



Linear orderings and powers of characterizable cardinals

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ARTICLE INFO

Article history:

Received 4 November 2010
 Received in revised form 25 August 2011
 Accepted 30 August 2011
 Available online 7 October 2011
 Communicated by I. Neeman

MSC:

primary 03C75
 03C30
 secondary 03C35
 03E10
 03C55

Keywords:

Infinitary logic
 Scott sentence
 Characterizable cardinal
 Dense linear ordering
 Cardinal power

ABSTRACT

We say that a countable model \mathcal{M} completely characterizes an infinite cardinal κ , if the Scott sentence of \mathcal{M} has a model in cardinality κ , but no models in cardinality κ^+ . If a structure \mathcal{M} completely characterizes κ , κ is called characterizable. In this paper, we concern ourselves with cardinals that are characterizable by linearly ordered structures (cf. Definition 2.1).

Under the assumption of GCH, Malitz completely resolved the problem by showing that κ is characterizable if and only if $\kappa = \aleph_\alpha$, for some $\alpha < \omega_1$ (cf. Malitz (1968) [7] and Baumgartner (1974) [1]). Our results concern the case where GCH fails.

From Hjorth (2002) [3], we can deduce that if κ is characterizable, then κ^+ is characterizable by a densely ordered structure (see Theorem 2.4 and Corollary 2.5).

We show that if κ is *homogeneously* characterizable (cf. Definition 2.2), then κ is characterizable by a densely ordered structure, while the converse fails (Theorem 2.3).

The main theorems are (1) If $\kappa > 2^\lambda$ is a characterizable cardinal, λ is characterizable by a densely ordered structure and λ is the least cardinal such that $\kappa^\lambda > \kappa$, then κ^λ is also characterizable (Theorem 5.4) and (2) if \aleph_α and κ^{\aleph_α} are characterizable cardinals, then the same is true for $\kappa^{\aleph_{\alpha+\beta}}$, for all countable β (Theorem 5.5).

Combining these two theorems we get that if $\kappa > 2^{\aleph_\alpha}$ is a characterizable cardinal, \aleph_α is characterizable by a densely ordered structure and \aleph_α is the least cardinal such that $\kappa^{\aleph_\alpha} > \kappa$, then for all $\beta < \alpha + \omega_1$, κ^{\aleph_β} is characterizable (Theorem 5.7). Also if κ is a characterizable cardinal, then κ^{\aleph_α} is characterizable, for all countable α (Corollary 5.6). This answers a question of the author in Souldatos (submitted for publication) [8].

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1. Structure of the paper

Throughout the whole paper, we work with countable languages \mathcal{L} and when we refer to a dense linear ordering we mean a dense linear ordering without endpoints. Abusing language, we will say that a structure is linearly ordered even in the case where only an infinite subset of the structure is linearly ordered. The first two sections provide some background material for the characterizable cardinals and for the dense linear orderings, respectively. Section 4 contains the construction that proves the following theorem.

Theorem 1.1. *If κ is a characterizable cardinal, then κ^{\aleph_1} is also a characterizable cardinal.*

This appears as Theorem 4.18 in Section 4 and it will be easily generalized to other $\lambda \geq \aleph_1$ in the last section.

2. Characterizable cardinals

This section provides the necessary background on characterizable cardinals.

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Definition 2.1. We say that a $\mathcal{L}_{\omega_1, \omega}$ sentence ϕ characterizes \aleph_α , or that \aleph_α is characterizable, if ϕ has models in cardinality \aleph_α , but not in cardinality $\aleph_{\alpha+1}$. If ϕ is the Scott sentence of a countable model \mathcal{M} (equivalently, if ϕ is any complete sentence), we say that \mathcal{M} completely characterizes \aleph_α , or that \aleph_α is completely characterizable by \mathcal{M} .

We say that a cardinal \aleph_α is characterized by a linear order, if there exists a countable model \mathcal{M} and a linear order $<$ defined on (a subset of) the countable structure \mathcal{M} , such that the Scott sentence of $(\mathcal{M}, <)$, say ϕ , characterizes \aleph_α and in addition, ϕ has a model \mathcal{N} where $<^{\mathcal{N}}$ is of size \aleph_α .

We denote by $\mathcal{CH}_{\omega_1, \omega}$, the set of all completely characterizable cardinals.

Definition 2.2. If P is a unary predicate symbol, we say that it is completely homogeneous for the \mathcal{L} -structure \mathcal{A} , if $P^{\mathcal{A}} = \{a \mid \mathcal{A} \models P(a)\}$ is infinite and every permutation of it extends to an automorphism of \mathcal{A} .

If κ is a cardinal, we will say that κ is homogeneously characterizable by (ϕ, P) , if ϕ is a complete $\mathcal{L}_{\omega_1, \omega}$ -sentence, P a unary predicate in the language of ϕ such that

- ϕ does not have models of power $> \kappa$,
- if \mathcal{M} is the (unique) countable model of ϕ , then $P^{\mathcal{M}}$ is infinite and P is completely homogeneous for \mathcal{M} and
- there is a model \mathcal{A} of ϕ such that $P^{\mathcal{A}}$ has cardinality κ .

If (ϕ, P) characterizes κ homogeneously and \mathcal{M}, P are as above, we write $(\mathcal{M}, P^{\mathcal{M}}) \models (\phi, P)$. Denote the set of all homogeneously characterizable cardinals by $\mathcal{HCH}_{\omega_1, \omega}$. Obviously, $\mathcal{HCH}_{\omega_1, \omega} \subset \mathcal{CH}_{\omega_1, \omega}$, but the inverse inclusion fails since $\aleph_0 \in \mathcal{CH}_{\omega_1, \omega} \setminus \mathcal{HCH}_{\omega_1, \omega}$ (cf. [3]).

In [7], Malitz proved that under the assumption of GCH, for every successor $\alpha < \omega_1$, \beth_α is homogeneously characterizable. Baumgartner improved this result in [1] by eliminating the GCH assumption. Notice that this result is optimal in the sense that if a $\mathcal{L}_{\omega_1, \omega}$ sentence has a model in \beth_{ω_1} , then it has models in all cardinalities (\beth_{ω_1} is the Hanf number for $\mathcal{L}_{\omega_1, \omega}$). It follows that under GCH a cardinal is characterizable (completely or not) if and only if it is one of the $\beth_\alpha = \aleph_\alpha$ for some countable α .

Further work on characterizable cardinals was done by Knight [5] and Hjorth in [3]. Knight proved that \aleph_1 is completely characterizable, while Hjorth extended this result to all \aleph_α 's, α countable.

Convention: For now on, we consider only completely characterizable cardinals, and we will refer to them as just characterizable cardinals.

Theorem 2.3. If $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$, then κ is characterized by a densely ordered structure.

Proof. Let (ϕ, P) witness the fact that $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ and \mathcal{M} be a countable model such that $(\mathcal{M}, P^{\mathcal{M}}) \models (\phi, P)$. Extend $\mathcal{L}(\phi)$, the language of ϕ , to include a new binary symbol $<$ and consider the new sentence ϕ' which is the conjunction of ϕ together with the sentence

$<$ is a dense linear order on $P^{\mathcal{M}}$ without endpoints.

Obviously, ϕ' does not have any models of cardinality $> \kappa$. We will prove that ϕ' is a complete sentence and that if \mathcal{M}' is the unique countable model of ϕ' , then $(\mathcal{M}', <^{\mathcal{M}'})$ characterize κ :

Toward completeness, let $\mathcal{M}_1, \mathcal{M}_2$ be two countable models of ϕ' . Since the reducts of $\mathcal{M}_1, \mathcal{M}_2$ on the language of ϕ both satisfy ϕ , they must be isomorphic. Call i such an isomorphism between \mathcal{M}_1 and \mathcal{M}_2 and let f be any bijection that maps the elements of $P^{\mathcal{M}_1}$ to the elements of $P^{\mathcal{M}_2}$. Then $i^{-1} \circ f$ is a permutation of $P^{\mathcal{M}_1}$ and by the homogeneity of P , it extends to an automorphism of \mathcal{M}_1 , say j . Then $i \circ j$ is a $\mathcal{L}(\phi)$ -isomorphism between \mathcal{M}_1 and \mathcal{M}_2 and $i \circ j$ agrees with f on $P^{\mathcal{M}_1}$. In other words, any bijection f between $P^{\mathcal{M}_1}$ and $P^{\mathcal{M}_2}$ extends to an $\mathcal{L}(\phi)$ -isomorphism between \mathcal{M}_1 and \mathcal{M}_2 . By the usual back-and-forth argument there is a bijection f between $(P^{\mathcal{M}_1}, <)$ and $(P^{\mathcal{M}_2}, <)$ that preserves $<$ and by extending this f we get an isomorphism between \mathcal{M}_1 and \mathcal{M}_2 that preserves both $\mathcal{L}(\phi)$ and $<$. This proves completeness.

Now, if \mathcal{A} is a model of ϕ such that $P^{\mathcal{A}}$ has size κ , then we can expand \mathcal{A} to a $\mathcal{L}(\phi) \cup \{<\}$ -structure \mathcal{A}' by defining $<$ to be a dense linear order on $P^{\mathcal{A}}$ without endpoints. By the definition of ϕ' , $\mathcal{A}' \models \phi'$ and this proves that ϕ' characterizes κ . \square

Notes: (1) The assumption $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$ is too strong, since \aleph_0 is characterizable by (ω, \in) , while $\aleph_0 \notin \mathcal{HCH}_{\omega_1, \omega}$. (2) It follows by the proof that for an uncountable model \mathcal{A} of ϕ we can define $<$ on $P^{\mathcal{A}}$ to be any dense linear order without endpoints. I.e. the models of ϕ' embed any dense linear ordering without endpoints of size up to κ .

In [3] Hjorth proves that every κ^+ is characterized by a densely ordered structure, for all $\kappa \in \mathcal{CH}_{\omega_1, \omega}$. Hjorth is actually concerned with the case that $\kappa = \aleph_\alpha$ and α countable, but his proof generalizes. The next theorems follows from his proof of Theorem 5.1 in [3].

Theorem 2.4 (Hjorth). If $\kappa \in \mathcal{CH}_{\omega_1, \omega}$, then at least one of the following is the case:

- (1) $\kappa^+ \in \mathcal{HCH}_{\omega_1, \omega}$ or,
- (2) there is a countable model \mathcal{M} in a language that contains a unary predicate P and a binary predicate $<$ whose Scott sentence $\phi_{\mathcal{M}}$

- (a) has no models of cardinality κ^{++} and
- (b) $\phi_{\mathcal{M}}$ has a model \mathcal{N} where $(P^{\mathcal{N}}, <^{\mathcal{N}})$ is a dense linear ordering without endpoints, it has size κ^+ , and every initial segment of this linear ordering has size κ .

