continuous mapping. Then (E, d_1, f) is sensitive if and only if (E, d_2, f) is sensitive.*

Proof. It suffices to show that (E, d_2, f) is sensitive if (E, d_1, f) is. Assume that $\delta_1 > 0$ is a sensitivity constant for (E, d_1, f). As d_1 and d_2 are compact-type metrics, there exist two metrics $\bar{d}_1$ and $\bar{d}_2$ of ωE satisfying $d_1 = \bar{d}_1|_{E\times E}$ and $d_2 = \bar{d}_2|_{E\times E}$. Since the identity mapping $i : (\omega E, \bar{d}_2) \to (\omega E, \bar{d}_1)$ is a homeomorphism, there exists $\delta_2 > 0$ such that if $\bar{d}_2(x, y) < \delta_2$, then $\bar{d}_1(i(x), i(y)) < \delta_1$, i.e., $\bar{d}_1(x, y) < \delta_1$. In particular, we have the following relation between d_1 and d_2: if $d_2(x, y) < \delta_2$, then $d_1(x, y) < \delta_1$ for any $x, y \in E$.

For any $x \in E$ and an open neighborhood $S_{d_2}(x, \epsilon)$ of x, this set $S_{d_2}(x, \epsilon)$ is also an open neighborhood of x under d_1. Because (E, d_1, f) is sensitive, there exists $y \in S_{d_2}(x, \epsilon)$ (i.e., $d_2(x, y) < \epsilon$) and $k \in N$ satisfying $d_1(f^k(x), f^k(y)) \geqslant \delta_1$. It follows from the above relation of d_1 and d_2 that $d_2(f^k(x), f^k(y)) \geqslant \delta_2$. This completes the proof. □

For any compact dynamical system (E, d, f), sensitivity, Li–Yorke's chaos, expansivity, and Bowen's entropy are metric independent (Walters [42]). Theorem 5.1 expands this result to locally compact systems: when E is non-compact HLCSC, sensitivity is metric independent within all compact-type metrics.

6. Examples

Finally, we give two examples to explore various dynamical properties related to the concepts introduced and results established in Sections 2–5.

Example 6.1. The dynamical system given in this example has the compact-type collective sensitivity. The space E is HLCSC and the metric d of E is of compact-type.

Let $(\Sigma(p),\sigma)$ be the full two-sided p-shift, where

$$\Sigma(p)=\{1,2,\dots,p\}^Z=\{s=(\dots,s_{-1},s_0,s_1,\dots)\mid s_n\in\{1,2,\dots,p\}\},$$

and $\sigma(s)=t$ where $t_n=s_{n+1}$. The metric $\bar{d}$ of $(\Sigma(p),\sigma)$ is defined by

$$\bar{d}(s,t)=\sum_{n=-\infty}^{\infty}\frac{\delta(s_n,t_n)}{2^{|n|}},$$

where $\delta(a,b)=1$ if $a\neq b$ and $\delta(a,b)=0$ if $a=b$. For properties of p-shift, we refer to [32,53]. For conditions of embedding dynamical systems into symbolic dynamical systems and conditions of subsystems of hyperspace systems topologically (semi-)conjugate to symbolic dynamical systems, we refer to [44,43].

Let $E=\Sigma(p)\setminus\{(\dots,1,1,1,\dots)\}$ and $f=\sigma|_E$. As $\Sigma(p)$ is compact, E is HLCSC and $d=\bar{d}|_{E\times E}$ is a compact-type metric. Since $(\Sigma(p),\sigma)$ is weakly mixing and (E,d,f) is a dense subsystem of $(\Sigma(p),\sigma)$, (E,d,f) is weakly mixing. Since $(\Sigma(p),\sigma)$ has a dense set of periodic points, so does (E,d,f). By Corollary 4.4, (E,d,f) is compact-type collectively sensitive, and of course collectively sensitive.

Hence, $(\mathcal{C},d_H,2^f)$ and $(\mathcal{F}_\infty,d_H,2^f)$ are sensitive by Theorem 2.3. $(2^E,\rho,2^f)$, $(\mathcal{C},\rho,2^f)$ and $(\mathcal{F}_\infty,\rho,2^f)$ are sensitive by Theorem 3.5.

