Symplectic group actions and covering spaces

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A B S T R A C T

For symplectic group actions which are not Hamiltonian there are two ways to define reduction. Firstly using the cylinder-valued momentum map and secondly lifting the action to any Hamiltonian cover (such as the universal cover), and then performing symplectic reduction in the usual way. We show that provided the action is free and proper, and the Hamiltonian holonomy associated to the action is closed, the natural projection from the latter to the former is a symplectic cover. At the same time we give a classification of all Hamiltonian covers of a given symplectic group action. The main properties of the lifting of a group action to a cover are studied.

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Introduction

There are many instances of symplectic group actions which are not Hamiltonian—i.e., for which there is no momentum map. These can occur both in applications [12] as well as in fundamental studies of symplectic geometry [1,2,4]. In such cases it is possible to define a "cylinder valued momentum map" [3], and then perform symplectic reduction with respect to this map [15,16]. An alternative approach is to lift to the universal cover, where the action is always Hamiltonian, and then to perform ordinary symplectic reduction. The principal purpose of this study is to relate the two procedures. In short, we show that under suitable hypotheses, the reduced space obtained from the universal cover is a symplectic cover of the one obtained from the cylinder valued momentum map.

In more detail, suppose a connected Lie group G acts on a connected manifold M, and let N be a cover of M. Then it may not be possible to lift the action of G, but there is a natural lift to universal covers giving an action of $\widetilde{G}$ on $\widetilde{M}$. This can then be used to define an action of $\widetilde{G}$ on the given cover N. This general construction is well known, but we were unable to find its principal properties in the literature, and consequently in Section 1 we establish the main results about these lifted actions. For example, since N can be written as a quotient of $\widetilde{M}$ by a subgroup of the group of deck transformations, we use this to determine exactly which subgroup of $\widetilde{G}$ acts trivially on N. We show that if the action on M is free and proper, then so is the appropriate lifted action on N. Further details on such lifted actions (including non-free actions) are available as notes [11].

In Section 2 we consider the case where M is a symplectic manifold, and G acts symplectically on M. We consider the covers of M for which the action is Hamiltonian. The "largest" Hamiltonian cover of M is of course its universal cover $\widetilde{M}$; we

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give an explicit expression for its momentum map (Proposition 2.3) and we use it to define a subgroup of the fundamental group of M whose corresponding set of subgroups classifies the Hamiltonian covers (Corollary 2.8). There is also a *"minimal"* such cover, denoted $\widehat{M}$ and which was first introduced in [14], where it is called the *universal covered space* of M; we give here a different interpretation of it as a quotient of the universal cover.

In Section 3, we consider the cylinder valued momentum map of [3] (where it is defined in a different manner, and called the "*moment réduit*"). In Theorem 3.4 we see that reduction can be carried out in two ways. One can either reduce M with respect to the cylinder valued momentum map or, alternatively, one can lift the action to the universal cover $\widetilde{M}$ (or on any other Hamiltonian cover) and then carry out (standard) symplectic reduction on it using its momentum map. The result is that the natural projection of this reduced space (inherited from the covering projection) yields the original reduced space; that is, both reduction schemes are equivalent up to the projection. If the original action is free and proper and its Hamiltonian holonomy is closed then both reduced spaces are symplectic manifolds, and the projection is in fact a symplectic cover. We also identify the deck transformation group of the cover.

We end both Sections 2 and 3 with the general example of a group acting by left translations on its cotangent bundle, with symplectic form equal to the sum of the canonical one and a magnetic term consisting of the pullback to the cotangent bundle of a left-invariant 2-form on the group. In particular we show that symplectic reduction via the cylinder-valued momentum map and Hamiltonian reduction via a standard momentum map yield the same result.

1. Lifting group actions to covering spaces

1.1. The category of covering spaces

We begin by recalling a few facts about covering spaces. Many of the details can be found in any introductory book on Algebraic Topology, for example Hatcher [6]. Let (M, z_0) be a connected manifold with a chosen base point z_0, and let $q_M : (\widetilde{M}, \tilde{z}_0) \to (M, z_0)$ be the universal cover. We realize the universal cover as the set of homotopy classes of paths in M with base point z_0. For definiteness, we take the base point in $\widetilde{M}$ to be the homotopy class $\tilde{z}_0$ of the trivial loop at z_0. Throughout, 'homotopic paths' will mean homotopy with fixed end-points, all paths will be parametrized by $t \in [0, 1]$, and for composition of paths $a * b$ means first do a and then b.

Any cover $p_N : (N, y_0) \to (M, z_0)$ has the same universal cover $(\widetilde{M}, \tilde{z}_0)$ as (M, z_0), and the covering map $q_N : (\widetilde{M}, \tilde{z}_0) \to (N, y_0)$ can be constructed as follows: Let $\tilde{z} \in \widetilde{M}$ and let $z(t)$ be a representative path in M, so $z(0) = z_0$. By the path lifting property of the covering map p_N, $z(t)$ can be lifted uniquely to a path $y(t)$ in (N, y_0). Then $q_N(\tilde{z}) = y(1)$.

Let $\mathfrak{C}$ be the category of all covers of (M, z_0). The morphisms are the covering maps. Since any element $(N, y_0) \in \mathfrak{C}$ also shares $\widetilde{M}$ as universal cover, it sits in a diagram,

$$(\widetilde{M}, \tilde{z}_0) \xrightarrow{q_N} (N, y_0) \xrightarrow{p_N} (M, z_0).$$

Note that the map $\widetilde{M} \to M$ can be written both as q_M and as $p_{\widetilde{M}}$.

It is well known that this category is isomorphic to the category of subgroups of the fundamental group $\pi_1(M, z_0)$ of M, where the morphisms are the inclusion homomorphisms of subgroups. The isomorphism is defined as follows. Let $p_N : (N, y_0) \to (M, z_0)$ be a cover. Then $\Gamma_N := p_{N*}(\pi_1(N, y_0))$ is the required subgroup of $\Gamma := \pi_1(M, z_0)$. Γ_N consists of the homotopy classes of closed paths in (M, z_0) whose lift to (N, y_0) is also closed, and the number of sheets of the cover p_N is equal to the index $\Gamma : \Gamma_N$. Note that since $\widetilde{M}$ is simply connected, $\Gamma_{\widetilde{M}}$ is trivial.

The inverse of this isomorphism can be defined using deck transformations. Let $\Gamma = \pi_1(M, z_0)$. Then Γ is the fibre of q_M over z_0, and it acts on $\widetilde{M}$ by deck transformations defined via the homotopy product: if $\gamma \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ then $\gamma * \tilde{z}$ gives the action of γ on $\tilde{z}$. Then given $\Gamma_1 < \Gamma$, define $N = \widetilde{M}/\Gamma_1$, and put $y_0 = \Gamma_1 \tilde{z}_0$. Then from the long exact sequence of homotopy, it follows that $\pi_1(N, y_0) \simeq \Gamma_1$. Furthermore, if $\Gamma_1 < \Gamma_2 < \Gamma$ then there is a well-defined morphism (covering map) $p : N_1 \to N_2$, where $N_j = \widetilde{M}/\Gamma_j$, obtained from noting that any Γ_1-orbit is contained in a unique Γ_2-orbit, so we put $p(\Gamma_1 \tilde{z}) = \Gamma_2 \tilde{z}$.

Let (N_1, y_1) be a cover of (M, z_0) with group Γ_1, and let $\Gamma_2 = \gamma \Gamma_1 \gamma^{-1}$ be a subgroup conjugate to Γ_1 (where $\gamma \in \Gamma$). Then $N_2 = \widetilde{M}/\Gamma_2$ is diffeomorphic to N_1, but the base point is now $y_2 = \Gamma_2 \tilde{z}_0$. A diffeomorphism is simply induced from the diffeomorphism $\tilde{z} \mapsto \gamma \cdot \tilde{z}$ of $\widetilde{M}$ (which does not in general map y_1 to y_2).

If $\Gamma_1 \lhd \Gamma$ (normal subgroup), then the cover (N, y_1) is said to be a *normal cover*. In this case the Γ-action (by deck transformations) on $\widetilde{M}$ descends to an action on N (with kernel Γ_1), and Γ/Γ_1 is the group of deck transformations of the cover $N \to M$. For a general cover, the group of deck transformations is isomorphic to $N_\Gamma(\Gamma_1)/\Gamma_1$, where $N_\Gamma(\Gamma_1)$ is the normalizer of Γ_1 in Γ. Only for normal covers does the group of deck transformations act transitively on the sheets of the cover. See [6] for examples.

Let us emphasize here that we view $\Gamma = \pi_1(M, z_0)$ both as a group acting on $\widetilde{M}$ by deck transformations, and as a discrete subset of $\widetilde{M}$—the fibre over z_0. In particular, for $\gamma \in \Gamma$, $\gamma * \tilde{z}_0 = \gamma$. In other words, $\tilde{z}_0$ is the identity element in Γ.

1.2. Lifting the group action

Now let G be a connected Lie group acting on the connected manifold M, and let $p_N : (N, y_0) \to (M, z_0)$ be a cover. To define the lifted action on N, we first describe the lift to $\widetilde{M}$ and then show it induces an action on N, using the cover $q_N : \widetilde{M} \to N$.

The action of G on M does not in general lift to an action of G on $\widetilde{M}$ but of the universal cover $\widetilde{G}$, which is also defined using homotopy classes of paths, with base point the identity element e. The covering map is denoted $q_G : \widetilde{G} \to G$. So if $\tilde{g}$ is represented by a path $g(t)$ then $q_G(\tilde{g}) = g(1)$. The product structure in $\widetilde{G}$ is given by pointwise multiplication of paths: if $\tilde{g}_1$ is represented by a path $g_1(t)$ and $\tilde{g}_2$ by $g_2(t)$, then $\tilde{g}_1 \tilde{g}_2$ is represented by the path $t \mapsto g_1(t) g_2(t)$.

Definition 1.1. Let $\tilde{g} \in \widetilde{G}$ be represented by a path $g(t)$ (with $g(0) = e$), and $\tilde{z} \in \widetilde{M}$ be represented by a path $z(t)$ (with $z(0) = z_0$). Then we define $\tilde{g} \cdot \tilde{z}$ to be $\tilde{y} \in \widetilde{M}$, where $\tilde{y}$ is the homotopy class represented by the path $t \mapsto g(t) \cdot z(t)$. It is readily checked that the homotopy class of this path depends only on the homotopy classes $\tilde{g}$ and $\tilde{z}$.

With this definition for the action of $\widetilde{G}$ on $\widetilde{M}$, it is clear that the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{G} \times \widetilde{M} & \longrightarrow & \widetilde{M} \\
\downarrow & & \downarrow \\
G \times M & \longrightarrow & M
\end{array}
\tag{1.1}
$$

where the vertical arrows are $q_G \times q_M$ and q_M respectively, and the horizontal arrows are the group actions. In particular,

$$
\tilde{y} = \tilde{g} \cdot \tilde{z} \quad \Longrightarrow \quad y = g \cdot z
\tag{1.2}
$$

where for $\tilde{z} \in \widetilde{M}$ we denote its projection to M by z, and similarly with elements of $\widetilde{G}$.