Cases 1 and 2 need not be exclusive the one to the other, but in either case we get

Corollary 2.5 (Hjorth). *If $\kappa \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$, then there is a countable densely ordered structure whose Scott sentence*

- (1) does not have any models in cardinality κ^{++} , but
- (2) does have a model which is densely ordered and has an increasing sequence of size κ^+ .

Proof. By Theorems 2.4 and 2.3. \square

Using Corollary 2.5 for the successor stages and taking unions at the limit stages we can conclude that

Corollary 2.6 (Hjorth). *If $\aleph_{\alpha} \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$, then $\aleph_{\alpha+\beta}$ is characterized by a densely ordered structure, for all $0 < \beta < \omega_1$. In particular, $\aleph_{\beta} \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$ is characterized by a densely ordered structure, for all countable ordinals β .*

The importance of characterizing cardinals by linearly ordered structures is emphasized by the next theorem. It is theorem 35 from [8].

Theorem 2.7. *Let ϕ be a complete sentence such that*

- (1) For every model \mathcal{M} of ϕ , $<^{\mathcal{M}}$ is a linear order.
- (2) ϕ does not have any models of cardinality λ^+ .
- (3) ϕ has a model \mathcal{M} with an $<^{\mathcal{M}}$ -increasing sequence of size λ .

Then 2^{λ} is characterizable.

Next we describe briefly a Fraïssé-type construction which we are going to use.

Definition 2.8. Let \mathcal{A} be a structure that contains \mathcal{M} and if $A_0 \subset \mathcal{A}$, then let $\langle A_0 \rangle$ be the substructure of \mathcal{A} that is generated by A_0 . We call finitely generated over \mathcal{M} the substructures of \mathcal{A} that have the form $\langle A_0 \rangle \cup \mathcal{M}$, where A_0 is a finite subset of $\mathcal{A} \setminus \mathcal{M}$. We write finitely generated/ \mathcal{M} .

If $B_0 = \langle A_0 \rangle \cup \mathcal{M}$, $B_1 = \langle A_1 \rangle \cup \mathcal{M}$ are finitely generated/ \mathcal{M} substructures of \mathcal{A} , we write $B_0 \subset B_1$ and we say that B_0 is a substructure of B_1 , if the same is true (in the usual sense) for $\langle A_0 \rangle$ and $\langle A_1 \rangle$. We also write $B_0 \cong B_1$ if there exists an isomorphism $i : B_0 \rightarrow B_1$ such that $i|_{\mathcal{M}} = id_{\mathcal{M}}$.

It is straightforward to extend the above definition in the case were we have finitely many $\mathcal{M}_0, \dots, \mathcal{M}_n$.

Fraïssé's theorem hold even for “finitely generated/ \mathcal{M} ” substructures (For a proof of Fraïssé's theorem one can consult [4]).

Theorem 2.9 (Fraïssé). *Fix a countable model \mathcal{M} and let $K(\mathcal{M})$ be a countable collection of finitely generated/ \mathcal{M} structures (up to isomorphism). If $K(\mathcal{M})$ has the Hereditary Property (HP), the Joint Embedding Property (JEP) and the Amalgamation Property (AP), then there is a countable structure \mathcal{F} which we will call the Fraïssé limit of $K(\mathcal{M})$, such that*

- (1) \mathcal{F} is unique up to isomorphism and contains \mathcal{M} ,
- (2) $K(\mathcal{M})$ is the collection of all finitely generated/ \mathcal{M} substructures of \mathcal{F} (up to isomorphism), and
- (3) every isomorphism between finitely generated/ \mathcal{M} substructures of \mathcal{F} extends to an automorphism of \mathcal{F} .

The converse is also true, i.e. if \mathcal{F} is a countable structure such that every isomorphism between finitely generated/ \mathcal{M} substructures of \mathcal{F} extends to an automorphism of \mathcal{F} , and $K(\mathcal{M})$ is the collection of all finitely generated/ \mathcal{M} substructures of \mathcal{F} , then $K(\mathcal{M})$ has the HP, the JEP and the AP.

Theorem 2.10 (Fraïssé). *Fix a model \mathcal{M} . Assume that \mathcal{A}, \mathcal{B} are two structures (not necessarily countable) that contain \mathcal{M} and such that*

- for every finitely generated/ \mathcal{M} substructures $C \subset D$ of \mathcal{A} (or of \mathcal{B}), and every embedding $f : C \mapsto \mathcal{A}$ ($f : C \mapsto \mathcal{B}$), there is an embedding $g : D \mapsto \mathcal{A}$ ($g : D \mapsto \mathcal{B}$) that extends f , and
- the collection of all finitely generated/ \mathcal{M} substructures of \mathcal{A} is the same as the collection of all finitely generated/ \mathcal{M} substructures of \mathcal{B} .

Then \mathcal{A} and \mathcal{B} are back-and-forth equivalent, equivalently $\mathcal{A} \equiv_{\infty, \omega} \mathcal{B}$.

In the cases we will work with JEP follows from AP. We now give a slightly different version of the above theorem that will be more fitting to work with

Theorem 2.11. *Fix a countable model \mathcal{M} . If $K(\mathcal{M})$ is a countable collection of finitely generated/ \mathcal{M} structures (up to isomorphism) and $K(\mathcal{M})$ has the HP, the JEP and the AP, then there is a unique (up to isomorphism) countable structure \mathcal{F} that contains \mathcal{M} and satisfies the conjunction of*

- (I) Every finitely generated/ \mathcal{M} substructure of \mathcal{F} is in $K(\mathcal{M})$.
 (II) For every A_0 finitely generated/ \mathcal{M} substructure of \mathcal{F} , if $A_1 \supset A_0$ and $A_1 \in K(\mathcal{M})$, then there exists some finitely generated/ \mathcal{M} substructure $B \subset \mathcal{F}$ and an isomorphism $i : B \cong A_1$, such that $A_0 \subset B$ and $i|_{A_0} = \text{id}$.

Moreover, if there is some $\mathcal{L}_{\omega_1, \omega}$ sentence ψ such that $A \in K(\mathcal{M})$ iff $A \models \psi$ (as it will be the case in our example), then the conjunction of (I) and (II) can be written as a $\mathcal{L}_{\omega_1, \omega}$ -sentence which is equivalent to the Scott sentence of \mathcal{F} and hence, it is complete.

Proof. Existence follows from Theorem 2.9. If \mathcal{F}_1 and \mathcal{F}_2 are both countable structures that contain \mathcal{M} and satisfy (I) and (II), then a standard back-and-forth argument establishes the isomorphism of \mathcal{F}_1 and \mathcal{F}_2 . \square

Notation: In case we want to indicate which class we are talking about, we will write $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$.

Corollary 2.12. If \mathcal{M} is countable and $\mathcal{M}' \cong \mathcal{M}$, then $\lim K(\mathcal{M}') \cong \lim K(\mathcal{M})$.

Theorem 2.11 can be extended even in the case which \mathcal{M} and $K(\mathcal{M})$ have cardinality $\kappa > \aleph_0$. The existence of \mathcal{F} in this case follows from the same diagonal argument as in the countable case, but the uniqueness of the Fraisse limit fails. However, all models of $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$ will be $\equiv_{\omega_1, \omega}$ -equivalent to each other (by Theorem 2.10). So, we get the following

Theorem 2.13. Let ψ be an $\mathcal{L}_{\omega_1, \omega}$ sentence. Assume that \mathcal{M} is a countable model with Scott sentence ϕ and \mathcal{N} is a model of ϕ (possibly uncountable) and let $K(\mathcal{M})$ be the collection of all finitely generated/ \mathcal{M} substructures that satisfy ψ and let $K(\mathcal{N})$ be the collection of all finitely generated/ \mathcal{N} substructures that also satisfy ψ . Moreover, assume that $K(\mathcal{M})$ and $K(\mathcal{N})$ both have the HP, the JEP and the AP. Then any model of $(I)_{K(\mathcal{N})}$ and $(II)_{K(\mathcal{N})}$ is $\equiv_{\infty, \omega}$ -equivalent to $\lim K(\mathcal{M})$.

Proof. By Theorem 2.11, $\lim K(\mathcal{M})$ exists and it is unique, and by the comments above, $K(\mathcal{N})$ has a limit which satisfies $(I)_{K(\mathcal{N})}$ and $(II)_{K(\mathcal{N})}$, but this limit may not be unique.

Since \mathcal{M} and \mathcal{N} satisfy the same Scott sentence ϕ , they are back-and-forth (or $\equiv_{\infty, \omega}$ -) equivalent. For this it follows that for any substructure $\mathcal{A} \in K(\mathcal{M})$ there is an substructure $\mathcal{B} \in K(\mathcal{N})$ such that \mathcal{A} and \mathcal{B} are back-and-forth equivalent, and vice versa. Using (I) and (II), for both $K(\mathcal{M})$ and $K(\mathcal{N})$, we can establish a back-and-forth equivalence for the Fraisse limits and this finishes the proof. \square

This proves that any Fraisse limit of $K(\mathcal{N})$ satisfies the Scott sentence of $\lim K(\mathcal{M})$. If \mathcal{M} is a countable model whose Scott sentence ϕ characterizes a certain cardinal κ , we will use the Scott sentence of $\lim K(\mathcal{M})$ to characterize some cardinal $\lambda \geq \kappa$. In order to construct a Fraisse limit of $K(\mathcal{N})$ we will use

Theorem 2.14. Assume that \mathcal{M} is a countable model whose Scott sentence ϕ characterizes an infinite cardinal κ , \mathcal{N} is a model of ϕ of cardinality $\leq \kappa$, $K(\mathcal{M})$ and $K(\mathcal{N})$ are as above and $\lambda \geq \kappa$. Moreover, assume that:

- (1) If A is a finitely generated/ \mathcal{N} structure, then there are $\leq \lambda$ many (non-isomorphic) structures in $K(\mathcal{N})$ that extend A , and
 (2) If \mathcal{G} is a structure such that

$$\mathcal{N} \subset \mathcal{G}, |\mathcal{G} \setminus \mathcal{N}| \leq \lambda, \mathcal{G} \text{ satisfies } (I)_{K(\mathcal{N})}$$

and for any A_0, A_1 are finitely generated/ \mathcal{N} structures with

$$A_0 \subset \mathcal{G}, A_1 \supset A_0 \text{ and } A_1 \in K(\mathcal{N}),$$

then there is another structure \mathcal{G}' that extends \mathcal{G} and

$$|\mathcal{G}' \setminus \mathcal{N}| \leq \lambda, \mathcal{G}' \text{ satisfies } (I)_{K(\mathcal{N})}$$

and there is some finitely generated/ \mathcal{N} structure $B \subset \mathcal{G}'$ and an isomorphism $i : B \cong A_1$, with $A_0 \subset B$ and $i|_{A_0} = \text{id}$.