Example 6.2. Let $E=(0,\infty)$, equipped with the subspace topology of R^1. Define $f:E\to E$ by $f(x)=\frac{1}{x^2}$, $x\in E$. Consider two metrics d_1 and d_2 on E: d_1 is the restriction of the usual metric of R^1, i.e., $d_1(x,y)=|x-y|$, and d_2 is defined as follows. First, define $h:E\to R^1$ by

$$h(x)=\begin{cases}1-\frac{1}{x}, & x\in(0,1),\\ x-1, & x\in[1,\infty).\end{cases}$$

Then define $g:R^1\to S^1$ as the stereographical projection with $g(0)=(0,0)\in S^1$, where S^1 is the circle $x^2+(y-1)^2=1$ (north pole $P=(0,2)$ removed) equipped with the metric d_S induced by the usual metric of R^2. As h and g are homeomorphisms, the composition $g\circ h:E\to S^1$ is a homeomorphism. Now, we define d_2 by $d_2(x,y)=d_S(g\circ h(x),g\circ h(y))$, $x,y\in E$.

(i) (E,d_1,f) is collectively sensitive and any $\delta>0$ is a collective sensitivity constant.

Let $\epsilon>0$. For any t distinct points $x_1<x_2<\dots<x_t$ of E, consider the four possible cases as follows.

Case 1. $0<x_1<x_2<\dots<x_t\leqslant 1$. If $x_t<1$, choose $y_i\in S_E(x_i,\epsilon)$ for each $1\leqslant i\leqslant t$ (so condition (i) of Definition 2.1 is satisfied) and for y_t we require $x_t<y_t<1$. Then it holds $1<f^n(y_t)<f^n(x_t)<f^n(x_{t-1})<\dots<f^n(x_1)$ for each odd $n\in N$, and thus $\lim_{n\to\infty}d_1(f^{2n-1}(x_t),f^{2n-1}(y_t))=\infty$. Consequently, $\lim_{n\to\infty}d_1(f^{2n-1}(x_1),f^{2n-1}(y_t))=\infty$ for all i $(1\leqslant i\leqslant t)$. The condition (ii) of Definition 2.1 is thus verified. If $x_t=1$, the above proof remains valid by noticing $f(x_t)=1$.

Case 2. $1\leqslant x_1<x_2<\dots<x_t$. This is similar to Case 1.

Case 3. $0<x_1<x_2<\dots<x_s<1<x_{s+1}<x_{s+2}<\dots<x_t$. Choose $y_i\in S_E(x_i,\epsilon)$ for each $1\leqslant i\leqslant t$ (so condition (i) of Definition 2.1 is satisfied) and for y_s we require $x_s<y_s<1$. Then it holds $0<f^n(x_t)<\dots<f^n(x_{s+1})<1<f^n(y_s)<f^n(x_s)<\dots<f^n(x_2)<f^n(x_1)$ for each odd $n\in N$. Since $\lim_{n\to\infty}d_1(f^{2n-1}(x_s),f^{2n-1}(y_s))=\infty$ and $\lim_{n\to\infty}d_1(f^{2n-1}(x_{s+1}),f^{2n-1}(y_s))=\infty$, we have $\lim_{n\to\infty}d_1(f^{2n-1}(x_i),f^{2n-1}(y_s))=\infty$ for all i $(1\leqslant i\leqslant t)$. The condition (ii) of Definition 2.1 is thus verified.

Case 4. $0<x_1<x_2<\dots<x_{s-1}<x_s=1<x_{s+1}<x_{s+2}<\dots<x_t$. This is similar to Case 3.

Hence, (E,d_1,f) is collectively sensitive and any $\delta>0$ is a collective sensitivity constant. It then follows from Theorem 2.3 that $(\mathcal{C},d_H,2^f)$ and $(\mathcal{F}_\infty,d_H,2^f)$ are all sensitive.

Noting that d_1 is not of compact-type, Theorem 3.5 cannot be applied to determine the sensitivity of $(2^E,\rho,2^f)$, $(\mathcal{C},\rho,2^f)$ and $(\mathcal{F}_\infty,\rho,2^f)$.

(ii) (E,d_2,f) is not sensitive.