Remark 1.2. A second approach to defining the action of $\widetilde{G}$ on $\widetilde{M}$ is as follows. The action of G gives rise to an action of the Lie algebra $\mathfrak{g}$. That is, to each $\xi \in \mathfrak{g}$ there is associated an infinitesimal generator vector field ξ_M on M. Let $N \to M$ be any cover. The covering map is a local diffeomorphism, so the vector fields ξ_M can be lifted to vector fields ξ_N on N. Because this covering map is a local diffeomorphism, this gives rise to an action of $\mathfrak{g}$ on N. Now $\mathfrak{g}$ is the Lie algebra of a unique simply connected Lie group $\widetilde{G}$. To see that the vector fields on N are complete, so defining an action of $\widetilde{G}$, one needs to compare the local actions on M and N. It is not hard to see that the two definitions of actions of $\widetilde{G}$ are equivalent.

Proposition 1.3. *The action of $\widetilde{G}$ on $\widetilde{M}$ commutes with the deck transformations. Furthermore, for each $\tilde{g} \in \pi_1(G, e)$ the homotopy class $g(t) \cdot z_0$ lies in the centre of $\pi_1(M, z_0)$.*

Proof. First note that if $g(t)$ is a path in G with $g(0) = e$, and $z(t)$ a path in M with $z(0) = z_0$ and $z(1) = z_1$, then the following three paths are homotopic:

$$
g(t) \cdot z(t), \qquad [g(t) \cdot z_0] * [g(1) \cdot z(t)], \qquad z(t) * [g(t) \cdot z_1].
\tag{1.3}
$$

Now let $\tilde{g} \in \widetilde{G}$, $\delta \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ with $q_M(\tilde{z}) = y \in M$. We want to show that $\tilde{g} \cdot (\delta \cdot \tilde{z}) = \delta \cdot (\tilde{g} \cdot \tilde{z})$. By (1.3) applied with $\gamma = \delta * \tilde{z}$, we have $\tilde{g} \cdot (\delta \cdot \tilde{z}) = [\delta * \tilde{z}] * [\tilde{g} \cdot y]$, while again by (1.3) applied with $\gamma = \tilde{z}$ we have $\delta \cdot (\tilde{g} \cdot \tilde{z}) = \delta * [\tilde{z} * (\tilde{g} \cdot y)]$. The result follows from the associativity of the homotopy product.

Finally let $\tilde{g} \in \pi_1(G, e)$ and $\delta \in \Gamma$. We want to show that $[\tilde{g} \cdot \tilde{z}_0] * \delta = \delta * [\tilde{g} \cdot \tilde{z}_0]$, where $\tilde{z}_0$ is the constant loop at x. By (1.3), $\delta * [\tilde{g} \cdot \tilde{z}_0] = \tilde{g} \cdot \delta = [\tilde{g} \cdot \tilde{z}_0] * \delta$ (since $g(1) = e$), as required. $\square$

Applying this to the left action of G on itself gives the well-known fact that $\pi_1(G, e)$ lies in the centre of $\widetilde{G}$. Consequently the following is a central extension:

$$
1 \to \pi_1(G, e) \to \widetilde{G} \xrightarrow{q_G} G \to 1.
\tag{1.4}
$$

Now we are in a position to define the action of $\widetilde{G}$ on an arbitrary cover (N, y_0) of (M, z_0). As in Section 1.1, let $\Gamma_N = p_{N*}(\pi_1(N, y_0)) < \Gamma$. So, $N \simeq \widetilde{M}/\Gamma_N$. That is, a point in N can be identified with a Γ_N-orbit of points in $\widetilde{M}$.

Definition 1.4. The $\widetilde{G}$-action on N is defined simply by

$$
\tilde{g} \cdot \Gamma_N \tilde{z} := \Gamma_N (\tilde{g} \cdot \tilde{z}).
$$

This is well-defined as the actions of $\widetilde{G}$ and Γ commute, by Proposition 1.3. It is clear too that the analogues of (1.1) and (1.2) hold with N in place of $\widetilde{M}$.

Proposition 1.5. *Let $p_N : (N, y_0) \to (M, z_0)$ be a covering map. The $\widetilde{G}$-orbits on N are the connected components of the inverse images under p_N of the orbits on M. More precisely, if $y \in p_N^{-1}(z) \subset N$ then $\widetilde{G} \cdot y$ is the connected component of $p_N^{-1}(G \cdot z)$ containing y. In particular if the G-orbits in M are closed, so too are the $\widetilde{G}$-orbits in N.*

Proof. Let $Z \subset M$ be any submanifold. Then $Z' := p_N^{-1}(Z)$ is a submanifold of N and the projection $p_N|_{Z'} : Z' \to Z$ is a cover, and if Z is closed so too is Z'. Moreover, if Z is G-invariant (hence $\widetilde{G}$-invariant), then by the equivariance of p_N so is Z', and if Z is a single orbit, then Z' is a discrete union of orbits: discrete because p_N is a cover. Since $\widetilde{G}$ is connected, the orbits are the connected components of Z'. □

1.3. The kernel of the lifted action

The natural action of $\widetilde{G}$ on $\widetilde{M}$ described above need not be effective, even if the action of G on M is, and the kernel is a subgroup of $\pi_1(G, e)$ which we describe in this section.

Let $\tilde{g} \in \pi_1(G, e)$ be represented by a path $g(t)$, with $g(1) = e$. The path $g(t)$ determines an element $[g(t) \cdot z_0]$ in the centre of $\pi_1(M, z_0)$. Moreover, homotopic loops in G give rise to homotopic loops in M, so this induces a well-defined homomorphism

$$a_{z_0} : \pi_1(G, e) \to \pi_1(M, z_0), \tag{1.5}$$

whose image lies in the centre of $\pi_1(M, z_0)$, by Proposition 1.3.

Proposition 1.6.

(i) *The kernel $K < \pi_1(G, e)$ of a_{z_0} is independent of z_0 and acts trivially on $\widetilde{M}$ and hence on every cover of M.*
(ii) *If (N, y_0) is a cover of (M, z_0), with associated subgroup Γ_N of $\pi_1(M, z_0)$, then $K_N := a_{z_0}^{-1}(\Gamma_N)$ is independent of the choice of base point y_0 in N, and acts trivially on N.*
(iii) *If G acts effectively on M then $G_N := \widetilde{G}/K_N$ acts effectively on N.*

Note that since the domain of a_{z_0} is $\pi_1(G, e)$ which is in the centre of $\widetilde{G}$, it follows that K_N is a normal subgroup of $\widetilde{G}$. And with the notation of the proposition, $K = K_{\widetilde{M}}$ since $\Gamma_{\widetilde{M}}$ is trivial. We will write $G' := \widetilde{G}/K$ for the group acting on $\widetilde{M}$.
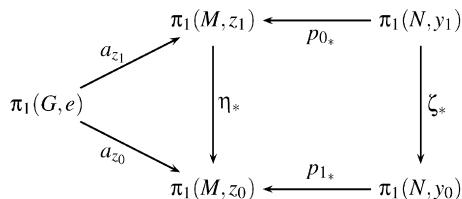
In particular, if a_{z_0} is trivial then $K = \pi_1(G, e)$ and the G-action on M lifts to an action of G on $\widetilde{M}$. That is, a_{z_0} is the obstruction to lifting the G-action. A particular case is where the action of G on M has a fixed point. If z_0 is such a fixed point then $a_{z_0} = 0$. More generally this is true if any (and hence every) G-orbit in M is contractible in M, since in that case too a_{z_0} is trivial. See also Remark 1.8.

Proof. (i) Let $z_0, z_1 \in M$ and let η be any path from z_0 to z_1 (recall we are assuming M is a connected manifold), and let $\tilde{g} \in \pi_1(G, e)$ with a representative path $g(t)$. For $T \in [0, 1]$ define $g^T(t) = g(Tt)$ (for $t \in [0, 1]$), so $g^T \in \widetilde{G}$. Then varying T defines a homotopy from η to $(g^T \cdot \tilde{z}_0) * (g(T)(\eta)) * ((g^T)^{-1} \tilde{z}_0')$. In particular, putting $T = 1$ shows that η is homotopic to $a_{z_0}(\tilde{g}) * \eta * a_{z_1}(\tilde{g}^{-1})$, or equivalently that

$$\eta * a_{z_1}(\tilde{g}^{-1}) * \bar{\eta} = a_{z_0}(\tilde{g}^{-1}),$$

where $\bar{\eta}$ is the reverse of the path η. This composition of paths defines the standard isomorphism $\eta_* : \pi_1(M, z_1) \to \pi_1(M, z_0)$. We have shown therefore that $a_{z_0} = \eta_* \circ a_{z_1}$, and so both have the same kernel. That K acts trivially on $\widetilde{M}$ follows from the definition of a_{z_0}: let $\tilde{z} \in \widetilde{M}$ and $\tilde{g} \in K$, then $\tilde{g} \cdot \tilde{z} = \tilde{g} \cdot (\tilde{z}_0 * \tilde{z}) = a_{z_0}(\tilde{g}) * \tilde{z} = \tilde{z}$ (using (1.3)).

(ii) Let $y_0, y_1 \in N$, let $z_j = p_N(y_j) \in M$ and let ζ be any path from y_0 to y_1, with η its projection to M. The result follows from the fact that the following diagram commutes (with $p_{(N, y_j)_*}$ written p_{j_*}):

$$
\begin{array}{ccc}
 & \pi_1(M, z_1) & \xleftarrow{ p_{0_*} } \pi_1(N, y_1) \\
{\scriptstyle a_{z_1}} \nearrow & \downarrow {\scriptstyle \eta_*} & \downarrow {\scriptstyle \zeta_*} \\
\pi_1(G, e) & & \\
{\scriptstyle a_{z_0}} \searrow & \downarrow & \downarrow \\
 & \pi_1(M, z_0) & \xleftarrow{ p_{1_*} } \pi_1(N, y_0)
\end{array}
$$

Writing $N = \widetilde{M}/\Gamma_N$, if $\tilde{g} \in a_{z_0}^{-1}(\Gamma_N)$ then $\tilde{g} \in K\Gamma_N$ and, $\tilde{g}\Gamma_N \tilde{z} \subset \Gamma_N K\tilde{z} = \Gamma_N \tilde{z}$ so $\tilde{g}$ acts trivially (using Proposition 1.3 and part (i)).

(iii) Suppose $\tilde{g} \in \widetilde{G}$ acts trivially on N, so for all $y \in N$, $\tilde{g} \cdot y = y$. Projecting to M, this implies that $g(1) \cdot z = z$ (for all $z \in M$) so $g(1) \in \cap_{z \in M} G_z = \{e\}$. Thus $\tilde{g} \in \pi_1(G, e)$.

To prove the statement, we first consider the case $N = \widetilde{M}$. If $\tilde{g} \notin K$ then $a_{z_0}(\tilde{g}) \neq \tilde{z}_0 \in \pi_1(M, z_0)$. Since $\pi_1(M, z_0)$ acts effectively (by deck transformations) on the fibre $q_M^{-1}(z_0) \simeq \pi_1(M, z_0) \subset \widetilde{M}$ it follows that $a_{z_0}(\tilde{g})$ acts non-trivially, which is in contradiction with the assumption that $\tilde{g}$ acts trivially.