Under the assumptions 1 and 2, we conclude that there is a structure \mathcal{G}^* with $\mathcal{N} \subset \mathcal{G}^*$, $|\mathcal{G}^*| = \lambda$ and \mathcal{G}^* satisfies $(I)_{K(\mathcal{N})}$ and $(II)_{K(\mathcal{N})}$. Then \mathcal{G}^* also satisfies the Scott sentence of $\lim K(\mathcal{M})$.

Proof. We construct \mathcal{G}^* by a diagonal argument. If \mathcal{G}_α is the structure at step α and $\mathcal{G}_\alpha \setminus \mathcal{N}$ has size $\leq \lambda$, then by assumption 1, there are $\leq \lambda$ many structures in $K(\mathcal{N})$ that extend some finitely generated/ \mathcal{N} substructure of \mathcal{G}_α . Using the second assumption we can ensure that we include a copy of each one of them into some \mathcal{G}_β , for $\beta > \alpha$. \square

The following is Theorem 27 from [8].

Theorem 2.15. If $\lambda \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$, then $\lambda^\omega \in \mathcal{H}\mathcal{C}\mathcal{H}_{\omega_1, \omega}$.

Theorem 2.16. If κ is a cardinal in $\mathcal{H}\mathcal{C}\mathcal{H}_{\omega_1, \omega}$ and P is a unary predicate in a countable language \mathcal{L} , then there is a sentence χ in $\mathcal{L}_{\omega_1, \omega}$ such that if $\mathcal{N} \models \chi$ then $\aleph_0 \leq |P^{\mathcal{N}}| \leq \kappa$.

Moreover, if \mathcal{M} is a countable \mathcal{L} -model with $|P^{\mathcal{M}}| = \aleph_0$ and

$$\mathcal{L} \cap \mathcal{L}(\chi) = P,$$

where $\mathcal{L}(\chi)$ is the language of χ , then there is a countable $\mathcal{L} \cup \mathcal{L}(\chi)$ -model \mathcal{M}' that extends \mathcal{M} and \mathcal{M}' satisfies χ .

If \mathcal{N} is a countable model isomorphic to \mathcal{M} and \mathcal{N}' and \mathcal{M}' are the $\mathcal{L} \cup \mathcal{L}(\chi)$ -extensions of \mathcal{N} and \mathcal{M} respectively that satisfy χ , then \mathcal{M} with the additional $\mathcal{L} \cup \mathcal{L}(\chi)$ -structure that inherits from \mathcal{M}' and \mathcal{N} with the $\mathcal{L} \cup \mathcal{L}(\chi)$ -structure that inherits from \mathcal{N}' are isomorphic. I.e. the structure that is added on \mathcal{M} by \mathcal{M}' is unique (up to isomorphism).

We will say that χ witnesses the fact that $|P(\cdot)| \leq \kappa$ can be expressed in $\mathcal{L}_{\omega_1, \omega}$.

Proof. Let \mathcal{M}_0 be a countable model with a homogeneous predicate $P^{\mathcal{M}_0}$ that witnesses that $\kappa \in \mathcal{HC}\mathcal{H}_{\omega_1, \omega}$. Let ϕ be the Scott sentence of \mathcal{M}_0 and let $M(\cdot)$ be a unary predicate not in $\mathcal{L}(\phi)$. Then take χ to be the conjunction of:

- (1) $M(\cdot)$ and $P(\cdot)$ are disjoint,
- (2) $P(\cdot)$ is infinite,
- (3) $M(\cdot) \cup P(\cdot) \models \phi$, and
- (4) $P(\cdot)$ is the homogeneous predicate of $M(\cdot) \cup P(\cdot)$ (cf. Definition 2.2).

Since $M(\cdot)$ together with $P(\cdot)$ satisfy ϕ , this restricts the size of $P(\cdot)$ to at most κ .

If \mathcal{M} is a countable model as in the assumption, then let $\mathcal{M}' = \mathcal{M} \cup (\mathcal{M}_0 \setminus P^{\mathcal{M}_0})$ and require that $(\mathcal{M}_0 \setminus P^{\mathcal{M}_0}) \cup P^{\mathcal{M}} \models \phi$. Since P is a homogeneous predicate, any permutation of $P^{\mathcal{M}}$ extends to an automorphism of the whole $(\mathcal{M}_0 \setminus P^{\mathcal{M}_0}) \cup P^{\mathcal{M}}$ structure and the result follows. \square

Note: The above proof relies heavily on the homogeneity of P . If this assumption is taken away it is possible for two different extensions $(\mathcal{M}_0 \setminus P^{\mathcal{M}_0}) \cup P^{\mathcal{M}}$ not to be isomorphic.

The following theorem is from [6]. Before we state it we need a definition. The interested reader should look [6] for more details.

Definition 2.17. Let $(A, <)$ be a linear ordering. Then

- $(A, <)$ is called *scattered*, if it does not embed $(\mathbb{Q}, <)$ (equivalently, $(A, <)$ does not contain a dense subset).
- A subset $B \subset A$ is called *an interval* if B contains at least two elements and

$$\forall x, y \in B \forall z \in A, x < z < y \Rightarrow z \in B.$$

- $(A, <)$ is called *self-additive*, if $A \equiv_{\omega_1, \omega} A + A$, where $A + A$ denotes the linear ordering that consists of two disjoint copies of $(A, <)$ placed the one on top of the other.

Theorem 2.18 (Landraitis). Let $(\mathcal{M}, <, P_i)_{i \in \omega}$ be a countable linearly ordered structure, P_i be a unary predicate for all i , and $\phi_{\mathcal{M}}$ be the Scott sentence of \mathcal{M} . Then:

- (1) $\phi_{\mathcal{M}}$ does not have any uncountable models iff every orbit of \mathcal{M} is scattered, and
- (2) $\phi_{\mathcal{M}}$ has a model of any cardinality iff \mathcal{M} has a self-additive interval, and
- (3) if neither case (1) nor case (2) happens, then $\phi_{\mathcal{M}}$ has models in all cardinalities $\leq 2^{\aleph_0}$, but no model in any cardinality above 2^{\aleph_0} .

All three cases do occur.

With the above theorem Landraitis gives a complete characterization of all infinite cardinals characterized by linearly ordered structures, but he works under the assumption that the language contains only unary predicate symbols. Our results do not use this restriction.

3. Dense linear orderings

In this section we provide some background definitions and theorems about dense linear orderings that we will use later. Most of the material here follows [2]. The reader who is familiar with it can skip to the next section.

Definition 3.1. For infinite cardinals $\kappa \leq \lambda$, let $D(\kappa, \lambda)$ iff there is a linear ordering of size λ with a dense set of size κ and we let $D(\kappa, \lambda, \mu)$ iff there is a linear ordering of size λ , character μ (see Definition 3.2 below for that) and with a dense set of size κ .

Let

$$Ded(\kappa) = \sup\{\lambda \mid D(\kappa, \lambda) \text{ holds}\}$$

and

$$Ded(\kappa, \mu) = \sup\{\lambda \mid D(\kappa, \lambda, \mu) \text{ holds}\}.$$

Definition 3.2. For a linear order $(M, <)$ and some $m \in M$, the *left character* of m is the least cardinal κ such that there is a cofinal function from κ to $\{n \mid n < m\}$, and the *right character* of m is the least κ such that there is a coinitial function from κ to $\{n \mid n > m\}$.

The *character* of m denoted $\chi(m)$ is the least of the left and right character.

The *character* of $(M, <)$ denoted $\chi(M, <)$ is the supremum of the cardinals $\{\chi(m) \mid m \in M\}$. If $(M, <)$ is a dense linear ordering, then $\chi(M, <)$ will always be infinite.

Definition 3.3. If $(M, <)$ is linear order and (L, U) is a partition of M with the property

$$\forall l \in L \forall u \in U, x < y,$$

then (L, U) is called a *Dedekind cut*. If neither L has a supremum, nor U an infimum, then the cut is also called a *gap*.

Definition 3.4. A linear ordering $(M, <)$ is *complete* if for every non-empty $M_0 \subset M$ that has an upper bound, M_0 has a least upper bound.

The dense linear ordering $(\overline{M}, <)$ is a *completion* of the dense linear ordering $(M, <)$ if \overline{M} contains M , \overline{M} is complete, $<$ is an extension of $<$ and M is dense in \overline{M} , in the sense that for every $x, y \in \overline{M}$ with $x < y$, there is some $m \in M$ such that $x < m < y$.

The completion of a linear order is unique up to isomorphism and it is easy to see that $\chi(M, <) = \chi(\overline{M}, <)$. We can redefine $D(\kappa, \lambda)$ (and $D(\kappa, \lambda, \mu)$) using completions:

- $D(\kappa, \lambda)$ holds iff there is a linear order of size κ whose completion has size $\geq \lambda$ and
- $D(\kappa, \lambda, \mu)$ holds iff there is a linear order of size κ and character μ whose completion has size $\geq \lambda$.

The following theorem is from [2]:

Theorem 3.5. ¹Let κ, λ be infinite cardinals.

- If μ is the least cardinal such that $\kappa < \lambda^\mu$, then $D(\kappa, \lambda^\mu, cf(\mu))$ holds.
- If $D(\kappa, \lambda, \mu)$ holds, then $\lambda \leq \kappa^\mu$.

Corollary 3.6. If μ is a regular cardinal and μ is the least such that $\kappa^\mu > \kappa$, then $Ded(\kappa, \mu) = \kappa^\mu$.

4. Powers of the form κ^{\aleph_1}

The main theorem in this section is

Theorem 4.1. If κ is a cardinal in $\mathcal{CH}_{\omega_1, \omega}$, then $\kappa^{\aleph_1} \in \mathcal{CH}_{\omega_1, \omega}$.

We will prove a seemingly weaker form of the theorem first.

Theorem 4.2. If κ is a cardinal in $\mathcal{CH}_{\omega_1, \omega}$ and \aleph_1 is the least cardinal such that $\kappa < \kappa^{\aleph_1}$, then $\kappa^{\aleph_1} \in \mathcal{CH}_{\omega_1, \omega}$.