Consider the Alexandroff compactification $\omega E=E\cup\{\omega\}$. d_2 as a compact-type metric can be extended to a metric $\bar{d}_2$ of ωE as follows: for $x,y\in E$, $\bar{d}_2(x,y)=d_2(x,y)$, and for $x\in E$, $\bar{d}_2(x,\omega)=d_S(g\circ h(x),P)$. Hence, d_2 is of compact-type.

For any $x\in E$ with $0<x<1$, we have $f^{2n}(x)=x^{2^{2n}}$ and $f^{2n-1}(x)=\frac{1}{x^{2^{2n-1}}}$. In R^1, $\lim_{n\to\infty}f^{2n}(x)=0$ and $\lim_{n\to\infty}f^{2n-1}(x)=+\infty$. Hence, in S^1, we have $\lim_{n\to\infty}g\circ h(f^{2n}(x))=P$ and $\lim_{n\to\infty}g\circ h(f^{2n-1}(x))=P$, implying $\lim_{n\to\infty}g\circ h(f^n(x))=P$. This shows that (E,d_2,f) is not sensitive. Of course, (E,d_2,f) is not compact-type collectively sensitive, neither collectively sensitive. Consequently, $(2^E,\rho,2^f)$, $(\mathcal{C},\rho,2^f)$ and $(\mathcal{F}_\infty,\rho,2^f)$ are not sensitive by Theorem 3.5; $(\mathcal{C},d_H,2^f)$ and $(\mathcal{F}_\infty,d_H,2^f)$ are not sensitive by Theorem 2.3.

(iii) This example also has other noticeable metric dependent properties regarding Li–Yorke's chaos, Bowen's entropy and expansivity. For relevant concepts, we refer to [32,9,10,8,42]. (E,d_1,f) has a Li–Yorke chaotic set $(0,1)$ (in the sense of $\limsup_{n\to\infty}d_1(f^n(x),f^n(y))>0$ for all $x,y\in(0,1)$, $x\neq y$, and $\liminf_{n\to\infty}d_1(f^n(x),f^n(y))=0$ for all $x,y\in(0,1)$), but (E,d_2,f) does not have one. Bowen's entropy for (E,d_1,f) is positive, but Bowen's entropy of (E,d_2,f) is zero. (E,d_1,f) is expansive, but (E,d_2,f) is not.

(iv) As a topological dynamical system, f is not chaotic in Devaney's sense of chaos. The only periodic point of the topological dynamical system (E, f) is 1, thus the set of periodic points is of course not dense in E. Moreover, f is not transitive. In fact, for any open sets $(a,b) \subseteq (1,\infty)$, choose c and d satisfying $f(b) < f(a) < c < d < 1$. Since $f^{2n}((a,b)) = (f^{2n}(a), f^{2n}(b)) \subseteq (1,\infty)$ for any $n \in N$, $f^{2n}((a,b)) \cap (c,d) = \emptyset$. Also, as $f^{2n-1}(b) < f^{2n-1}(a) \leqslant f(a)$, $(f^{2n-1}(b), f^{2n-1}(a)) \cap (c,d) = \emptyset$, i.e., $f^{2n-1}((a,b)) \cap (c,d) = \emptyset$. Therefore, (E, f) is not transitive. Moreover, Example 6.2 shows that transitivity and dense periodic points are sufficient but not necessary conditions of sensitivity (see Theorem II, Section 1.2).

Appendix A

A.1. Related dynamical terminologies

Let (E, d, f) be a dynamical system. Let N denote the set of all positive integers. f is (topologically) transitive if for any pair of non-empty open subsets U and V of E, there exists $k \in N$ such that $f^k(U) \cap V \neq \emptyset$. f is (topologically) mixing if for any pair of non-empty open subsets U and V of E there exists $n \in N$ such that $f^k(U) \cap V \neq \emptyset$ for all $k \in N$ with $k \geqslant n$. f is (topologically) weakly mixing if for any non-empty open subsets U_1, U_2, V_1 and V_2 of E, there exists $k \in N$ such that $f^k(U_1) \cap V_1 \neq \emptyset$ and $f^k(U_2) \cap V_2 \neq \emptyset$.

A point $p \in E$ is periodic for f if $f^k(p) = p$ for some $k \in N$.