Now suppose $\tilde{g} \in \widetilde{G}$ acts trivially on N. We have $\tilde{g} \Gamma_N \tilde{z}_0 = \Gamma_N \tilde{z}_0$, so that $\tilde{g} \in \Gamma_N K = a_{z_0}^{-1}(\Gamma_N)$ as required. $\square$

Proposition 1.7. *Let N be any cover of M. If the action of G on M is free and proper then so is the action of G_N on N.*

Proof. First suppose G acts freely on M, and let $y = \Gamma_N \tilde{z} \in p_N^{-1}(z_0) \subset N$. We need to show that the isotropy group $\widetilde{G}_y$ for the $\widetilde{G}$ action on N is equal to K_N. Now, $\tilde{g} \cdot y = \tilde{g} \Gamma_N \tilde{z} = \tilde{g} \Gamma_N \gamma \tilde{z}_0$, for some $\gamma \in \Gamma$, as Γ acts transitively on the fibre over z_0 in $\widetilde{M}$. So $\tilde{g} \cdot y = y$ if and only if, $\tilde{g} \Gamma_N \gamma \tilde{z}_0 = \Gamma_N \gamma \tilde{z}_0$. However, the action of $\tilde{g}$ commutes with that of Γ so this reduces to $a_{z_0}(\tilde{g}) \in \Gamma_N$ as required for the freeness of the G_N-action.

To show the G_N-action is proper, we need to show that the action map $\Phi_N : G_N \times N \to N \times N$ is closed and has compact fibres. The fibre $\Phi_N^{-1}(x, y) = \{(g, y) \in G_N \times N \mid g \cdot x = y\}$. If this is non-empty, and $h \cdot x = y$ then $\Phi_N^{-1}(x, y) \simeq h(G_N)_x$, which is a single element of G_N as the action is free.

To see that the action map is closed, consider a sequence (g_i, x_i) in $G_N \times N$ for which $(g_i \cdot x_i, x_i)$ converges to (y, z). Then of course $x_i \to z$. We claim that $g_i \cdot z \to y$. This is because,

$$d(g_i \cdot z, y) \leqslant d(g_i \cdot z, g_i \cdot x_i) + d(g_i \cdot x_i, y) = d(z, x_i) + d(g_i \cdot x_i, y),$$

where d is the G_N-invariant metric on N defined above. Both terms on the right tend to 0 so that $d(g_i \cdot z, y) \to 0$ as required.

Now, by Proposition 1.5 the G_N-orbits in N are closed and hence there is an $g \in G_N$ with $y = g \cdot z$. That is, $g_i \cdot z \to g \cdot z$. Consequently, $g_i (G_N)_z \to g(G_N)_z$ in $G_N/(G_N)_z$. By taking a slice to the proper $(G_N)_z$-action on G, this can be rewritten as $g_i h_i \to g$ in G_N, for some sequence $h_i \in (G_N)_z$. Since $(G_N)_z$ is compact, (h_i) has a convergent subsequence, $h_{i_k} \to h$. Then $g_{i_k} \to gh^{-1}$. It follows therefore that $(g_{i_k}, x_{i_k}) \to (gh^{-1}, z)$ and $\Phi_N(gh^{-1}, z) = (y, z)$. $\square$

Remark 1.8. D. Gottlieb [5] considered the images in $\pi_1(M, z_0)$ of "cyclic homotopies" of a space, which includes the image of a_{z_0} as a particular case. He showed in particular that image(a_{z_0}) lies in the subgroup $P(M, z_0)$ of $\pi_1(M, z_0)$ consisting of those loops which act trivially on all homotopy groups $\pi_k(M, z_0)$. Furthermore, he showed that if M is homotopic to a compact polyhedron, and the Euler characteristic $\chi(M) \neq 0$, then image$(a_{z_0}) = 0$, which implies by what we proved above that every group action on such a space lifts (as an action of G) to its universal cover.

1.4. Orbit spaces and covers for free actions

It will be useful for Section 3 to compare the orbit spaces M/G and $\widetilde{M}/\widetilde{G}$ (or $\widetilde{M}/G'$ where $G' = \widetilde{G}/K$) when the G-action is free and proper, and more generally with N/G_N when N is a normal cover of M.

Let N be a normal cover of M (see the end of Section 1.1), with associated group Γ_N. Then there is an action of $G_N \times \Gamma$ on N (the action of Γ by deck transformations factors through one of Γ/Γ_N, and commutes with the G_N-action, by Proposition 1.3).

Proposition 1.9. *Let G act freely and properly on M. Then the natural map $q'_M : \widetilde{M}/G' \to M/G$ is a covering map, with deck transformation group equal to* coker(a_{z_0}) *acting transitively on the fibres.*

More generally, if $p_N : N \to M$ is a normal cover then $p'_N : N/G_N \to M/G$ is a normal cover with deck transformation group coker$(a_{z_0})/\Gamma_N \simeq \Gamma/(\text{image}(a_{z_0})\Gamma_N)$.

Proof. Since G acts freely and properly on M then G_N acts freely and properly on N, so both M/G and N/G_N are smooth manifolds. Moreover, since N is a normal cover of M, it follows that $\Delta_N := \Gamma/\Gamma_N$ acts freely and transitively on the fibres of the covering map, and so $M \simeq N/\Delta_N$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\ q_N\ } & N & \xrightarrow{\ p_N\ } & M \\
{\scriptstyle \pi_{\widetilde{M}}}\downarrow & & {\scriptstyle \pi_N}\downarrow & & {\scriptstyle \pi_M}\downarrow \\
\widetilde{M}/G' & \xrightarrow{\ q'_N\ } & N/G_N & \xrightarrow{\ p'_N\ } & M/G
\end{array}
\tag{1.6}
$$

Since the covers q_N and p_N are local diffeomorphisms, it follows that slices to the $\widetilde{G}$-actions can be chosen in $\widetilde{M}$, N and M in a way compatible with the covers. Consequently, the lower horizontal maps in the diagram are also covers (the same is true if the cover N is not normal).

First consider the cover $q'_M : \widetilde{M}/G' \longrightarrow M/G$. Since the action of Γ on $\widetilde{M}$ commutes with the action of G', it descends to an action on $\widetilde{M}/G'$. Moreover, since $\widetilde{M}/\Gamma \simeq M$, so

$$(\widetilde{M}/G')/\Gamma \simeq \widetilde{M}/(G' \times \Gamma) \simeq M/G.$$

(All diffeomorphisms $\simeq$ are natural.) Furthermore, since Γ acts transitively on the fibres of $\widetilde{M} \to M$, so it does on the fibres of $\widetilde{M}/G' \to M/G$.

We claim that the isotropy subgroup of the action of Γ for any point in $\widetilde{M}/G'$ is $\Gamma' = \text{image}(a_{z_0})$. Indeed, for the action of $G' \times \Gamma$ on $\widetilde{M}$ the isotropy subgroup of $\tilde{x}$ is

$$H = \left\{ (\tilde{g}, \gamma) \mid \tilde{g} \cdot \gamma \cdot \tilde{x} = \tilde{x} \right\}.$$

Clearly then, $(\tilde{g}, \gamma) \in H$ implies in particular $\tilde{g} \in \pi_1(G, e)$, and for such $\tilde{g}$, $(\tilde{g}, \gamma) \cdot \tilde{x} = a_{z_0}(\tilde{g}) * \gamma * \tilde{x}$ and so $(\tilde{g}, \gamma) \in H$ iff $a_{z_0}(\tilde{g}) = \gamma^{-1}$. Thus $\gamma \in \Gamma$ acts trivially on $\widetilde{M}/G'$ if and only if $\exists \tilde{g} \in G'$ such that $a_{z_0}(\tilde{g}^{-1}) = \gamma$, as required for the claim. Consequently, for the cover q'_M, the deck transformation group is $\Gamma/\text{image}(a_{z_0}) = \text{coker}(a_{z_0})$, and this acts transitively on the fibres.

The same argument as above can be used for the more general normal cover $p_N : N \to M$, with G' replaced by G_N and Γ by Γ/Γ_N. $\quad\square$

Remark 1.10. If N is a cover of M but not a normal cover, then as pointed out in the proof N/G is still a cover of M/G. Moreover, the fibre still has cardinality $\text{coker}(a_{z_0})/\Gamma_N$, but the latter is not in this case a group.

Notice that as G acts freely and properly on M, then $\widetilde{M}/G'$ is a connected and simply connected manifold (simply connected because G' is connected). Consequently, $\widetilde{M}/G'$ is the (a) universal cover of M/G.

2. Hamiltonian covers

For the remainder of the paper, we assume the manifold M is endowed with a symplectic form ω and the Lie group G acts by symplectomorphisms. Notice that any cover $p_N : N \to M$ of M is also symplectic with form $\omega_N := p_N^* \omega$ and that, moreover, the lifted action of $\widetilde{G}$ (or G_N) on N is also symplectic. It follows that the category of all symplectic covers of (M, ω) coincides with the category of all covers of M. Furthermore, the deck transformations on $\widetilde{M}$ are also symplectic.

Symplectic Lie group actions are linked at a very fundamental level with the existence of *momentum maps*. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. We recall that a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ for the symplectic G-action on (M, ω) is defined by the condition that its components $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$, $\xi \in \mathfrak{g}$, are Hamiltonian functions for the infinitesimal generator vector fields $\xi_M(m) := \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot m$. The existence of a momentum map for the action is by no means guaranteed; however, it could be that the lifted action to a cover has this feature. For example, if the cover is simply connected (as is $\widetilde{M}$), the action necessarily has a momentum map associated. This remark leads us to the following definitions.

Definition 2.1. Let (M, z_0, ω) be a connected symplectic manifold endowed with an action of the connected Lie group G. We say that the smooth cover $p_N : (N, y_0) \to (M, z_0)$ of (M, z_0) is a *Hamiltonian cover* of (M, z_0, ω) if N is connected and the lifted action of $\widetilde{G}$ (or G_N) on (N, ω_N) has a momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ associated.

Note that we keep the base points in the notation as the choice of momentum map depends on the base point.

If the G-action on M is already Hamiltonian, then every cover is naturally a Hamiltonian cover, so the interesting case is where the symplectic action on M is not Hamiltonian.

The connectedness hypothesis on N assumed in the previous definition implies that any two momentum maps of the G_N-action on N differ by a constant element in $\mathfrak{g}^*$. We will assume that $\mathbf{J}_N$ is chosen so that $\mathbf{J}_N(y_0) = 0$. (This choice should perhaps be denoted $\mathbf{J}_{(N,y_0)}$, but we will refrain from the temptation!)

Definition 2.2. Let (M, z_0, ω) be a connected symplectic manifold and G a Lie group acting symplectically thereon. Let $\mathfrak{H}$ be the category whose objects $\text{Ob}(\mathfrak{H})$ are the pairs

$$\big(p_N : (N, y_0, \omega_N) \to (M, z_0, \omega), \mathbf{J}_N \big),$$

where p_N is a Hamiltonian cover of (M, z_0, ω) and $\mathbf{J}_N : N \to \mathfrak{g}^*$ is the momentum map for the lifted $\widetilde{G}$- (or G_N-) action on N satisfying $\mathbf{J}_N(y_0) = 0$, and whose morphisms $\text{Mor}(\mathfrak{H})$ are the smooth maps $p : (N_1, y_1, \omega_1) \to (N_2, y_2, \omega_2)$ that satisfy the following properties:

(i) p is a $\widetilde{G}$-equivariant symplectic covering map

(ii) the following diagram commutes:
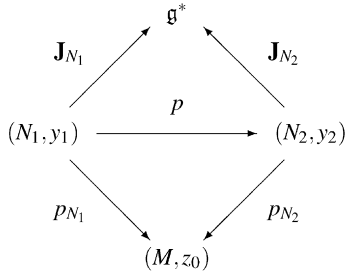
We will refer to $\mathfrak{H}$ as the category of *Hamiltonian covers* of (M, z_0, ω).