Before we depart in proving this theorem we make some comments:

- The assumption that \aleph_1 is the least cardinal such that $\kappa < \kappa^{\aleph_1}$, is not as restrictive as it seems. If $\kappa < \kappa^{\aleph_0} = \kappa^{\aleph_1}$, then κ^{\aleph_1} is in $\mathcal{CH}_{\omega_1, \omega}$ by [Theorem 2.15](#). If $\kappa \leq \kappa^{\aleph_0} < \kappa^{\aleph_1}$, then we can apply [Theorem 4.2](#) to κ^{\aleph_0} and conclude again that $\kappa^{\aleph_1} \in \mathcal{CH}_{\omega_1, \omega}$. So, [Theorem 4.2](#) suffices to prove [Theorem 4.1](#).
- If $\kappa \in \mathcal{CH}_{\omega_1, \omega}$ and \aleph_1 is the least cardinal such that $\kappa < \kappa^{\aleph_1}$, then $\kappa = \kappa^{\aleph_0}$ and by [Theorem 2.15](#), $\kappa \in \mathcal{HCH}_{\omega_1, \omega}$. We will make use of this fact in the proof. Also, by the same theorem κ^{\aleph_1} is also in $\mathcal{HCH}_{\omega_1, \omega}$.
- There is nothing special about \aleph_1 . If λ is a cardinal that is characterized by a densely ordered structure and λ is the least such that $\kappa < \kappa^\lambda$, then we will prove in the next section that $\kappa^\lambda \in \mathcal{CH}_{\omega_1, \omega}$.

The first goal is to construct a linear order whose character is carefully controlled. In particular, we will require that it stays bounded by \aleph_1 . Notice that the size of the linear order will not be bounded at this point.

The idea behind the construction is to try to mimic the behavior of the lexicographic order defined on κ^{ω_1} . For $x \neq y \in \kappa^{\omega_1}$ let f be the function that maps (x, y) to the least ordinal $\alpha \in \omega_1$ such that $x(\alpha) \neq y(\alpha)$. Then we can define the lexicographic order:

$$x \triangleleft y \text{ iff } x(\alpha) < y(\alpha), \text{ for } \alpha = f(x, y).$$

Under this definition, for three distinct elements $x, y, z \in \kappa^{\omega_1}$ with $x \triangleleft y \triangleleft z$, we can have only three possibilities

- (1) Either $f(x, y) = f(x, z) = f(y, z)$, or
- (2) $f(x, y) = f(x, z) < f(y, z)$, or
- (3) $f(x, z) = f(y, z) < f(x, y)$.

This property is the one that drives the whole construction.

By [Corollary 2.5](#), there is a linearly ordered structure $\mathcal{M} = (M, <, \dots)$ that characterizes \aleph_1 and let ϕ its Scott sentence. Let \mathcal{L} be the language that extends the language of ϕ and contains the unary predicate symbols V, M, N , the binary predicate \triangleleft , the binary function symbol f and let $K(\mathcal{M})$ be the collection of all countable \mathcal{L} -structures \mathcal{A} that satisfies the conjunction of:

- (1) $V(\mathcal{A}) \cup M(\mathcal{A})$ is a partition of the space. $V(\mathcal{A})$ is finite, while $M(\mathcal{A})$ is infinite and $M(\mathcal{A}) = \mathcal{M}$.
- (2) \triangleleft is a linear order on $V(\mathcal{A})$, not to be confused with $<$ the linear order on $M(\mathcal{A}) = \mathcal{M}$.

¹ Although [Theorem 3.5](#) appears in [2] the results are much older. Baumgartner attributes the first part of [Theorem 3.5](#) to Sierpinski and the second part to Tarski.

- (3) For every $x, y \in V(\mathcal{A}), x \neq y, f(x, y) = f(y, x) \in M(\mathcal{A})$.
- (4) If $x \triangleleft y \triangleleft z$ are three distinct elements of $V(\mathcal{A})$, then one of the three is the case:
 - 4(a) $f(x, y) = f(x, z) = f(y, z)$, or
 - 4(b) $f(x, y) = f(x, z) < f(y, z)$, or
 - 4(c) $f(x, z) = f(y, z) < f(x, y)$.
- (5) For all $x \in \mathcal{A}, N(x)$ implies that $x \in V(\mathcal{A})$. For some $x \in V(\mathcal{A})$, we will say that it is 1-colored if $N(x)$, and we will say that it is 0-colored otherwise.

Before we proceed we need some work. We start by the following observation on property (4).

Observation 4.3. (a) Property (4) can be formulated equivalently as:

If $x \triangleleft y \triangleleft z$, then $f(x, z) = \min\{f(x, y), f(y, z)\}$.

In many cases we will use this equivalent formulation.

(b) If $f(x, y) = f(x, z) < f(y, z)$, then the only way to violate (4) is if either $y \triangleleft x \triangleleft z$, or $z \triangleleft x \triangleleft y$.

Definition 4.4. If $\mathcal{A} \in K(\mathcal{M}), x, y \in V(\mathcal{A})$ and $m \in M(\mathcal{A})$, define $x \sim_m y$ iff

$$x = y \text{ or } f(x, y) > m.$$

Lemma 4.5. \sim_m is an equivalence relation and if $m_1 < m_2$, then \sim_{m_2} is a refinement of \sim_{m_1} .

Proof. Transitivity is the only one that needs some work. Assume $x \sim_m y$ and $y \sim_m z$. Then $f(x, y) > m$ and $f(y, z) > m$. Since the triplet x, y, z satisfies property (4), $f(x, z)$ cannot be less than both $f(x, y)$ and $f(y, z)$ and the result follows.

If $m_1 < m_2$ and $x \sim_{m_2} y$, then $x = y$ or $f(x, y) > m_2 > m_1$. \square

We will denote by $[x]_m$ the equivalence class of x under \sim_m .

Lemma 4.6. If $\mathcal{A} \in K(\mathcal{M}), x, y, y' \in V(\mathcal{A}), x \not\sim_m y$ and $y' \sim_m y$, then

$$x \triangleleft y \text{ iff } x \triangleleft y'.$$

Proof. Since \sim_m is an equivalence relation $x \not\sim_m y'$ and it suffices to prove that $x \triangleleft y$ implies $x \triangleleft y'$. Assume otherwise, i.e. $y' \triangleleft x \triangleleft y$. By property (4), $f(y, y')$ is equal to at least one of the $f(x, y)$ and $f(y', x)$. Since $y' \sim_m y, f(y, y') > m$. Combining these two we conclude that either $f(x, y) > m$ or $f(y', x) > m$, which means that $x \sim_m y$ or $x \sim_m y'$. Contradiction. \square

Definition 4.7. If $x, y \in V(\mathcal{A})$, we write $[x]_m \triangleleft [y]_m$, if for some (all) $x' \in [x]_m$ and some $y' \in [y]_m, x' \triangleleft y'$.

In view of the above lemma, the definition is well-defined and it makes the set of all equivalence classes $\{[x]_m | x \in V(\mathcal{A})\}$ into a linearly ordered set. The linear order of $V(\mathcal{A})$ and the linear order on the set $\{[x]_m | x \in V(\mathcal{A})\}$ are not the same, but the one arises naturally from the other, so we will use \triangleleft for both of them.

Now we are ready to prove the following

Lemma 4.8. $K(\mathcal{M})$ has the HP, the JEP and the AP.

Proof. HP is immediate and JEP follows from AP.

For AP, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K(\mathcal{M})$ and $\mathcal{A} \subset \mathcal{B}, \mathcal{C}$. We keep $N(\cdot)$ as it is on \mathcal{B} and \mathcal{C} . All the work is to extend \triangleleft and f appropriately, so that $\mathcal{B} \cup \mathcal{C}$ becomes a structure in $K(\mathcal{M})$. For all that follows fix some $b \in V(\mathcal{B}) \setminus V(\mathcal{A})$. Let

$$m_0 = \max\{f(a, b) | a \in V(\mathcal{A})\},$$

$$A_0 = \{a \in V(\mathcal{A}) | f(a, b) = m_0\},$$

$$L = \{a \in A_0 | a \triangleleft b\} \text{ and } l = \sup L,$$

$$U = \{a \in A_0 | b \triangleleft a\} \text{ and } u = \inf U.$$

Notice that L or U maybe empty, but at least one of them is not empty. If L for instance, is empty, then let $l = -\infty$ and if U is empty, then $u = \infty$. If both L, U are not empty, then $f(l, u) = m_0$.

Case 1. If $l \neq -\infty$ and $c \in V(\mathcal{C})$ with $[c]_{m_0} \triangleleft [l]_{m_0}$ or $[c]_{m_0} = [l]_{m_0}$, then define

$$f(b, c) = \min\{f(l, c), f(l, b)\}$$

and

$$c \triangleleft b.$$

If $l \triangleleft c$ and $f(l, c) < m_0$, then let

$$f(b, c) = f(l, c)$$

and

$$b \triangleleft c.$$

Case 2. If $u \neq \infty$ we work similarly. In particular, if $[u]_{m_0} \triangleleft [c]_{m_0}$ or $[c]_{m_0} = [u]_{m_0}$, then define

$$f(b, c) = \min\{f(u, c), f(u, b)\}$$

and

$$b \triangleleft c.$$

If $c \triangleleft u$ and $f(u, c) < m_0$, then let

$$f(b, c) = f(u, c)$$

and

$$c \triangleleft b.$$

Note: If both $l \neq -\infty$ and $u \neq \infty$, then there is some overlap between case 1 and case 2. We will prove that the definitions agree in this case (see Claim 1).

Case 3. The only elements of $V(\mathcal{C})$ that were not considered in the above two cases are the $c \in V(\mathcal{C})$ such that $f(c, l) = m_0$ and/or $f(c, u) = m_0$, and $l \triangleleft c \triangleleft u$. Let C_0 be the set of all these c 's. If $C_0 = \emptyset$, we are done, otherwise we have to do some more work:

Split C_0 (arbitrarily) into two disjoint sets D, E such that $D \cup E = C_0$ and for all $d \in D$ and for all $e \in E$, $d \triangleleft e$. Notice that we allow the possibility that one of the D, E is empty. Define $d \triangleleft b \triangleleft e$, for all $d \in D$ and for all $e \in E$. Let $d_0 = \sup D$ and $e_0 = \inf E$. If both d_0, e_0 exist, choose one of them arbitrarily, say d_0 , and choose some $m_1 \geq f(d_0, e_0)$ and let $f(b, d_0) = m_1$ and for all other $c \in C_0$, let $f(b, c) = \min\{f(b, d_0), f(c, d_0)\}$. If only one of d_0, e_0 exist, say d_0 , then choose some arbitrary $m_1 \geq m_0$, let $f(b, d_0) = m_1$ and for all other $c \in C_0$, again let $f(b, c) = \min\{f(b, d_0), f(c, d_0)\}$.

First we verify that the above definition is well-defined.