If E is non-metrizable, above concepts remain valid for the topological dynamical system (E, f). However, when sensitivity is of concern, E must be metrizable.

f has sensitive dependence on initial conditions (sensitivity) if there exists a constant $\delta > 0$ such that for every point x and every open neighborhood U_x of x, there exist $y \in U_x$ and $k \in N$ such that $d(f^k(x), f^k(y)) \geqslant \delta$. Such a δ is called a sensitivity constant [1,2,53].

If Λ is an invariant set of f, i.e., $f(\Lambda) \subseteq \Lambda$, $(\Lambda, f|_\Lambda)$ is said to be a subsystem of (E, f).

A.2. The Hausdorff metric, Vietoris topology and hit-or-miss topology

Let E be any topological space. Let $\mathcal{F}(E)$, $\mathcal{G}(E)$, and $\mathcal{K}(E)$ denote respectively the sets of all closed, open and compact subsets of E, abbreviated as $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$ ($\emptyset \in \mathcal{F}$, $\emptyset \in \mathcal{G}$ and $\emptyset \in \mathcal{K}$).

The hit-or-miss topology τ_f (also known as H-topology [14], Fell topology [5,6,29], Choquet–Matheron topology [38], or weak Vietoris topology [49]) on $\mathcal{F}$ is generated by the subbase

$$\mathcal{F}^K, \quad K \in \mathcal{K}; \qquad \mathcal{F}_G, \quad G \in \mathcal{G}, \tag{2}$$

where $\mathcal{F}^K = \{F \in \mathcal{F} \mid F \cap K = \emptyset\}$ and $\mathcal{F}_G = \{F \in \mathcal{F} \mid F \cap G \neq \emptyset\}$.

A topological base of τ_f is [25]

$$\mathcal{F}^K_{G_1,\dots,G_n}, \quad K \in \mathcal{K}, G_i \in \mathcal{G}\ (1 \leqslant i \leqslant n),\ n \geqslant 0, \tag{3}$$

where $\mathcal{F}^K_{G_1,\dots,G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \dots \cap \mathcal{F}_{G_n}$. Note that $\mathcal{F}^\emptyset = \mathcal{F}$ and $\mathcal{F}^K_{G_1,\dots,G_n}$ means $\mathcal{F}^K$ when $n = 0$.

The Vietoris topology τ_v on 2^E (also known as finite topology [26]; another notation $CL(E)$ is also used, e.g., [5,18,27]), the family of all *non-empty* closed subsets of E, is generated by the base [41,26,12]

$$\mathcal{V}(U_1, U_2, \dots, U_n) = \left\{ F \in 2^E \;\middle|\; F \subseteq \bigcup_{i=1}^n U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leqslant n \right\}, \tag{4}$$

where $U_1, U_2, \dots, U_n$ are open subsets of E. Clearly, $2^E = \mathcal{F} \setminus \{\emptyset\}$. Alternatively, a subbase of the Vietoris topology on 2^E can be obtained from (2) by replacing $\mathcal{F}$ by 2^E and compact subsets K by closed subsets F. It follows from these definitions that the Vietoris topology is finer than the subspace topology on 2^E induced by the hit-or-miss topology.

The Hausdorff metric d_H (induced topology τ_h) on the family of all *non-empty bounded* closed subsets of a metric space (E, d) is defined by [12]

$$d_H(A, B) = \max\left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \tag{5}$$

or equivalently by

$$d_H(A, B) = \inf\left\{ \epsilon\colon\ S(A, \epsilon) \supseteq B,\ S(B, \epsilon) \supseteq A \right\}, \tag{6}$$

where $S(A, \epsilon) = \{x \in E \mid d(x, A) < \epsilon\}$ is an ϵ-neighborhood of A in E; likewise, $S(B, \epsilon)$ is an ϵ-neighborhood of B.

Properties of the hit-or-miss topology and relations among hit-or-miss topology, Vietoris topology and Hausdorff metric topology were summarized in our recent papers [47,51].

Let E be HLCSC. So far, three compatible metrics of the hit-or-miss topology are known. One is on $\mathcal{F}$, constructed by using the Alexandroff compactification of E (see Watson [48], Rockafellar and Wets [33,34], Wei and Wang [50]), another is on 2^E, constructed bypassing the Alexandroff compactification of E (see Lechicki and Levi [20]), and the third is on $\mathcal{F}$, also constructed bypassing the Alexandroff compactification of E (Wei and Wang [51,46]).

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