It should be clear that the ingredients ω_N and $\mathbf{J}_N$ are both uniquely determined by $p_N : (N, y_0) \to (M, z_0)$ (given the symplectic form on M), so $\mathfrak{H}$ is in fact a (full) subcategory of the category of all covers of (M, z_0).

The category of the Hamiltonian covers of a symplectic manifold acted upon symplectically by a Lie algebra was studied in [14]. We will now use the developments in Section 1 to recover those results in the context of group actions. The study that we carry out in the following paragraphs sheds light on the *universal covered space* introduced in [14] and additionally will be of much use in Section 3 where we will spell out in detail the interplay between Hamiltonian covers and symplectic reduction.

2.1. The momentum map on the universal cover

We now start by giving an expression for the momentum map associated to the $\widetilde{G}$-action on the universal cover $\widetilde{M}$ of M. As far as this momentum map is concerned, it does not matter if we consider the $\widetilde{G}$ or the G' action (defined after Proposition 1.6) since both have the same Lie algebra and the momentum map depends only on the infinitesimal part of the action. Recall that the *Chu map* $\Psi : M \to Z^2(\mathfrak{g})$ is defined by

$$\Psi(z)(\xi, \eta) := \omega(z)\big(\xi_M(z), \eta_M(z)\big). \tag{2.1}$$

for $\xi, \eta \in \mathfrak{g}$.

Proposition 2.3. *Let (M, ω) be a connected symplectic manifold acted upon symplectically by the connected Lie group G. Then, the $\widetilde{G}$-action on $(\widetilde{M}, \widetilde{\omega} := q_M^* \omega)$ has a momentum map associated $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ that can be expressed as follows: realize $\widetilde{M}$ as the set of homotopy classes of paths in M with base point z_0. Let $\tilde{x} \in \widetilde{M}$ and $x(t)$ an element in the homotopy class $\tilde{x}$. Then, for any $\xi \in \mathfrak{g}$*

$$\langle \mathbf{J}(\tilde{x}), \xi \rangle = \int_{[0,1]} x^*(\mathbf{i}_{\xi_M} \omega) = \int_0^1 \omega\big(x(t)\big)\big(\xi_M\big(x(t)\big), \dot{x}(t)\big)\, \mathrm{d}t. \tag{2.2}$$

If $\tilde{x} \in \pi_1(M, z_0)$ and $\tilde{y} \in \widetilde{M}$ then $\tilde{x} * \tilde{y} \in \widetilde{M}$ and

$$\mathbf{J}(\tilde{x} * \tilde{y}) = \mathbf{J}(\tilde{x}) + \mathbf{J}(\tilde{y}). \tag{2.3}$$

The non-equivariance cocycle $\sigma_{\mathbf{J}} : \widetilde{G} \to \mathfrak{g}^*$ of $\mathbf{J}$ is given by

$$\langle \sigma_{\mathbf{J}}(\tilde{g}), \xi \rangle = \int_0^1 \Psi(z_0)(\xi_t, \eta_t)\, \mathrm{d}t, \tag{2.4}$$

for any $\xi \in \mathfrak{g}$, $\tilde{g} \in \widetilde{G}$, and $g(t)$ a curve in the homotopy class of $\tilde{g}$, where $\xi_t = \mathrm{Ad}_{g(t)^{-1}} \xi$ and $\eta_t = (T_e L_{g(t)})^{-1} \dot{g}(t)$, and Ψ is the Chu map defined in (2.1) above.

The non-equivariance cocycle is used to define an affine action of $\widetilde{G}$ on $\mathfrak{g}^*$ with respect to which the momentum map is equivariant, namely

$$\tilde{g} \cdot \mu = \mathrm{Ad}_{g^{-1}}^* \mu + \sigma_{\mathbf{J}}(\tilde{g}). \tag{2.5}$$

Momentum maps are only defined up to a constant; the one in (2.2) is normalized to vanish on the trivial homotopy class $\tilde{z}_0$ at z_0. The expression (2.2) is closely related to the one in [10] for the momentum map of the action of a group G on the fundamental groupoid of a symplectic G-manifold; see Remark 2.5 below.

Proof. Let $\alpha := \mathbf{i}_{\xi_M}\omega$. Since this 1-form on M is closed, it follows that $\int x^*\alpha$ depends only on the homotopy class (indeed homology class) of x; that is, $\mathbf{J}(\tilde{x})$ is well-defined by (2.2).

To show that $\mathbf{J}$ is a momentum map for the $\widetilde{G}$-action on $\widetilde{M}$, we use the Poincaré Lemma on the closed form α. Cover the image of $x(t)$ in M by contractible well-chained open sets (open in M), $U_1, \ldots, U_n$, with $x(0) = z_0 \in U_1$ and $x(1) \in U_n$. We can enumerate these sets consecutively along the curve $x(t)$, and let $z_j = x(t_j) \in U_j \cap U_{j+1}$ lie on the curve and $z_n = x(1)$.

On each U_j we can write $\alpha = \mathrm{d}\phi_j$ for some function ϕ_j (in fact a local momentum for ξ_M). Then on $U_i \cap U_j$, $\mu_{i,j} := \phi_i - \phi_j$ is constant. Now, with $I = [0,1]$ and $I_j = [t_j, t_{j+1}]$ we have

$$\int_I x^*\alpha = \sum_j \int_{I_j} x^*\mathrm{d}\phi_j = \sum_j \big(\phi_j(z_{j+1}) - \phi_j(z_j)\big) = \phi_n(z_n) - \phi_1(z_0) - \sum_{j=1}^{n-1} \mu_{j+1,j}. \tag{2.6}$$

The covering map $q_M : \widetilde{M} \to M, \tilde{x} \mapsto x(1)$ identifies the tangent space $T_{\tilde{x}}\widetilde{M}$ with $T_{x(1)}M$. Let $\tilde{v} \in T_{\tilde{x}}\widetilde{M}$ arbitrary and $v = T_{\tilde{x}}q_M(\tilde{v})$. Thus, differentiating (2.6) at $\tilde{x}$ in the direction $\tilde{v} \in T_{\tilde{x}}\widetilde{M}$ gives

$$\mathrm{d}\Big(\int x^*\alpha\Big)(\tilde{v}) = \mathrm{d}\phi_n\big(x(1)\big)(v) = \alpha\big(x(1)\big)(v) = \omega(\xi_M, v) = \widetilde{\omega}(\xi_{\widetilde{M}}, \tilde{v}),$$

as required. The identity (2.3) follows from a straightforward verification.

We conclude by computing the non-equivariance cocycle $\sigma_{\mathbf{J}}$. By definition, for any $\tilde{g} \in \widetilde{G}$ and $\xi \in \mathfrak{g}$

$$\sigma_{\mathbf{J}}(\tilde{g}) = \mathbf{J}(\tilde{g} \cdot \tilde{x}) - \mathrm{Ad}^*_{\tilde{g}^{-1}}\mathbf{J}(\tilde{x}),$$

for any $\tilde{x} \in \widetilde{M}$. Take $\tilde{x} = \tilde{z}_0$ and use (2.2). The formula for $\sigma_{\mathbf{J}}$ then follows by recalling that $\mathbf{J}(\tilde{z}_0) = 0$ and that the G-action on M is symplectic. $\square$

Remark 2.4. If the Chu map vanishes at one point, then $\mathbf{J}$ is clearly coadjoint-equivariant. This happens if there is an isotropic orbit in M (and hence in $\widetilde{M}$).

Remark 2.5. Let $\Pi(M)$ be the fundamental groupoid of M, which has a natural symplectic structure and Hamiltonian action of G derived from those on M, as described by Mikami and Weinstein, [10]. The relationship between the momentum map $\mathcal{J} : \Pi(M) \to \mathfrak{g}^*$ defined in [10] and ours is as follows (we thank Rui Loja Fernandes for explaining this to us). Given the base point $z_0 \in M$ there is a natural cover $\widetilde{M} \times \widetilde{M} \to \Pi(M)$ (with fibre $\pi_1(M, z_0)$). The momentum map $\mathcal{J}$ lifts to one on $\widetilde{M} \times \widetilde{M}$, and our momentum map is the restriction of this lift to the first factor $\widetilde{M} \times \{\tilde{z}_0\}$.

Conversely, given our momentum map $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$, the map:

$$\widetilde{M} \times \widetilde{M} \to \mathfrak{g}^*, \quad (\tilde{x}, \tilde{y}) \mapsto \mathbf{J}(\tilde{x}) - \mathbf{J}(\tilde{y})$$

descends to the quotient by $\pi_1(M, z_0)$ and yields the momentum map $\mathcal{J} : \Pi(M) \to \mathfrak{g}^*$.

2.2. The Hamiltonian holonomy and Hamiltonian covers

Definition 2.6. Let (M, z_0, ω) be a connected symplectic manifold with symplectic action of the connected Lie group G. Let $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ be the momentum map defined in Proposition 2.3. The *Hamiltonian holonomy* $\mathcal{H}$ of the G-action on (M, ω) is defined as $\mathcal{H} = \mathbf{J}(\Gamma)$, and for an arbitrary symplectic cover $p_N : N \to M$, the holonomy group is $\mathcal{H}_N := \mathbf{J}(\Gamma_N)$, where $\Gamma = \pi_1(M, z_0)$ and $\Gamma_N = (p_N)_*(\pi_1(N, y_0))$ (as in Section 1).

Proposition 2.7. *The symplectic cover $p_N : (N, y_0) \to (M, z_0)$ is Hamiltonian if and only if $\mathcal{H}_N = 0$.*

Proof. If the $\widetilde{G}$-action on N is Hamiltonian, then the momentum map is well-defined. This means that if γ is any closed loop in N, then $\mathbf{J}(\bar{\gamma}) = 0$, where $\bar{\gamma} \in \pi_1(M, z_0)$ is the image under $(p_N)_*$ of the homotopy class of γ. Conversely, if $\mathcal{H}_N = 0$ then the map $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ descends to a map $\mathbf{J}_N : \widetilde{M}/\Gamma_N \to \mathfrak{g}^*$, and as described in Section 1, $N \simeq \widetilde{M}/\Gamma_N$ as covers of M. $\square$

Let us emphasize that if $p_N : (N, y_0) \to (M, z_0)$ is a Hamiltonian cover, then the momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ is defined uniquely by the following diagram.

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ \mathbf{J}\ } & \mathfrak{g}^* \\
{\scriptstyle q_N}\downarrow & & \downarrow{\scriptstyle =} \\
N & \xrightarrow{\ \mathbf{J}_N\ } & \mathfrak{g}^*
\end{array} \tag{2.7}$$

As we pointed out in Section 1, the subgroups of the fundamental group $\Gamma = \pi_1(M, z_0)$ classify the covers of M. In a similar vein, the following result shows that the subgroups of the subgroup Γ_0 of Γ play the same role with respect to the Hamiltonian covers of the symplectic G-manifold (M, ω).