Claim 1. *If $l \neq -\infty$ and $u \neq \infty$, then cases (1) and (2) of the above definition do not contradict each other.*

Proof. Notice that cases 1 and 2 overlap for all the c 's such that $f(l, c) < m_0$ or $f(u, c) < m_0$. If $f(l, c) < m_0$, then $f(u, c) = f(l, c) < m_0 = f(l, u)$. By Observation 4.3, either $c \triangleleft l \triangleleft u$, or $l \triangleleft u \triangleleft c$. Consider the first case and the second is dealt with symmetrically. If $c \triangleleft l$, then $[c]_{m_0} \triangleleft [l]_{m_0}$ and case 1 of the definition gives $f(b, c) = \min\{f(l, c), f(l, b)\} = \min\{f(l, c), m_0\} = f(l, c)$ and $c \triangleleft b$. For the same c , case 2 of the definition gives $f(b, c) = f(u, c)$ and $c \triangleleft b$. Since $f(u, c) = f(l, c)$, the two definitions completely agree. \square

Next we have to verify that $\mathcal{B} \cup \mathcal{C}$ under the above definition satisfies property (4). The proof splits into many cases given by corresponding claims. We deal only with the case that $l \neq -\infty$. We can prove similar claims for the case that $u \neq \infty$, but are quite similar and we leave the details to the reader. So, for all the following claims assume that $l \neq -\infty$.

Claim 2. *If $c_0, c_1 \in C_0$, then $f(c_0, c_1) \geq m_0$.*

Proof. Assume $c_0 \triangleleft c_1$. Then $l \triangleleft c_0 \triangleleft c_1$ and $f(l, c_0) = f(l, c_1) = m_0$. So, by property (4), $m_0 = f(l, c_0) = f(l, c_1) \leq f(c_0, c_1)$. \square

Claim 3. *If $c \in V(\mathcal{C})$ are such that $f(l, c) < m_0$, then $f(b, c) = f(l, c)$ and $c \triangleleft l$ iff $c \triangleleft b$.*

Proof. If $l \triangleleft c$, then the result is immediate by case 1. If $c \triangleleft l$, then $[c]_{m_0} \triangleleft [l]_{m_0}$ and case 1 again gives $c \triangleleft b$ and $f(b, c) = \min\{f(l, c), f(l, b)\} = \min\{f(l, c), m_0\} = f(l, c)$, which concludes the proof. \square

Claim 4. *If $c_0, c_1 \in V(\mathcal{C})$ is such that $f(l, c_0), f(l, c_1) < m_0$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. By Claim 3, $f(b, c_i) = f(l, c_i)$ and $c_i \triangleleft l$ iff $c_i \triangleleft b$, for $i = 0, 1$. Then property (4) for b, c_0, c_1 follows from the corresponding property for l, c_0, c_1 .

Claim 5. *If $c_0, c_1 \in V(\mathcal{C})$ are such that $f(l, c_0) < m_0 < f(l, c_1)$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. By Claim 3, $f(b, c_0) = f(l, c_0) < m_0$ and either $c_0 \triangleleft l \triangleleft b$ or $l \triangleleft b \triangleleft c_0$. Since $m_0 < f(l, c_1)$, it is $[c_1]_{m_0} = [l]_{m_0}$ and by case 1, $c_1 \triangleleft b$ and $f(b, c_1) = \min\{f(l, c_1), f(l, b)\} = m_0$. If $c_0 \triangleleft l \triangleleft b$, then $c_0 \triangleleft c_1 \triangleleft b$. Otherwise, it would be $c_1 \triangleleft c_0 \triangleleft l$ and by Observation 4.3, $m_0 < f(l, c_1) = \min\{f(l, c_0), f(c_0, c_1)\} \leq f(l, c_0) < m_0$. Contradiction. Thus, in either case c_0 is the minimum or the maximum of the three elements b, c_0, c_1 . It suffices to prove $f(l, c_0) = f(c_0, c_1)$, because then $f(b, c_0) = f(l, c_0) = f(c_0, c_1) < m_0 = f(b, c_1)$ and we have property (4).

If $c_0 \triangleleft c_1 \triangleleft l \triangleleft b$ or $l \triangleleft c_1 \triangleleft b \triangleleft c_0$, then by Observation 4.3, $f(l, c_0) = \min\{f(l, c_1), f(c_0, c_1)\} = f(c_0, c_1)$, since $f(l, c_1) > m_0 > f(l, c_0)$. If $c_0 \triangleleft l \triangleleft c_1 \triangleleft b$ or $c_1 \triangleleft l \triangleleft b \triangleleft c_0$, then, by Observation 4.3 again, $f(c_0, c_1) = \min\{f(l, c_0), f(l, c_1)\} = f(l, c_0)$ and we are done. \square

Claim 6. *If $c_0, c_1 \in V(\mathcal{C})$ are such that $m_0 < f(l, c_0), f(l, c_1)$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. Without loss of generality assume that $c_0 \triangleleft c_1$. By assumption $[c_0]_{m_0} = [l]_{m_0} = [c_1]_{m_0}$ and by case 1, $l \triangleleft c_0 \triangleleft c_1 \triangleleft b$ and $f(b, c_0) = f(b, c_1) = \min\{f(l, b), f(l, c_0)\} = \min\{f(l, b), f(l, c_0)\} = m_0$. It suffices to prove that $m_0 < f(c_0, c_1)$. By Observation 4.3, $f(l, c_1) = \min\{f(l, c_0), f(c_0, c_1)\}$ and by assumption, both $f(l, c_0), f(l, c_1)$ are greater than m_0 . So, it must also be that $f(c_0, c_1) > m_0$. \square

Claim 7. *If $c_0, c_1 \in C_0$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. Without loss of generality assume that $c_0 \triangleleft c_1$ and

$$f(b, c_0) = \min\{f(b, d_0), f(c_0, d_0)\}$$

and

$$f(b, c_1) = \min\{f(b, d_0), f(c_1, d_0)\}.$$

The proof splits into 3 cases:

- (1) $c_0, c_1 \in D$. Then $c_0 \triangleleft c_1 \triangleleft d_0 \triangleleft b$. By **Observation 4.3**, $f(c_0, d_0) = \min\{f(c_0, c_1), f(c_1, d_0)\}$. If $f(c_0, d_0) = f(c_1, d_0) \leq f(c_0, c_1)$, then $f(b, c_0) = f(b, c_1) \leq f(c_0, d_0) \leq f(c_0, c_1)$ and property (4) is satisfied. If $f(c_0, d_0) = f(c_0, c_1) < f(c_1, d_0)$, then either $f(b, c_0) = f(b, c_1) = f(b, d_0) < f(c_0, d_0) = f(c_0, c_1)$, or $f(b, c_0) = f(c_0, d_0) = f(c_0, c_1) < f(b, d_0)$. In the latter case, since also $f(c_0, c_1) < f(c_1, d_0)$, we conclude $f(b, c_0) = f(c_0, c_1) < \min\{f(b, d_0), f(c_1, d_0)\} = f(b, c_1)$ and thus, in both cases property (4) is satisfied.
- (2) $c_0 \in D$ and $c_1 \in E$. Then $c_0 \triangleleft d_0 \triangleleft b \triangleleft e_0 \triangleleft c_1$ and by **Observation 4.3**, $f(c_1, d_0) = \min\{f(d_0, e_0), f(c_1, e_0)\} \leq f(d_0, e_0) \leq m_1 = f(b, d_0)$. By definition $f(b, c_1) = \min\{f(b, d_0), f(c_1, d_0)\} = f(c_1, d_0)$. By **Observation 4.3** again, $f(c_0, c_1) = \min\{f(c_0, d_0), f(c_1, d_0)\}$. If $f(c_0, c_1) = f(c_1, d_0) < f(c_0, d_0)$, then $f(c_0, c_1) = f(c_1, d_0) = f(b, c_1) \leq f(b, d_0)$, which implies $f(c_0, c_1) = f(b, c_1) < \min\{f(c_0, d_0), f(b, d_0)\} = f(b, c_0)$ by definition and gives property (4). If $f(c_0, c_1) = f(c_0, d_0) \leq f(c_1, d_0)$, then $f(c_0, d_0) \leq f(c_1, d_0) \leq f(b, d_0)$. By definition $f(b, c_0) = \min\{f(b, d_0), f(c_0, d_0)\} = f(c_0, d_0) = f(c_0, c_1) \leq f(c_1, d_0)$ and again property (4) is satisfied.
- (3) $c_0, c_1 \in E$. Then $d_0 \triangleleft b \triangleleft c_0 \triangleleft c_1$. As in the previous case, we can prove that $f(b, c_i) = f(c_i, d_0)$, $i = 0, 1$. Since $d_0 \triangleleft c_0 \triangleleft c_1$, by **Observation 4.3**, $f(c_1, d_0) = \min\{f(c_0, c_1), f(c_0, d_0)\}$. So, either $f(b, c_1) = f(c_1, d_0) = f(c_0, c_1) \leq f(c_0, d_0) = f(b, c_0)$, or $f(b, c_1) = f(c_1, d_0) = f(c_0, d_0) = f(b, c_0) < f(c_0, c_1)$, and in both cases property (4) is satisfied. \square

Claim 8. *If $c_0, c_1 \in V(\mathcal{C})$ and $f(l, c_0) = f(l, c_1) = m_0$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. Without loss of generality assume that $c_0 \triangleleft c_1$. If $c_0, c_1 \in C_0$, then the result is from the previous claim. Otherwise, we have to consider two cases:

- (1) Assume that $c_0 \triangleleft l \triangleleft c_1$, i.e. $c_1 \in C_0$, while $c_0 \notin C_0$. Then $c_0 \triangleleft l \triangleleft b$ and by **Observation 4.3**, $f(b, c_0) = \min\{f(l, c_0), f(b, l)\} = m_0$. By definition, $f(b, c_1) = \min\{f(b, d_0), f(c_1, d_0)\}$ and by definition again $f(b, d_0) = m_1 \geq f(d_0, e_0) \geq m_0$, while by **Claim 2**, $f(c_1, d_0) \geq m_0$. So, $f(b, c_1) \geq m_0$. By **Observation 4.3** for $c_0 \triangleleft l \triangleleft c_1$, $f(c_0, c_1) = \min\{f(l, c_0), f(l, c_1)\} = m_0$. Overall, $f(b, c_0) = f(c_0, c_1) = m_0 \leq f(b, c_1)$ and we have property (4).
- (2) Assume that $c_0 \triangleleft c_1 \triangleleft l \triangleleft b$, i.e. both $c_0, c_1 \notin C_0$. Then $[c_i]_{m_0} \triangleleft [l]_{m_0}$ and by definition $f(b, c_i) = \min\{f(l, c_i), f(l, b)\} = m_0$, for both $i = 0, 1$. By **observation 4.3**, $m_0 = f(c_0, l) = \min\{f(c_0, c_1), f(c_1, l)\} \leq f(c_0, c_1)$. Combining all these, $f(b, c_0) = f(b, c_1) = m_0 \leq f(c_0, c_1)$ which gives property (4). \square

Claim 9. *If $c_0, c_1 \in V(\mathcal{C})$ and $f(l, c_0) < m_0 = f(l, c_1)$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. Without loss of generality assume that $c_0 \triangleleft l$ (the other case is handled similarly). We split into two cases:

- (1) $c_1 \notin C_0$. Then $c_0 \triangleleft c_1 \triangleleft l$. Otherwise, it would be $c_1 \triangleleft c_0 \triangleleft l$ and by **Observation 4.3**, $m_0 = f(c_1, l) = \min\{f(c_0, l), f(c_1, c_0)\} \leq f(l, c_0) < m_0$. Contradiction.
So, $c_0 \triangleleft c_1 \triangleleft l$. By definition, $f(b, c_0) = \min\{f(l, b), f(l, c_0)\} = \min\{m_0, f(l, c_0)\} = f(l, c_0) < m_0$. A similar argument proves that $f(b, c_1) = m_0$ and **Observation 4.3** for $c_0 \triangleleft c_1 \triangleleft l$ implies $f(l, c_0) = \min\{f(l, c_1), f(c_0, c_1)\}$, while $f(l, c_0) < m_0 = f(l, c_1)$. So, it must be $f(l, c_0) = f(c_0, c_1) < f(l, c_1)$. Combining all these, $f(b, c_0) = f(l, c_0) = f(c_0, c_1) < m_0 = f(b, c_1)$ and we are done.
- (2) $c_1 \in C_0$. Then $c_0 \triangleleft l \triangleleft c_1$ and $f(b, c_0) = \min\{f(b, l), f(l, c_0)\} = f(l, c_0)$, while $f(b, c_1) = \min\{f(b, d_0), f(c_1, d_0)\}$. By **Claim 2**, $f(c_1, d_0)$ must be greater or equal to m_0 and by definition again $f(b, d_0) \geq f(d_0, e_0) \geq m_0$, which combined gives $f(b, c_1) \geq m_0$. **Observation 4.3** for $c_0 \triangleleft l \triangleleft c_1$ gives $f(c_0, c_1) = \min\{f(l, c_0), f(l, c_1)\} = f(l, c_0) = f(b, c_0) < m_0 \leq f(b, c_1)$ and this concludes the claim. \square

Claim 10. *If $c_0, c_1 \in V(\mathcal{C})$ and $f(l, c_0) = m_0 < f(l, c_1)$, then the triplet b, c_0, c_1 satisfies property (4).*

Proof. Since $f(l, c_1) > m_0$, $c_1 \sim_{m_0} l$ and $[c_1]_{m_0} = [l]_{m_0}$ and by definition $c_1 \triangleleft b$ and $f(b, c_1) = \min\{f(l, c_1), f(l, b)\} = \min\{f(l, c_1), m_0\} = m_0$. By property (4) for c_0, l, c_1 , $f(c_0, c_1) = f(l, c_0) = m_0 < f(l, c_1)$.

If $c_0 \triangleleft l \triangleleft b$, then by definition, $f(b, c_0) = \min\{f(l, c_0), f(l, b)\} = m_0$ and $f(b, c_0) = f(c_0, c_1) = f(b, c_1) = m_0$ and we have the result. If $l \triangleleft c_0$, then either $c_0 \in C_0$, or otherwise $u \neq \infty$ and $u \triangleleft c_0$. If $c_0 \in C_0$, then $f(b, c_0) = \min\{f(b, d_0), f(c_0, d_0)\}$ and both $f(b, d_0), f(c_0, d_0)$ are $\geq m_0$. Therefore, $f(b, c_0) \geq m_0$ and $f(c_0, c_1) = m_0 = f(b, c_1) \leq f(b, c_0)$, which gives property (4). If $u \neq \infty$ and $u \triangleleft c_0$, then by definition $f(b, c_0) = \min\{f(u, c_0), f(b, u)\} = m_0$ and then, $f(b, c_0) = m_0 = f(b, c_1) = f(c_0, c_1)$, which concludes the proof of the claim. \square

The above claims prove that $\mathcal{B} \cup \mathcal{C}$ satisfies property (4) in all cases and the proof of the lemma is also concluded. \square

Note: The proof of the above lemma would have been simpler, if we had defined f and \triangleleft differently on C_0 . An easy example is to let $b \triangleleft c$, for all $c \in C_0$. The reason we went through all this work is because in the proof of [Theorem 4.13](#) we will need a similar construction, namely we will need to make a Dedekind cut on C_0 and place b appropriately in the cut. The details of the proof of [Theorem 4.13](#) follow closely the proof we just did and we will omit it as it tends to be very repetitive. The interested reader should be able to fill in all the details following the example of the proof above.

Now, by [Theorem 2.11](#) there is a Fraïssé limit of $K(\mathcal{M})$, call it \mathcal{F} and let $\phi_{\mathcal{F}}$ be its Scott sentence. By the same theorem we know that the $\phi_{\mathcal{F}}$ is equivalent to the conjunction of $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$ and the following lemma gives us another equivalence.

Lemma 4.9. *If $\phi_{\mathcal{F}}$ is the Scott sentence of the Fraïssé limit of $K(\mathcal{M})$, then $\phi_{\mathcal{F}}$ is equivalent to the conjunction of the following:*

$(I)_{K(\mathcal{M})}$ (cf. [Theorem 2.11](#))

- (1*) for all $x \in V(\mathcal{F})$ and for all $m \in M(\mathcal{F})$, there exist z_1, z_2 such that $z_1 \triangleleft x \triangleleft z_2$, $f(x, z_1) = f(x, z_2) = m$ and we can require z_1, z_2 to be 0-, or 1-colored (not necessarily the same). In particular, \triangleleft restricted to the subset that contains the 0-colored (respectively the 1-colored) elements of $V(\mathcal{F})$ is a linear ordering without endpoints.
- (2*) for all $x \triangleleft y \in V(\mathcal{F})$ and for all $m \geq f(x, y)$, $m \in M(\mathcal{F})$, there exist some z_1, z_2 such that $x \triangleleft z_1, z_2 \triangleleft y$, $f(x, z_1) = m$ and $f(y, z_2) = m$ and again, we can require z_1, z_2 to be 0-, or 1-colored.

Also note that z_1 is not required to be different than z_2 , although in most cases they will be different.

Proof. By [Theorem 2.11](#), $\phi_{\mathcal{F}}$ is equivalent to the conjunction of $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$. So we have to prove that $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$ are equivalent to the conjunction of $(I)_{K(\mathcal{M})}$ and (1*) and (2*).

The direction $(I)_{K(\mathcal{M})} \wedge (II)_{K(\mathcal{M})} \rightarrow (I)_{K(\mathcal{M})} \wedge (1^*) \wedge (2^*)$ is immediate, because if $x \triangleleft y \in V(\mathcal{F})$, the structure generated by x, y is in $K(\mathcal{M})$ and then we can use $(II)_{K(\mathcal{M})}$ to extend this structure to a structure that contains the desired z_1, z_2 for both (1*) and (2*).

The direction $(I)_{K(\mathcal{M})} \wedge (1^*) \wedge (2^*) \rightarrow (I)_{K(\mathcal{M})} \wedge (II)_{K(\mathcal{M})}$ needs some more work. Assume that \mathcal{A}_0 is a finitely generated/ \mathcal{M} substructure of \mathcal{F} and $\mathcal{A}_1 \supset \mathcal{A}_0$ with $\mathcal{A}_1 \in K(\mathcal{M})$. We need to find a finitely generated/ \mathcal{M} substructure \mathcal{F}_1 of \mathcal{F} with $\mathcal{A}_0 \subset \mathcal{F}_1$ and some isomorphism $i: \mathcal{F}_1 \cong \mathcal{A}_1$ with $i|_{\mathcal{A}_0} = id$. We work by induction on $n = |\mathcal{A}_1 \setminus \mathcal{A}_0| = |V(\mathcal{A}_1) \setminus V(\mathcal{A}_0)|$.

If $n = 0$, the result is obvious and it suffices to prove the result for $n = 1$. Let $V(\mathcal{A}_1) = V(\mathcal{A}_0) \cup \{a\}$ and we will find some element $z \in \mathcal{F}$ such that $\mathcal{A}_1 \cong \mathcal{A}_0 \cup \{z\}$. If $a \triangleleft a_0$, or $a_0 \triangleleft a$, for all $a_0 \in V(\mathcal{A}_0)$, we can find this z using (1*). Otherwise let a_0 be the maximum element in $V(\mathcal{A}_0)$ such that $a_0 \triangleleft a$ and a_1 be the minimum element in $V(\mathcal{A}_0)$ such that $a \triangleleft a_1$, and we can find the desired z using (2*) on the tuple a_0, a_1 . \square

Before we prove anything else we prove the following

Theorem 4.10. *If \mathcal{G} is a model of $\phi_{\mathcal{F}}$, then $(V(\mathcal{G}), \triangleleft)$ is a linear order with character $\chi(V(\mathcal{G}), \triangleleft) = cf(M(\mathcal{G}), <)$.*

Proof. Since $M(\mathcal{G})$ is a model of ϕ , it has size $\leq \aleph_1$ and the same is true for its cofinality. Without loss of generality we will assume that $cf(M(\mathcal{G}), <) = \aleph_1$. Fix a cofinal sequence $\{m_\alpha \in M(\mathcal{G}) \mid \alpha < \omega_1\}$ of length \aleph_1 . It is not hard to see that

$$\mathcal{F} \models \forall a \in V(\mathcal{F}) \forall m \in \mathcal{M} \exists b \in V(\mathcal{F}) (b \triangleleft a \wedge f(a, b) = m). \quad (4.1)$$

Therefore, the same sentence is true for \mathcal{G} .

Fix some $a \in V(\mathcal{G})$. For every $\alpha \in \aleph_1$, find some $b_\alpha \triangleleft a$ given by (4.1), such that $f(a, b_\alpha) = m_\alpha$. Then the sequence $(b_\alpha \mid \alpha < \aleph_1)$ is cofinal in $\{c \mid c \triangleleft a\}$. If $c \triangleleft a$ and $f(a, c) = m$, there is some α such that $m \leq m_\alpha$. Then $f(b_{\alpha+1}, c) = \min\{f(a, c), f(a, b_{\alpha+1})\} = m$, which implies that $c \triangleleft b_{\alpha+1}$. So, the character of $V(\mathcal{G})$ is at most \aleph_1 .