Define,

$$\Gamma_0 := \mathbf{J}^{-1}(0) \cap q_M^{-1}(z_0) < \pi_1(M, z_0); \tag{2.8}$$

that is, $\Gamma_0 = \ker(\mathbf{J}_{|\Gamma} : \Gamma \to \mathfrak{g}^*)$. It follows that $\Gamma_0 \lhd \Gamma$.

Corollary 2.8. *The symplectic cover $p_N : (N, y_0) \to (M, z_0)$ is Hamiltonian if and only if $\Gamma_N < \Gamma_0$. Consequently, $\mathfrak{H}$ is isomorphic to the category of subgroups of Γ_0.*

Recall that the category $\mathfrak{S}(\Gamma)$ of subgroups of a group Γ is the category whose objects are the subgroups, and whose morphisms are the inclusions of one subgroup into another. We have therefore shown that $\mathfrak{H} \simeq \mathfrak{S}(\Gamma_0)$. Explicitly, the isomorphism is given by

$$\mathfrak{H} \longrightarrow \mathfrak{S}(\Gamma_0)$$
$$\left(p_N : (N, y_0) \to (M, z_0), \mathbf{J}_N\right) \longmapsto \Gamma_N = (p_N)_*\left(\pi_1(N, y_0)\right). \tag{2.9}$$

2.3. The universal Hamiltonian covering and covered spaces

As it was shown in the previous section, the Hamiltonian covers of a symplectic G-manifold (M, ω) are characterized by the subgroups of Γ_0. The cover associated to the smallest possible subgroup, that is, the trivial group, is obviously the simply connected universal cover $\widetilde{M}$ of M. It is easy to check that this object satisfies in the category $\mathfrak{H}$ of Hamiltonian covers, the same universality property that it satisfies in the general category of covering spaces, that is, $(p_{\widetilde{M}} : \widetilde{M} \to M, \mathbf{J}) \in \mathrm{Ob}(\mathfrak{H})$ and for any other Hamiltonian cover $(p_N : N \to M, \mathbf{J}_N)$ of (M, ω) there exists a morphism $q_N : (\widetilde{M}, \widetilde{\omega}) \to (N, \omega_N)$ in $\mathrm{Mor}(\mathfrak{H})$. Moreover, any other element in $\mathrm{Ob}(\mathfrak{H})$ that has this universality property is isomorphic to $(p_{\widetilde{M}} : \widetilde{M} \to M, \mathbf{J})$ (we have suppressed the dependence on base points $z_0, y_0, \tilde{z}_0$ in this discussion; if they are included the morphisms become unique—see Remark 2.10 below).

A difference between the general category of covering spaces and the category of Hamiltonian covers arises when we look at the cover associated to the biggest possible subgroup of Γ_0, that is, Γ_0 itself. Unlike the situation found for general covers, where the biggest possible subgroup that one considers is the fundamental group Γ and it is associated to the trivial (identity) cover, the cover associated to Γ_0 is non-trivial (unless M is already Hamiltonian) and has an interesting universality property that is "dual" to the one exhibited by the universal cover. Define $\widehat{M} := \widetilde{M}/\Gamma_0$; it follows from the corollary above that this Hamiltonian cover is *minimal*. It was first introduced under a different guise in [14], where it is called the *universal covered space* of (M, ω), and defined using a holonomy bundle associated to a flat $\mathfrak{g}^*$-valued connection. Recall from Section 1.1 that a cover $N \to M$ is said to be normal if Γ_N is a normal subgroup of Γ. Since Γ_0 is the kernel of a homomorphism $\Gamma \to \mathcal{H}$, it follows that $\widehat{M}$ is a normal cover of M. By Proposition 1.6, the group $\widehat{G} := \widetilde{G}/a_{z_0}^{-1}(\Gamma_0)$ acts effectively on $\widehat{M}$ (as always, we assume that G acts effectively on M).

Proposition 2.9. *$\widehat{M}$ is a Hamiltonian normal cover of M with the universal property that for any given Hamiltonian cover $p_N : N \to M$ of M there is a Hamiltonian cover $\hat{p}_N : N \to \widehat{M}$.*

Proof. Since we have shown that $\mathfrak{H} \simeq \mathfrak{S}(\Gamma_0)$, this property of $\widehat{M}$ in $\mathfrak{H}$ follows from the corresponding property of Γ_0 in $\mathfrak{S}(\Gamma_0)$; namely that for every subgroup Γ_1 of Γ_0 there is an inclusion $\Gamma_1 \hookrightarrow \Gamma_0$. $\quad\square$

Remark 2.10. $(\widetilde{M}, \tilde{z}_0)$ and $(\widehat{M}, \hat{z}_0)$ are initial and final objects in the category of Hamiltonian covers of (M, z_0) with base points; this of course corresponds to the fact that 1 and Γ_0 are initial and final objects in the category $\mathfrak{S}(\Gamma_0)$.

2.4. The connection in $M \times \mathfrak{g}^*$ and a model for the universal covered space

The universal covered space $\widehat{M}$ was introduced in [14] (though there it is denoted $\widetilde{M}$) using a connection in $M \times \mathfrak{g}^*$ proposed in [3]. Here we briefly review that definition, and show that it is equivalent to the one given above.

Let (M, ω) be a connected paracompact symplectic manifold and let G be a connected Lie group that acts symplectically on M. Consider the Cartesian product $M \times \mathfrak{g}^*$ and let $\pi : M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fibre bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*, +)$ as Abelian structure group. The group $(\mathfrak{g}^*, +)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (z, \mu) := (z, \mu - \nu)$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ be the connection one-form defined by

$$\left\langle \alpha(z, \mu)(v_z, \nu), \xi \right\rangle := (\mathbf{i}_{\xi_M} \omega)(z)(v_z) - \langle \nu, \xi \rangle, \tag{2.10}$$

where $(z, \mu) \in M \times \mathfrak{g}^*$, $(v_z, \nu) \in T_z M \times \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$.

The connection α is flat. For $(z_0, 0) \in M \times \mathfrak{g}^*$, let $\widehat{M}' := (M \times \mathfrak{g}^*)(z_0, 0)$ be the holonomy bundle through $(z_0, 0)$ and let $\mathcal{H}(z_0, 0)$ be the holonomy group of α with reference point $(z_0, 0)$ (which is an Abelian zero-dimensional Lie subgroup of $\mathfrak{g}^*$ by the flatness of α); in other words, $\widehat{M}'$ is the maximal integral leaf of the horizontal distribution associated to α that contains the point $(z_0, 0)$ and it is hence endowed with a natural initial submanifold structure with respect to $M \times \mathfrak{g}^*$. See for example Kobayashi and Nomizu [7] for standard definitions and properties of flat connections and holonomy bundles.

The principal bundle $(\widehat{M}', M, \hat{p}, \mathcal{H}) := (\widehat{M}', M, \pi|_{(M \times \mathfrak{g}^*)(z_0, 0)}, \mathcal{H}(z_0, 0))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$. A straightforward verification shows that $\mathcal{H}(z_0, 0)$ coincides with the Hamiltonian holonomy $\mathcal{H}$ introduced in Definition 2.6. In this sense, the momentum map $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ establishes a relationship between the deck transformation groups of the universal cover of M and of the holonomy bundle $\hat{p} : \widehat{M}' \to M$. Moreover, the holonomy bundle $\widehat{M}'$ can be expressed using $\mathbf{J}$ as

$$\widehat{M}' = \left\{ \left(q_M(\tilde{x}), \mathbf{J}(\tilde{x}) \right) \mid \tilde{x} \in \widetilde{M} \right\}. \tag{2.11}$$

This expression allows one to check easily that $(\widehat{M}', M, \hat{p}, \mathcal{H})$ is actually a Hamiltonian cover of M with the symplectic form $\widehat{\omega}' := \hat{p}^* \omega$. The $G_{\widehat{M}'}$-action on $\widehat{M}'$ is symplectic and is induced by the $\widetilde{G}$-action on $\widehat{M}'$ given by

$$\tilde{g} \cdot (x, \mu) = \left(g \cdot x, \mathbf{J}(\tilde{g} \cdot \tilde{x}) \right) = \left(g \cdot x, \sigma_{\mathbf{J}}(\tilde{g}) + \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\tilde{x}) \right), \tag{2.12}$$

where $(x, \mu) \in \widehat{M}'$, $g = p_{\widetilde{G}}(\tilde{g})$, and $\tilde{x}$ is such that $p_{\widetilde{M}}(\tilde{x}) = x$, and $\mathbf{J}(\tilde{x}) = \mu$. The $G_{\widehat{M}'}$-action on $\widehat{M}'$ has a momentum map $\hat{\mathbf{J}} : \widehat{M}' \to \mathfrak{g}^*$ given by $\hat{\mathbf{J}}(x, \mu) = \mu$.

Proposition 2.11. *The universal covered space* $\widehat{M} = \widetilde{M}/\Gamma_0$ *is symplectomorphic to* $\widehat{M}'$.

Proof. The required symplectomorphism is implemented by the map

$$\Theta : \widetilde{M}/\Gamma_0 \longrightarrow \widehat{M}'$$

$$[\tilde{x}] \longmapsto \left(x(1), \mathbf{J}(\tilde{x}) \right).$$

This map is well defined since by (2.3), the smooth map $\theta : \widetilde{M} \longrightarrow \widehat{M}'$ given by $\tilde{x} \longmapsto (x(1), \mathbf{J}(\tilde{x}))$ is Γ_0 invariant and hence it drops to the smooth map Θ. The map θ is an immersion since for any $v_{\tilde{x}} \in T_{\tilde{x}} \widetilde{M}$ such that $0 = T_{\tilde{x}} \theta \cdot v_{\tilde{x}} = (T_{\tilde{x}} p_{\widetilde{M}} \cdot v_{\tilde{x}}, T_{\tilde{x}} \mathbf{J} \cdot v_{\tilde{x}})$, we have that $T_{\tilde{x}} p_{\widetilde{M}} \cdot v_{\tilde{x}} = 0$ and hence $v_{\tilde{x}} = 0$, necessarily. Given that Γ_0 is a discrete group, the projection $\widetilde{M} \to \widetilde{M}/\Gamma_0$ is a local diffeomorphism and hence Θ is also an immersion. Additionally, by (2.11), the map Θ is also surjective. We conclude by showing that Θ is injective. Let $\tilde{x}, \tilde{y} \in \widetilde{M}$ be such that $\Theta([\tilde{x}]) = \Theta([\tilde{y}])$. This implies that

$$x(1) = y(1) \quad \text{and that} \quad \mathbf{J}(\tilde{x}) = \mathbf{J}(\tilde{y}). \tag{2.13}$$

The first equality in (2.13) implies that $\tilde{x} * \bar{\tilde{y}} \in \pi_1(M, z_0)$, where $\bar{\tilde{y}}$ is the homotopy class associated to the reverse path $\bar{y}$ of y. Moreover, by the second equality in (2.13), it is easy to check that $\mathbf{J}(\tilde{x} * \bar{\tilde{y}}) = 0$, and hence $\tilde{x} * \bar{\tilde{y}} \in \Gamma_0$. Since $(\tilde{x} * \bar{\tilde{y}}) * \tilde{y} = \tilde{x}$ we can conclude that $[\tilde{x}] = [\tilde{y}]$, as required. Consequently, Θ being a smooth bijective immersion, it is necessarily a diffeomorphism. A straightforward verification shows that $\Theta \in \mathrm{Mor}(\mathfrak{H})$, which concludes the proof. $\square$

2.5. Example

We apply the ideas developed in this section to the left action of a Lie group G on its cotangent bundle, but with a modified symplectic form.

Let G be a connected Lie group, and let $\theta : \mathfrak{g} \to \mathfrak{g}^*$ be a symplectic cocycle which is not a coboundary, so it represents a non-zero element of $H^1_s(\mathfrak{g}, \mathfrak{g}^*)$ (the subscript meaning *symplectic* cocycles; that is, θ is skew-symmetric—see [17] for details). One can also view θ as a real-valued 2-cocycle $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by putting $\Sigma(\xi, \eta) := \langle \theta(\xi), \eta \rangle$. Indeed, $H^2(\mathfrak{g}, \mathbb{R}) \cong H^1_s(\mathfrak{g}, \mathfrak{g}^*)$.

Let $g(t)$ $(t \in [0, 1])$ be a differentiable path in G and define,

$$\Theta\left(g(\cdot)\right) = \int_0^1 \mathrm{Ad}^*_{g(t)^{-1}} \theta\left(g(t)^{-1} \dot{g}(t)\right) dt. \tag{2.14}$$

It is well known (and easy to check) that Θ depends only on the homotopy class of the path $g(t)$ (relative to the end points), so by restricting to $g(0) = e$, Θ defines a map $\Theta : \widetilde{G} \longrightarrow \mathfrak{g}^*$. Moreover, one can also check that Θ is a 1-cocycle on $\widetilde{G}$, and so defines a well-defined element of $H^1(\widetilde{G}, \mathfrak{g}^*)$.

Let $\Gamma_0 < \pi_1(G, e)$ be the kernel of the restriction of Θ to the subgroup $\pi_1(G, e)$ of $\widetilde{G}$. Then for any subgroup $\Gamma_1 < \Gamma_0$, Θ descends to a 1-cocycle $\Theta_1 \in H^1(G_1, \mathfrak{g}^*)$, where $G_1 = \widetilde{G}/\Gamma_1$. In particular, write $\widehat{G} = \widetilde{G}/\Gamma_0$. (The notation Γ_0 is justified in the corollary below.)

Now consider the action of G on T^*G by lifting left multiplication. Given the 2-cocycle Σ associated to θ, define a closed differential 2-form B_θ on G to be the left-invariant 2-form whose value at e is Σ. Write $\pi : T^*G \to G$, and on $M = T^*G$ consider the symplectic form

$$\Omega_\theta = \Omega_{\text{canon}} - \pi^* B_\theta. \tag{2.15}$$

where Ω_{canon} is the canonical cotangent bundle symplectic form.

We claim that the action of G on M is symplectic, and is Hamiltonian if and only if $\Gamma_0 = \pi_1(G, e)$. More generally, we claim that whenever $\Gamma_1 < \Gamma_0$ the lift of the action to T^*G_1 is Hamiltonian.

Proposition 2.12. *The action of $\widetilde{G}$ on $\widetilde{M} = T^*\widetilde{G} \cong \widetilde{G} \times \mathfrak{g}^*$ with symplectic form given by (2.15) is Hamiltonian, with momentum map given by*

$$\mathbf{J}^\theta(\tilde{g}, \mu) = \mathrm{Ad}^*_{g^{-1}} \mu + \Theta(\tilde{g}),$$

where $g = \tilde{g}(1)$, and we have identified the Lie algebras of G and $\widetilde{G}$. The non-equivariance cocycle of this momentum map is simply Θ.

If $\theta = \delta \nu$ for some $\nu \in \mathfrak{g}^*$ (i.e. θ represents zero in $H^1(\mathfrak{g}, \mathfrak{g}^*)$), then the action on T^*G is Hamiltonian with momentum map $\mathbf{J}(g, \mu) = \mathrm{Ad}^*_{g^{-1}} \mu + \nu$.

Proof. The action is symplectic because B_θ is left-invariant. For the momentum map, the first term of the right-hand side in (2.15) is the standard expression due to Ω_{canon}. For the second term, one needs to check that

$$-\iota_{\xi_{\widetilde{M}}} \pi^* B_\theta = \langle \mathrm{d}\Theta, \xi \rangle.$$

Each side of this is an invariant function, so it suffices to check the equality at the identity element. Now, $\iota_{\xi_{\widetilde{M}}} \pi^* B_\theta = \iota_{\xi_G} B_\theta$ and at the identity this is $\iota_\xi \Sigma$. On the other hand $\langle \mathrm{d}\Theta(e)(\eta), \xi \rangle = \langle \theta(\eta), \xi \rangle = -\Sigma(\xi, \eta)$.

For the non-equivariance cocycle $\sigma \in H^1(G, \mathfrak{g}^*)$,

$$\sigma(h) = \mathbf{J}^\theta\big(h \cdot (e, 0)\big) - \mathrm{Ad}^*_{h^{-1}} \mathbf{J}^\theta(e, 0) = \mathbf{J}^\theta(h, 0) - 0 = \Theta(h). \qquad \square$$

Notice that $\mathbf{J}^\theta(e, 0) = 0$, so this choice of momentum map agrees with the one of Proposition 2.3 if we take $z_0 = (e, 0)$ as base point.

Corollary 2.13. *The group $\Gamma_0 < \pi_1(G, e)$ defined in (2.8) coincides with the group Γ_0 defined above in terms of Θ. Consequently, given any subgroup $\Gamma_1 < \pi_1(G, e)$, the action of G_1 on T^*G_1 is Hamiltonian if and only if $\Gamma_1 < \Gamma_0$.*

Proof. Following the notation of Section 2.2, we can take $z_0 = (e, 0) \in M = T^*G$, and $q_M = q_G \times \mathrm{id}$ on $\widetilde{M} = T^*\widetilde{G} \simeq \widetilde{G} \times \mathfrak{g}^*$. Then $q_M^{-1}(z_0) = \pi_1(G, e) \times \{0\}$ and

$$\Gamma_0 := \big(\mathbf{J}^\theta\big)^{-1}(0) \cap \big(\pi_1(G, e) \times \{0\}\big) = \Theta^{-1}(0) \cap \pi_1(G, e),$$

as required. The rest of the statement follows from Corollary 2.8. $\square$

Notice that with $\widehat{G} = \widetilde{G}/\Gamma_0$, $T^*\widehat{G}$ is the universal covered space for the given symplectic action of G, and it depends on the choice of θ.

Example 2.14. Let $G = \mathbb{T} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be a d-dimensional torus, so $\widetilde{G} = \mathbb{R}^d$ and $\pi_1(G, e) = \mathbb{Z}^d$, and $\mathfrak{g} = \mathbb{R}^d$ can be identified with $\widetilde{G}$. For this case, $H^1_s(\mathfrak{t}, \mathfrak{t}^*)$ is the space of all skew-symmetric linear maps $\mathfrak{t} \to \mathfrak{t}^*$. Let θ be such a map. Then $\Theta : \widetilde{G} \to \mathfrak{t}^*$ can be identified with θ, and the subgroup $\Gamma_0 < \mathbb{Z}^d$ is $\Gamma_0 = \ker(\theta) \cap \mathbb{Z}^d$. In particular, if $\theta : \mathfrak{t} \to \mathfrak{t}^*$ is invertible then $\Gamma_0 = 0$ and the only Hamiltonian cover is the universal cover $\mathbb{R}^d$. The same occurs if $\ker \theta$ is "sufficiently irrational". If, on the other hand, $\ker \theta$ contains some but not all points of the integer lattice, then $\widehat{G}$ will be a cylinder; that is a product $\mathbb{T}^r \times \mathbb{R}^{d-r}$ for some r with $1 \leqslant r \leqslant d - 1$. The Hamiltonian holonomy is $\mathcal{H} = \theta(\mathbb{Z}^d) \subset \mathfrak{t}^*$, which may or may not be closed in $\mathfrak{t}^*$, depending on the "irrationality" of $\ker \theta$. In all cases, the momentum map on the cover $T^*\mathbb{R}^d$ is given by $\mathbf{J}(u, \mu) = \mu + \Theta(u)$.

Example 2.15. Consider the group G that is a central extension of $\mathbb{R}^2$ by S^1 with cocycle $\frac{1}{2}\omega$. That is, as sets $G = S^1 \times \mathbb{R}^2$, with multiplication

$$(\alpha, u)(\beta, v) = \left(\alpha + \beta + \frac{1}{2}\bar{\omega}(u, v), u + v\right), \tag{2.16}$$

where ω is the standard symplectic form on $\mathbb{R}^2$, and $\frac{1}{2}\bar{\omega}(u, v) = \frac{1}{2}\omega(u, v) \bmod 1 \in S^1 = \mathbb{R}/\mathbb{Z}$. The universal cover of G is the Heisenberg group H, with the same multiplication rule but with ω in place of $\bar{\omega}$. We identify $\mathfrak{g}$ with $\mathbb{R} \times \mathbb{R}^2$, and correspondingly $\mathfrak{g}^* \simeq \mathbb{R}^* \times (\mathbb{R}^2)^*$. One finds that

$$H_s^1(\mathfrak{g}, \mathfrak{g}^*) \simeq \left\{ \begin{pmatrix} 0 & \sigma \\ -\sigma^T & 0 \end{pmatrix} \middle| \sigma \in L(\mathbb{R}^2, \mathbb{R}^*) \right\}.$$

Now fix any non-zero such σ and let θ be the corresponding element of $H^1(\mathfrak{g}, \mathfrak{g}^*)$. The integral of θ on H given by (2.14) is,

$$\Theta(\alpha, u) = \begin{pmatrix} \sigma(u) \\ -\alpha\sigma - \frac{1}{2}\sigma(u)\iota_u\omega \end{pmatrix}.$$

Note that Θ does not descend to a function on G. The momentum map on T^*H is given by

$$\mathbf{J}\left((\alpha, u), \begin{pmatrix} \psi \\ \nu \end{pmatrix}\right) = \mathrm{Ad}^*_{(\alpha, u)^{-1}} \begin{pmatrix} \psi \\ \nu \end{pmatrix} + \Theta(\alpha, u) = \begin{pmatrix} \psi + \sigma(u) \\ \nu - \alpha\sigma - (\psi + \frac{1}{2}\sigma(u))\iota_u\omega \end{pmatrix}.$$

The Hamiltonian holonomy is therefore

$$\mathcal{H} = \mathbf{J}(\mathbb{Z}, 0) = \begin{pmatrix} 0 \\ \mathbb{Z}\sigma \end{pmatrix},$$

which is closed. The cylinder-valued momentum map on T^*G takes values in $C = \mathfrak{g}^*/\mathcal{H} \simeq \mathbb{R} \times \mathbb{R} \times S^1$.

We continue these examples at the end of the next section, where we consider symplectic reduction for such actions.