On the other hand, if $(c_n)_{n \in \omega}$ is an \triangleleft -increasing countable sequence, such that

$$c_0 \triangleleft c_1 \triangleleft \dots \triangleleft a,$$

then by [Observation 4.3](#), $f(c_n, a) = \min\{f(c_{n+1}, a), f(c_n, c_{n+1})\} \leq f(c_{n+1}, a)$. Therefore the sequence $(f(c_n, a) \mid n \in \omega)$ is increasing and by the assumption on the cofinality of $(M(\mathcal{G}), <)$, this sequence cannot be cofinal. Hence, it is bounded above and we can find as above some $\alpha \in \aleph_1$ such that $c_n \triangleleft b_\alpha$, for all n . Then b_α is an upperbound of $(c_n)_{n \in \omega}$ and $(c_n)_{n \in \omega}$ cannot be cofinal in $\{c \in V(\mathcal{G}) \mid c \triangleleft a\}$, which proves that the left character of (any) $a \in V(\mathcal{G})$ is equal to \aleph_1 . We can repeat the proof for the right character being equal to \aleph_1 and this concludes the proof of the theorem. \square

Lemma 4.11. *If \mathcal{G} is a model that satisfies $(I)_{K(\mathcal{M})}$ and $x \in V(\mathcal{G})$ and $Y \subset V(\mathcal{G})$ are such that for all $y \in Y$, $y \triangleleft x$ and for all $m \in M(\mathcal{G})$ there exists $y \in Y$ such that $f(x, y) > m$, then x is the supremum of Y .*

Quite symmetrically, if $x \triangleleft y$, for all $y \in Y$, and for all $m \in M(\mathcal{G})$ there exists some y such that $f(x, y) > m$, then x is the infimum of Y .

Proof. Toward contradiction, assume there is some z such that for all $y \in Y$, $y \triangleleft z \triangleleft x$ and let $m = f(x, z)$. By assumption find some $y \in Y$ such that $f(x, y) > m$. Then the triplet x, y, z contradicts property (4) of $K(\mathcal{M})$ since $y \triangleleft z \triangleleft x$ and $f(y, x) > m = f(x, z)$.

For the second part, the proof is symmetrical. \square

Definition 4.12. Assume \mathcal{G} is a structure that satisfies $(I)_{K(\mathcal{M})}$ and (D, E) is a Dedekind cut, i.e. a partition of $V(\mathcal{G})$ such that $d \triangleleft e$, for all $d \in D$ and for all $e \in E$. If there exists some $d \in D$ such that the set $\{f(d, e) \mid e \in E\}$ is cofinal in $\{f(d', e) \mid d' \in D, e \in E\}$ and the set $\{f(d, d') \mid d' \in D, d \triangleleft d'\}$ is coinital in $\{m \in \mathcal{M} \mid m > f(d', e) \text{ for all } d' \in D, e \in E\}$ and the set $\{f(d', e) \mid d' \in D, e \in E\}$ does not have a supremum (equivalently, the set $\{m \in \mathcal{M} \mid m > f(d', e) \text{ for all } d' \in D, e \in E\}$ does not have an infimum), then we will call (D, E) an *irremovable gap*.

Symmetrically, if there exists some $e \in E$ such that the set $\{f(d, e) \mid d \in D\}$ is cofinal in $\{f(d, e') \mid d \in D, e' \in E\}$ and the set $\{f(e, e') \mid e' \in E, e' \triangleleft e\}$ is coinital in $\{m \in \mathcal{M} \mid m > f(d', e) \text{ for all } d' \in D, e \in E\}$ and the set $\{f(d, e') \mid d \in D, e' \in E\}$ does not have a supremum (equivalently, the set $\{m \in \mathcal{M} \mid m > f(d', e) \text{ for all } d' \in D, e \in E\}$ does not have an infimum), then (D, E) is an *irremovable gap*.

If (D, E) consists of a gap on $V(\mathcal{G})$ that is not irremovable, we will call it a *removable gap*.

If (D, E) is an irremovable gap, it follows that neither $\sup D$ exists nor $\inf E$, and moreover, we cannot extend $V(\mathcal{G})$ by adding an element b such that for all $d \in D$ and for all $e \in E$, $d \triangleleft b \triangleleft e$. The reason for that is that if d witnesses the fact that (D, E) is irremovable, then by **Observation 4.3**, $f(b, d)$ has to be smaller or equal than all the elements of the set $\{f(d, d') \mid d' \in D, d \triangleleft d'\}$ and bigger or equal to all the elements of the set $\{f(d, e) \mid e \in E\}$, which by the assumptions above cannot happen.

Also notice that the sets

$$M_1 = \{m \in \mathcal{M} \mid m \leq f(d', e) \text{ for some } d' \in D, e \in E\}$$

and

$$M_2 = \{m \in \mathcal{M} \mid m > f(d', e) \text{ for all } d' \in D, e \in E\}$$

consist of a Dedekind cut on $(\mathcal{M}, <)$. If neither M_1 has a supremum nor M_2 an infimum, then the cut is a gap and this gap gives rise to irremovable gaps on $(V(\mathcal{G}), \triangleleft)$ (one for every element of $V(\mathcal{G})$). Therefore, if $(\mathcal{M}, <)$ is a complete order, then there are no irremovable gaps on $(V(\mathcal{G}), \triangleleft)$. Otherwise the number of irremovable gaps is bounded by $|V(\mathcal{G})|$ times the number of gaps on $(\mathcal{M}, <)$. If the character of $(\mathcal{M}, <)$ is μ , $\mu \leq \aleph_1 = |\mathcal{M}|$, then the number of gaps on $(\mathcal{M}, <)$ is $\leq \aleph_1^\mu \leq 2^{\aleph_1}$.

Theorem 4.13. Assume \mathcal{G} is a structure that satisfies $(I)_{K(\mathcal{M})}$, $|\mathcal{G} \setminus \mathcal{M}| \leq \kappa$, and (D, E) is a removable gap on $V(\mathcal{G})$. Also assume that $\mathcal{A}_0 \subset \mathcal{G}$, $\mathcal{A}_1 = \mathcal{A}_0 \cup \{b'\}$ and \mathcal{A}_1 is a structure in $K(\mathcal{M})$. Then there exists another structure \mathcal{G}' which extends \mathcal{G} , \mathcal{G}' satisfies $(I)_{K(\mathcal{M})}$, $|\mathcal{G}' \setminus \mathcal{M}| \leq \kappa$, there exists some element $b \in \mathcal{G}'$ such that $d \triangleleft b \triangleleft c$, for all $d \in D$ and all $e \in E$, and there exists an isomorphism $i : \mathcal{A}_0 \cup \{b\} \cong \mathcal{A}_1$, with $i|_{\mathcal{A}_0} = \text{id}$.

We will say that \mathcal{G}' removes that gap (D, E) by adding b' .

Proof. As in **Lemma 4.8**, let $d_0 = \sup D$ and $e_0 = \inf E$ and notice that d_0, e_0 may not exist. We distinguish the following cases:

- d_0 exists and there is some $m \in M(\mathcal{G})$ such that for all $e \in E, f(d_0, e) \leq m$.
Then choose some arbitrary $m_1 \geq m_0$, let $f(b, d_0) = m_1$ and for all other $c \in C_0$, let $f(b, c) = \min\{f(b, d_0), f(c, d_0)\}$.
 - Symmetrically, if e_0 exists and there is some $m \in M(\mathcal{G})$ such that for all $d \in D, f(e_0, d) \leq m$, then choose some arbitrary $m_1 \geq m_0$, let $f(b, e_0) = m_1$ and $f(b, c) = \min\{f(b, e_0), f(c, e_0)\}$, for all other $c \in C_0$.
 - If both d_0, e_0 exist, then choose one of them arbitrarily, say d_0 , and choose some $m_1 \geq f(d_0, e_0)$ and let $f(b, d_0) = m_1$ and $f(b, c) = \min\{f(b, d_0), f(c, d_0)\}$, for all other $c \in C_0$.
- These first three cases are similar to the ones we encountered in the proof of Amalgamation. The next ones are new:
- d_0 exists and for all $m \in M(\mathcal{G})$ there exists $e \in E, f(d_0, e) \geq m$.
Then by **Lemma 4.11** d_0 is the infimum of E and in this case (D, E) is not a gap.
 - Symmetrically, if e_0 exists and for all $m \in M(\mathcal{G})$ there exists $d \in D, f(e_0, d) \geq m$, then e_0 is the supremum of D and again (D, E) is not a gap.
 - If for every $d \in D$ the set $\{f(d, e) \mid e \in E\}$ is not cofinal in $\{f(d', e) \mid d' \in D, e \in E\}$, then for every d there exists some $d' \in D$ and $e' \in E$ such that for every $e \in E, f(d, e) < f(d', e')$. In this case define $f(b, d) = f(d, d')$.
Similarly, if for every $e \in E$ the set $\{f(d, e) \mid d \in D\}$ is not cofinal in $\{f(d, e') \mid d \in D, e' \in E\}$, then there exists some $e' \in E$ and some $d' \in D$ such that for every $d \in D, f(d, e) < f(d', e')$. Define $f(b, e) = f(e, e')$.
 - If there exists some $d \in D$ such that the set $\{f(d, e) \mid e \in E\}$ is cofinal in $\{f(d', e) \mid d' \in D, e \in E\}$ and there exists some $s \in \mathcal{M}$ greater than all $f(d', e), d' \in D, e \in E$ and smaller than all $f(d, d'), d' \in D, d \triangleleft d'$, then let $f(b, d) = s$ and for every other $c \in C_0$, let $f(b, c) = \min\{f(b, d), f(c, d)\}$.
 - Symmetrically, if there exists some $e \in E$ such that the set $\{f(d, e) \mid d \in D\}$ is cofinal in $\{f(d, e') \mid d \in D, e' \in E\}$ and there exists some s greater than all $f(d, e'), d \in D, e' \in E$ and smaller than all $f(e, e'), e' \in E, e' \triangleleft e$, then let $f(b, e) = s$ and for every other $c \in C_0$, let $f(b, c) = \min\{f(b, e), f(c, e)\}$.

In all these cases we have to prove that f was defined in such a way that $\mathcal{G}' = \mathcal{G} \cup \{b\}$ also satisfies $(I)_{K(\mathcal{M})}$. The details of the proof follow the proof of **Lemma 4.8** and are left to the reader. \square

Theorem 4.14. Let \mathcal{M} be a model of ϕ of cardinality \aleph_1 . Then for every linear ordering $(L, <)$ that has cardinality $|L| = \kappa \geq \aleph_1$, there is some model \mathcal{G}^* of cardinality κ that satisfies $\phi_{\mathcal{F}}$, $M(\mathcal{G}^*) = \mathcal{M}$, and there exists a one-to-one function $F : L \rightarrow V(\mathcal{G}^*)$ such that

$$x < y \text{ iff } F(x) \triangleleft F(y)$$

and

for all $x \in L$, $F(x)$ is 0-colored.

In other words, every linear ordering of size $\geq \aleph_1$ can be embedded into a model of $\phi_{\mathcal{F}}$ of the same cardinality.