3. Symplectic reduction and Hamiltonian covers

Symplectic reduction is a well studied process that prescribes how to construct symplectic quotients out of the orbit spaces associated to the symplectic symmetries of a given symplectic manifold. Even though it is known how to carry this out for fully general symplectic actions [15], the implementation of this procedure is particularly convenient in the presence of a standard momentum map, that is, when the Hamiltonian holonomy is trivial (this is the so-called symplectic or Meyer–Marsden–Weinstein reduction [8,9]). Unlike the situation encountered in the general case with a non-trivial Hamiltonian holonomy, the existence of a standard momentum map implies the existence of a unique canonical symplectic reduced space. In the light of this remark the notion of Hamiltonian cover appears as an interesting and useful object for reduction. More specifically, one may ask whether, given a symplectic action on a symplectic manifold with non-trivial holonomy and with respect to which we want to reduce, we could lift the action to a Hamiltonian cover, perform reduction there with respect to a standard momentum map, and then project down the resulting space. How would this compare with the potentially complicated reduction in the original manifold? The main result in this section shows that indeed both processes yield essentially the same result. Furthermore, we show that this projection down is a cover.

3.1. The cylinder valued momentum map

Recall the definition of the holonomy of a symplectic action of G on M given in Definition 2.6: namely, $\mathcal{H} = \mathbf{J}(\Gamma)$, where as always, $\Gamma = \pi_1(M, z_0)$. Using this definition, Eq. (2.3) can be expressed by saying that $\mathbf{J}$ is equivariant with respect to Γ acting as deck transformations on $\widetilde{M}$ and as translations by elements of $\mathcal{H}$ on $\mathfrak{g}^*$. It follows that $\mathbf{J}$ descends to another map with values in $\mathfrak{g}^*/\mathcal{H}$. However, in general this is a difficult object to use as $\mathcal{H}$ is not necessarily a *closed* subgroup of $\mathfrak{g}^*$. To circumvent this, we proceed as follows.

Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\overline{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\overline{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$). Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be the projection. Define $\mathbf{K} : M \to C$ to be the map that makes the following diagram commutative:

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \\
q_M \downarrow & & \downarrow \pi_C \\
M & \xrightarrow{\mathbf{K}} & C = \mathfrak{g}^*/\overline{\mathcal{H}}
\end{array}
\tag{3.1}
$$

In other words, $\mathbf{K}$ is defined by $\mathbf{K}(z) = \pi_C(\mathbf{J}(\tilde{z}))$, where $\tilde{z} \in \widetilde{M}$ is any path with endpoint z. We will refer to $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ as a *cylinder valued momentum map* associated to the symplectic G-action on (M, ω). This object was introduced in [3] using the connection described in Section 2.4, where it is called the "*moment réduit*".

Any other choice of Hamiltonian cover in place of $\widetilde{M}$ would render the same Hamiltonian holonomy group $\mathcal{H}$ and the same cylinder valued momentum map. If one chose a different base point $z_1 \in M$ in place of z_0 the holonomy group would remain the same, but the cylinder valued momentum map would differ from $\mathbf{K}$ by a constant in $\mathfrak{g}^*/\overline{\mathcal{H}}$.

Elementary properties

The cylinder valued momentum map is a strict generalization of the standard (Kostant–Souriau) momentum map since the G-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map is a standard momentum map. The cylinder valued momentum map satisfies Noether's Theorem; that is, for any G-invariant function $h \in C^\infty(M)^G$, the flow F_t of its associated Hamiltonian vector field X_h satisfies the identity $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$. Additionally, using diagram (3.1) and identifying $T_z M$ and $T_{\tilde{z}} \widetilde{M}$ via $T_{\tilde{z}} q_M$, one has that for any $v_z \in T_z M$, $T_z \mathbf{K}(v_z) = T_\mu \pi_C(\nu)$, where $\mu = \mathbf{J}(\tilde{z}) \in \mathfrak{g}^*$ and $\nu = T_{\tilde{z}} \mathbf{J}(v_z) \in \mathfrak{g}^*$.

Consequently, $T_z \mathbf{K}(v_z) = 0$ is equivalent to $T_{\tilde{z}} \mathbf{J}(v_z) \in \mathrm{Lie}(\overline{\mathcal{H}}) \subset \overline{\mathcal{H}}$, or equivalently $\mathbf{i}_{v_z} \omega \in \mathrm{Lie}(\overline{\mathcal{H}})$, so that

$$\ker T_z \mathbf{K} = \left[\left(\mathrm{Lie}(\overline{\mathcal{H}}) \right)^\circ \cdot z \right]^\omega.$$

Here $\mathrm{Lie}(\overline{\mathcal{H}}) \subset \mathfrak{g}^*$ is the Lie algebra of $\overline{\mathcal{H}}$, and $\mathrm{Lie}(\overline{\mathcal{H}})^\circ$ its annihilator in $\mathfrak{g}$, and the upper index ω denotes the ω-orthogonal complement of the set in question. The notation $\mathfrak{k} \cdot m$ for any subspace $\mathfrak{k} \subset \mathfrak{g}$ has the usual meaning: namely the vector subspace of $T_z M$ formed by evaluating all infinitesimal generators η_M at the point $z \in M$ for all $\eta \in \mathfrak{k}$. Furthermore, $\mathrm{range}(T_z \mathbf{K}) = T_\mu \pi_C((\mathfrak{g}_z)^\circ)$ (the Bifurcation Lemma).

Equivariance properties of the cylinder valued momentum map

There is a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map is G-equivariant. This action is constructed by noticing first that since G is connected it follows (see [15]) that the Hamiltonian holonomy $\mathcal{H}$ is pointwise fixed by the coadjoint action, that is, $\mathrm{Ad}^*_{g^{-1}} h = h$, for any $g \in G$ and any $h \in \mathcal{H}$. Hence, the coadjoint action on $\mathfrak{g}^*$ descends to a well defined action $\mathcal{A}d^*$ on $\mathfrak{g}^*/\overline{\mathcal{H}}$ defined so that for any $g \in G$, $\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$. With this in mind, we define $\bar{\sigma}_{\mathbf{K}} : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ by

$$\bar{\sigma}_{\mathbf{K}}(g, z) := \mathbf{K}(g \cdot z) - \mathcal{A}d^*_{g^{-1}} \mathbf{K}(z).$$

Since M is connected by hypothesis, it can be shown that $\bar{\sigma}_{\mathbf{K}}$ does not depend on the point $z \in M$ and hence it defines a map $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ which is a group valued one-cocycle: for any $g, h \in G$, it satisfies the equality $\sigma_{\mathbf{K}}(gh) = \sigma_{\mathbf{K}}(g) + \mathcal{A}d^*_{g^{-1}} \sigma_{\mathbf{K}}(h)$. This guarantees that the map

$$\Phi : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \longrightarrow \mathfrak{g}^*/\overline{\mathcal{H}}$$
$$\left(g, \pi_C(\mu) \right) \longmapsto \mathcal{A}d^*_{g^{-1}} \left(\pi_C(\mu) \right) + \sigma_{\mathbf{K}}(g),$$

defines a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map $\mathbf{K}$ is G-equivariant; that is, for any $g \in G$, $z \in M$, we have

$$\mathbf{K}(g \cdot z) = \Phi\left(g, \mathbf{K}(z) \right).$$

We will refer to $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ as the *non-equivariance one-cocycle* of the cylinder valued momentum map $\mathbf{K} : M \to \underline{\mathfrak{g}}^*/\overline{\mathcal{H}}$ and to Φ as the *affine G-action* on $\mathfrak{g}^*/\overline{\mathcal{H}}$ induced by $\sigma_{\mathbf{K}}$. The infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ are given by the expression

$$\xi_{\mathfrak{g}^*/\overline{\mathcal{H}}}\left(\pi_C(\mu) \right) = -T_\mu \pi_C\left(\Psi(z)(\xi, \cdot) \right), \tag{3.2}$$

for any $\xi \in \mathfrak{g}$, where $\mathbf{K}(z) = \pi_C(\mu)$, and $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map defined in (2.1).

The non-equivariance cocycles $\sigma_{\mathbf{J}} : \widetilde{G} \to \mathfrak{g}^*$ and $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ are related by

$$\pi_C \circ \sigma_{\mathbf{J}} = \sigma_{\mathbf{K}} \circ q_G. \tag{3.3}$$

Proposition 3.1. *If the action of G has an isotropic orbit then the cylinder valued momentum map for this action can be chosen coadjoint equivariant.*

Proof. This follows from Remark 2.4. Let $z_0 \in M$ be a point in the isotropic orbit and construct a universal cover $\widetilde{M}$ of M by taking homotopies of curves with a fixed endpoint starting at z_0. Let $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ be the momentum map for the $\widetilde{G}$-action on $\widetilde{M}$ introduced in Proposition 2.3. Since the G-orbit containing z_0 is isotropic, the integrand in (2.4) is identically zero and hence $\sigma_{\mathbf{J}} = 0$ (see Remark 2.4). Therefore by (3.3) the non-equivariance cocycle $\sigma_{\mathbf{K}}$ satisfies $\sigma_{\mathbf{K}} \circ q_G = \pi_C \circ \sigma_{\mathbf{J}} = 0$. $\quad\square$

Remark 3.2. For any Hamiltonian cover $p_N : N \to M$ of (M, ω) there exists a momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ for the $\widetilde{G}$ (and also G_N) action on N such that $\mathbf{J}_N \circ q_N = \mathbf{J}$ and $\sigma_{\mathbf{J}_N} = \sigma_{\mathbf{J}}$, where $q_N : \widetilde{M} \to N$ is the $\widetilde{G}$-equivariant cover such that $p_N \circ q_N = q_M$. Consequently, there is a commutative diagram analogous to (3.1) with N and $\mathbf{J}_N$ in place of $\widetilde{M}$ and $\mathbf{J}$.

3.2. Reductions

The following result establishes a crucial relationship between the deck transformation group of $q_M : \widetilde{M} \to M$, that is, $\Gamma := \pi_1(M, z_0)$, and the deck transformation group of $\hat{p} : \widehat{M} \to M$, that is $\mathcal{H} \simeq \Gamma/\Gamma_0$.

Proposition 3.3. *Let G be a connected Lie group acting symplectically on the symplectic manifold (M, ω) with Hamiltonian holonomy $\mathcal{H}$ and let* $\mathbf{J} : \widetilde{M} \to M$ *be the momentum map for the lifted action on $(\widetilde{M}, \tilde{z}_0)$ defined in Proposition 2.3. Then, for any $\mu \in \mathfrak{g}^*$*

$$q_M^{-1}\big(q_M\big(\mathbf{J}^{-1}(\mu)\big)\big) = \mathbf{J}^{-1}(\mu + \mathcal{H}). \tag{3.4}$$

More generally, for any Hamiltonian cover $p_N : (N, y_0) \to (M, z_0)$ of (M, z_0, ω), let $\mathbf{J}_N : N \to \mathfrak{g}^*$ be the momentum map discussed in Remark 3.2. Then, for any $\mu \in \mathfrak{g}^*$

$$p_N^{-1}\big(p_N\big(\mathbf{J}_N^{-1}(\mu)\big)\big) = \mathbf{J}_N^{-1}(\mu + \mathcal{H}). \tag{3.5}$$

Proof. Since Γ acts transitively on the fibres of q_M, (3.4) is equivalent to

$$\mathbf{J}^{-1}(\mu + \mathcal{H}) = \Gamma \cdot \mathbf{J}^{-1}(\mu).$$

By Proposition 2.3, if $\mathbf{J}(\tilde{z}) = \mu$ and $\gamma \in \Gamma$ then $\mathbf{J}(\gamma \cdot \tilde{z}) = \mu + \nu$ for some $\nu \in \mathcal{H}$; that is, $\gamma \cdot \tilde{z} \in \mathbf{J}^{-1}(\mu + \mathcal{H})$. Conversely, given $\nu \in \mathcal{H}$ there is a $\gamma \in \Gamma$ for which $\mathbf{J}(\gamma \cdot \tilde{z}) = \mu + \nu$ so proving the statement.