Proof. Let \mathcal{G} be the \mathcal{L} -structure such that $(V(\mathcal{G}), \triangleleft) \cong (L, <)$ and for every $x, y \in V(\mathcal{G})$, x, y are both 0-colored and $f(x, y) = m$, for some fixed $m \in M(\mathcal{G})$. It is immediate that this \mathcal{G} satisfies $(I)_{K(\mathcal{M})}$ and we have to extend \mathcal{G} to a structure \mathcal{G}^* that satisfies both $(I)_{K(\mathcal{M})}$ and $(II)_{K(\mathcal{M})}$. Since for any finitely generated \mathcal{M} substructure of \mathcal{G} there are $\aleph_1 \leq \kappa$ many structures in $K(\mathcal{M})$ that extend it, we can use Theorems 2.14 and 4.13 to find the desired \mathcal{G}^* . \square

Corollary 4.15. For every infinite cardinal κ , there is a model of $\phi_{\mathcal{F}}$ that has size κ .

Theorem 4.16. Let \mathcal{G} be a model of $\phi_{\mathcal{F}}$ and $\overline{V(\mathcal{G})}$ the \triangleleft -completion of $V(\mathcal{G})$. Then there is some model $\overline{\mathcal{G}}$ such that

- If U is the set of irremovable gaps of $V(\mathcal{G})$ (cf. Definition 4.12), then $V(\overline{\mathcal{G}}) = \overline{V(\mathcal{G})} \setminus U$,
- for all $x \in V(\overline{\mathcal{G}}) \setminus V(\mathcal{G})$, x is 1-colored, i.e. no new 0-colored elements are introduced,
- the function $f^{\overline{\mathcal{G}}}$ restricted on $\mathcal{G} \times \mathcal{G}$ agrees with the function $f^{\mathcal{G}}$ and
- $\overline{\mathcal{G}} \models \phi_{\mathcal{F}}$.

The model $\overline{\mathcal{G}}$ we will call the completion of \mathcal{G} .

In particular, if $|\mathcal{G}| = \kappa > 2^{\aleph_1}$, then the completion of \mathcal{G} has cardinality equal to $|\overline{V(\mathcal{G})}|$.

Proof. Since \mathcal{G} satisfies $\phi_{\mathcal{F}}$, by Theorem 4.9, \mathcal{G} satisfies $(I)_{K(\mathcal{M})}$ and (1^*) and (2^*) . Since (1^*) and (2^*) are density requirements and $V(\mathcal{G})$ is dense in $V(\overline{\mathcal{G}})$, $\overline{\mathcal{G}}$ satisfies (1^*) and (2^*) too. So, it remains to show that $\overline{\mathcal{G}}$ satisfies $(I)_{K(\mathcal{M})}$ and this follows from Theorem 4.13. We remove all the removable gaps, one at a time, by applying Theorem 4.13 and we make sure that all the elements we add are 1-colored.

If $\kappa > 2^{\aleph_1}$, then there exist at most $\kappa \cdot \aleph_1^{\mu} \leq \kappa \cdot 2^{\aleph_1} = \kappa$ many irremovable gaps (see comments after Definition 4.12), where μ is the character of $(\mathcal{M}, <)$ and $\mu \leq \aleph_1$, and the result follows. \square

Theorem 4.17. If $\kappa \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$ and \aleph_1 is the least cardinal such that $\kappa^{\aleph_1} > \kappa$, then $\kappa^{\aleph_1} \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$.

Proof. If $\kappa \leq 2^{\aleph_1}$, then $\kappa^{\aleph_1} = 2^{\aleph_1}$ and the result follows from Theorem 2.7 and Corollaries 2.5 and 2.6. So assume that $\kappa > 2^{\aleph_1}$. By the assumption on κ and by Theorem 2.15 we can assume that $\kappa \in \mathcal{H}\mathcal{C}\mathcal{H}_{\omega_1, \omega}$. Let ψ be the conjunction of $\phi_{\mathcal{F}}$ together with the sentence that expresses the fact that the set of the 0-colored elements of $V(\cdot)$ has cardinality $\leq \kappa$. Since $\kappa \in \mathcal{H}\mathcal{C}\mathcal{H}_{\omega_1, \omega}$ we can express this fact by a sentence in $\mathcal{L}_{\omega_1, \omega}$ by Theorem 2.16.

By Theorem 4.14, $\phi_{\mathcal{F}}$ has a model \mathcal{G}^* that embeds the linear ordering $\kappa^{<\aleph_1}$ into the set of the 0-colored elements. By assumption $\kappa^{<\aleph_1} = \kappa$ and we can assume that \mathcal{G}^* also has cardinality κ . From this it follows that \mathcal{G}^* is also a model of ψ . By Theorem 4.16 there is a model $\overline{\mathcal{G}^*}$ of cardinality equal to the cardinality of $|\kappa^{<\aleph_1}| = \kappa^{\aleph_1}$ that introduces no new 0-colored elements. Therefore, $\overline{\mathcal{G}^*}$ is also a model of ψ . Since by Theorem 4.10 any model of ψ (and $\phi_{\mathcal{F}}$) has character $\leq \aleph_1$, and any model of ψ has a dense subset of cardinality $\leq \kappa$ (the set of the 0-colored elements), by Theorem 3.5 there is no model of ψ of cardinality $> \kappa^{\aleph_1}$ which concludes the proof. \square

Theorem 4.18. If $\kappa \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$, then $\kappa^{\aleph_1} \in \mathcal{C}\mathcal{H}_{\omega_1, \omega}$.

Proof. If $\kappa^{\aleph_0} = \kappa^{\aleph_1}$, the result follows from 2.15. Otherwise use the previous theorem for κ^{\aleph_0} . \square

5. Powers of the form κ^λ

There is nothing particular about \aleph_1 that cannot be generalized to any uncountable cardinal λ that is characterized by a densely ordered structure. The proofs of the following theorems follow from the proof of the corresponding theorems in the previous section by replacing \aleph_1 with λ . Thus we have the following:

Let $(\mathcal{M}, <)$ be a countable densely ordered structure that characterizes λ and $K(\mathcal{M})$ be defined as above (see Theorem 4.2). Then $K(\mathcal{M})$ has the HP, the JEP and the AP and there exists a Fraisse limit for $K(\mathcal{M})$ which we will call \mathcal{F} and let $\phi_{\mathcal{F}}$ be its Scott sentence.

Theorem 5.1. If \mathcal{G} is a model of $\phi_{\mathcal{F}}$, then $(V(\mathcal{G}), \triangleleft)$ is a dense linear ordering with character $\chi(V(\mathcal{G}), \triangleleft) = cf(M(\mathcal{G}), <)$.

Since the character of $(M(\mathcal{G}), <)$ is bounded by $|M(\mathcal{G})| \leq \lambda$, the set of gaps on $(M(\mathcal{G}), <)$ will have size at most 2^λ . If λ is a singular cardinal, then $\chi(V(\mathcal{G}), \triangleleft) = cf(M(\mathcal{G}), <) < \lambda$ and we may get strict inequality.

Theorem 5.2. *If λ and $\phi_{\mathcal{F}}$ are as above, then for every linear ordering $(L, <)$ with cardinality $|L| = \kappa \geq \lambda$, there exists some model \mathcal{G}^* of $\phi_{\mathcal{F}}$ of cardinality κ and a one-to-one function $F : L \rightarrow \mathcal{G}^*$ such that*

$$x < y \text{ iff } F(x) \triangleleft F(y)$$

and

for all $x \in L$, $F(x)$ is 0-colored.

In other words, every linear ordering of size $\geq \lambda$ can be embedded into a model of $\phi_{\mathcal{F}}$ of the same cardinality.

If \mathcal{G} is a model of $\phi_{\mathcal{F}}$ of size κ , then the set of irremovable gaps on $V(\mathcal{G})$ has size $\leq \kappa \cdot 2^\lambda$.

Theorem 5.3. *If \mathcal{G} is a model of $\phi_{\mathcal{F}}$ of cardinality $\kappa > 2^\lambda$, then there is a model of ϕ of cardinality equal to $|\overline{V(\mathcal{G})}|$.*

Theorem 5.4. *Assume that $\kappa > 2^\lambda$ is a characterizable cardinal, λ is a cardinal characterizable by a densely ordered structure and λ is the least cardinal such that $\kappa^\lambda > \kappa$. Then $\kappa^\lambda \in \mathcal{CH}_{\omega_1, \omega}$.*

Observe here that we allow the possibility that λ is countable. In this case the result follows from [Theorem 2.15](#). In the case that $\kappa \leq 2^\lambda$, the characterizability of $\kappa^\lambda = 2^\lambda$ follows from [Theorem 2.7](#), but we require the extra assumption that there exists a model with an increasing sequence of size λ . If λ is a successor of a characterizable cardinal (as it is in the case of \aleph_1), such a sequence is guaranteed by [Corollary 2.5](#), but in the general case this question is open (see [Open Question 2](#)). So, we have

Theorem 5.5. *If \aleph_α and κ^{\aleph_α} are both in $\mathcal{CH}_{\omega_1, \omega}$, then $\kappa^{\aleph_{\alpha+\beta}} \in \mathcal{CH}_{\omega_1, \omega}$, for all countable ordinals β .*

Proof. By induction on β . If $\kappa^{\aleph_{\alpha+\beta}} = 2^{\aleph_{\alpha+\beta}}$, then use [Corollary 2.5](#) and [Theorem 2.7](#). Otherwise, use [Corollary 2.6](#) and [Theorem 5.4](#). \square

Corollary 5.6. *If $\kappa \in \mathcal{CH}_{\omega_1, \omega}$, then $\kappa^{\aleph_\alpha} \in \mathcal{CH}_{\omega_1, \omega}$, for all countable ordinals α .*

Theorem 5.7. *Assume that $\kappa > 2^{\aleph_\alpha}$ is a characterizable cardinal, \aleph_α is a cardinal characterizable by a densely ordered structure and \aleph_α is the least cardinal such that $\kappa^{\aleph_\alpha} > \kappa$. Then κ^{\aleph_β} is in $\mathcal{CH}_{\omega_1, \omega}$, for all $\beta < \alpha + \omega_1$.*

Proof. By [Theorems 5.4](#) and [5.5](#). \square

To the best of our knowledge the following questions are open:

Open Question 1. *Assume that κ is in $\mathcal{CH}_{\omega_1, \omega}$. Does it follow that κ is characterized by the Scott sentence of a densely ordered structure?*

Assume that ϕ characterizes κ . Extend the language of ϕ to \mathcal{L}_1 by including $<$ and assume that the conjunction of ϕ and “ $<$ is a dense linear order” has more than \aleph_0 -many non-isomorphic countable models. Does it follow that one of them characterizes κ ?

Open Question 2. *Assume that ϕ is the Scott sentence of a densely ordered structure that characterizes κ . Can we find another densely ordered structure with Scott sentence ψ such that*

- ψ characterizes κ and
- ψ has a model with an increasing sequence of size κ ?

If the answer is positive, what if we require that ψ has a model with cofinality κ ?

Open Question 3. *Similarly as above, if $\mu \leq \kappa$, can we find some ψ such that*

- ψ characterizes κ and
- ψ has a model of size κ and character μ ?

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