In order to prove (3.5) let $q_N : \widetilde{M} \to N$ be the $\widetilde{G}$-equivariant cover such that $p_N \circ q_N = q_M$. This equality and the surjectivity of q_N imply that for any set $A \subset N$, $p_N(A) = q_M(q_N^{-1}(A))$. Now, the relations $\mathbf{J}_N \circ q_N = \mathbf{J}$ and (3.4) imply that $q_M(q_N^{-1}(\mathbf{J}_N^{-1}(\mu + \mathcal{H}))) = q_M(q_N^{-1}(\mathbf{J}_N^{-1}(\mu)))$ and hence $p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H})) = p_N(\mathbf{J}_N^{-1}(\mu))$, as required. ☐

The main result of this section shows that when the Hamiltonian holonomy is closed reduction behaves well with respect to the lifting of the action to any Hamiltonian cover. More explicitly, we show that in order to carry out reduction one can either stay in the original manifold and use the cylinder valued momentum map or one can lift the action to a Hamiltonian cover, perform ordinary symplectic (Marsden–Weinstein) reduction there and then project the resulting quotient. The two strategies yield closely related results. Notice that if the Hamiltonian holonomy of the action $\mathcal{H}$ is not closed in $\mathfrak{g}^*$, the reduced spaces obtained via the cylinder valued momentum map are in general not symplectic but Poisson manifolds [15].

For the remainder of this section we assume the Hamiltonian holonomy $\mathcal{H}$ to be a closed subset of $\mathfrak{g}^*$, and we write $\tilde{g} \cdot \mu$ for the *modified* coadjoint action of G' or $\widetilde{G}$ on $\mathfrak{g}^*$, and similarly $g \cdot [\mu]$ for the inherited action on $\mathfrak{g}^*/\mathcal{H}$. We also write $\Gamma' := \text{image}(a_{z_0})$, where a_{z_0} is defined in (1.5).

Let N be any Hamiltonian cover of M, and consider the diagram for N analogous to (3.1); of course particular cases of interest are $N = \widetilde{M}$ and $N = \widehat{M}$. As $\mathcal{H}$ is closed, the image of $\mathbf{J}_N^{-1}(\mu + \mathcal{H})$ under p_N is precisely $\mathbf{K}^{-1}([\mu])$, by the definition of $\mathbf{K}$. Reduction of each defines a map

$$(p_N)_\mu : N_\mu \longrightarrow M_{[\mu]}.$$

In the case that $N = \widetilde{M}$, we denote the projection by $(q_M)_\mu : \widetilde{M}_\mu \to M_{[\mu]}$.

For each $\mu \in \mathfrak{g}^*$ define

$$\Gamma_\mu = \Gamma \cap \mathbf{J}^{-1}\big(\sigma_\mu(\widetilde{G})\big)$$

where $\sigma_\mu : \widetilde{G} \to \mathfrak{g}^*$ is the 1-cocycle $\sigma_\mu = \sigma_{\mathbf{J}} + \delta\mu$ and $\delta\mu(\tilde{g}) = \delta\mu(g) = \text{Ad}^*_{g^{-1}}\mu - \mu$ is the coboundary associated to μ. Note that for all $\mu \in \mathfrak{g}^*$, $\Gamma' < \Gamma_\mu$. Indeed, given $\tilde{g} \in \pi_1(G, e)$, $\mathbf{J}(\tilde{g} \cdot \tilde{z}_0) = \sigma(\tilde{g}) = \sigma_\mu(\tilde{g})$ as required; the last equality holds because for $\tilde{g} \in \pi_1(G, e)$, $\delta\mu(\tilde{g}) = 0$.

Furthermore, we have that $\Gamma_\mu \supset \Gamma_0 = \mathbf{J}^{-1}(0) \cap \Gamma$. Since both Γ' and Γ_0 are normal subgroups of Γ (and hence of Γ_μ), with Γ' being in the centre, it follows that, for all $\mu \in \mathfrak{g}^*$, the product

$$\Gamma'\Gamma_0 \lhd \Gamma_\mu. \tag{3.6}$$

Theorem 3.4. *Suppose the action of G on (M, ω) is free and proper, and the holonomy group $\mathcal{H}$ is closed. Then the map $(q_M)_\mu : \widetilde{M}_\mu \to M_{[\mu]}$ is a cover, with transitive deck transformation group isomorphic to*

$$\Gamma_{\mu,\text{red}} := \Gamma_\mu/\Gamma'.$$

More generally, if N is a normal Hamiltonian cover of M then $(p_N)_\mu$ is a normal cover, with the deck transformation group

$$\Gamma_\mu/(\Gamma_N \Gamma').$$

Proof. We approach this from the point of view of orbit reduction; that is we consider

$$M_{[\mu]} = \mathbf{K}^{-1}\big(G \cdot [\mu]\big)/G \subset M/G, \quad \text{and} \quad \widetilde{M}_\mu = \mathbf{J}^{-1}(\widetilde{G} \cdot \mu)/\widetilde{G} \subset \widetilde{M}/\widetilde{G}.$$

In both cases, the G or $\widetilde{G}$ actions are the coadjoint action modified by the cocycle $\sigma_\mathbf{K}$ and $\sigma_\mathbf{J}$, respectively. It is well-known that for proper actions, point and orbit reductions are equivalent (for a proof, see Theorem 6.4.1 of [13]), and the equivalence respects the projections induced by $\widetilde{M} \to M$.

Consider then the following commutative diagrams:

$$\begin{array}{ccc}
\mathbf{J}^{-1}(\widetilde{G} \cdot \mu) & \xrightarrow{\pi_{\widetilde{M}}} & \widetilde{M}_\mu \\
{\scriptstyle q_M}\downarrow & & \downarrow{\scriptstyle q'_M} \\
\mathbf{K}^{-1}(G \cdot [\mu]) & \xrightarrow{\pi_M} & M_{[\mu]}
\end{array}
\qquad \subset \qquad
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\pi_{\widetilde{M}}} & \widetilde{M}/G' \\
{\scriptstyle q_M}\downarrow & & \downarrow{\scriptstyle q'_M} \\
M & \xrightarrow{\pi_M} & M/G
\end{array}
\tag{3.7}$$

The maps in the left-hand diagram are just restrictions of those in the right-hand one.

First we claim that $q_M : \mathbf{J}^{-1}(\widetilde{G} \cdot \mu) \to \mathbf{K}^{-1}(G \cdot [\mu])$ is a cover whose group of covering transformations is Γ_μ defined above. The result then follows from Proposition 1.9, but with Γ replaced by Γ_μ, since $\Gamma' < \Gamma_\mu$.

To prove the claim, we know from Proposition 3.3 that $q_M^{-1}(\mathbf{K}^{-1}([\mu])) = \mathbf{J}^{-1}(\mu + \mathcal{H})$. Saturating by $\widetilde{G}$, we have

$$q_M^{-1}\big(\mathbf{K}^{-1}(G \cdot [\mu])\big) = \mathbf{J}^{-1}\big(\widetilde{G} \cdot (\mu + \mathcal{H})\big),$$

and this is a cover with group Γ (that of the cover $\widetilde{M} \to M$).

Now let $z \in M$ be such that $\mathbf{K}(z) = [\mu]$ (so in particular $z \in \mathbf{K}^{-1}(G \cdot [\mu])$), and let $Z = q_m^{-1}(z)$ be the fibre over z. If $\tilde{z} \in Z$ then $Z = \Gamma \cdot \tilde{z}$, and $\mathbf{J}(\Gamma \cdot \tilde{z}) = \mu + \mathcal{H}$, so we choose $\tilde{z} \in Z$ such that $\mathbf{J}(\tilde{z}) = \mu$.

We now show that $Z \cap \mathbf{J}^{-1}(\widetilde{G} \cdot \mu) = \Gamma_\mu \cdot \tilde{z}$. To this end, let $\tilde{z}_1 \in Z$. Then $\exists \gamma \in \Gamma$ such that $\tilde{z}_1 = \gamma \cdot \tilde{z}$, so

$$\mathbf{J}(\tilde{z}_1) = \mathbf{J}(\tilde{z}) + \mathbf{J}(\gamma) = \mu + \mathbf{J}(\gamma).$$

Then $\mu + \mathbf{J}(\gamma) \in \widetilde{G} \cdot \mu$ if and only if $\exists \tilde{g} \in \widetilde{G}$ such that

$$\mu + \mathbf{J}(\gamma) = \tilde{g} \cdot \mu = \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(\tilde{g}),$$

so that $\mathbf{J}(\gamma) = \delta\mu(\tilde{g}) + \sigma(\tilde{g}) = \sigma_\mu(\tilde{g})$; that is, $\gamma \in \Gamma_\mu$, as required.

The proof of the second part of the theorem, with a general normal cover N, is identical, given that $N = \widetilde{M}/\Gamma_N$. $\quad\square$

Corollary 3.5. *The cover $\widehat{M}_\mu \to M_{[\mu]}$ has cover transformation group $\Gamma_\mu/\Gamma_0 \Gamma'$. This is trivial if $\mathbf{J}(\Gamma') = \mathcal{H} \cap \sigma_\mathbf{J}(\widetilde{G})$, in which case the cover is a symplectomorphism.*

Remark 3.6. If the Hamiltonian holonomy is not closed but the action is still free and proper, the reduced spaces $M_{[\mu]}$ and $\widetilde{M}_\mu$ are Poisson manifolds [15], and the natural map $p_\mu : \widetilde{M}_\mu \to M_{[\mu]}$ is a surjective Poisson submersion.

3.3. Example

We continue the example of G acting on T^*G with symplectic form modified by a cocycle θ, as discussed in Section 2.5. In this case, $\Gamma = \pi_1(G, e)$ and $a_{z_0} : \pi_1(G, e) \to \Gamma$ is the identity, so $\Gamma' = \Gamma$ and it follows that $\Gamma_\mu = \Gamma$ for all $\mu \in \mathfrak{g}^*$.

Write $M = T^*G$ and $\widetilde{M} = T^*\widetilde{G}$ and assume that the Hamiltonian holonomy $\mathcal{H} = \Theta(\Gamma) \subset \mathfrak{g}^*$ is closed. It follows from Theorem 3.4 that the projection $\widetilde{M}_\mu \to M_{[\mu]}$ is a cover with trivial (and transitive) deck transformation group, so is in fact a symplectomorphism. Indeed the same is true for any intermediate cover G_1 for which the action on T^*G_1 is Hamiltonian. In particular, we find that for the left action of G on T^*G with modified symplectic form, Hamiltonian reduction for a Hamiltonian lift and symplectic reduction via the cylinder valued momentum map yield the same result.

The well-known statement that the symplectic reduced spaces for the canonical left action of G on T^*G coincide with the coadjoint orbits [8] remains true when both the symplectic structure and the action on $\mathfrak{g}^*$ are modified by a cocycle Θ (see for example [13]). The statement above shows that this remains true for cylinder valued momentum maps, where the orbits are those of $\widetilde{G}$ in $\mathfrak{g}^*$ rather than those of G in C.

Example 3.7. Returning to Example 2.14 on the torus, given $\theta \in H^1_s(\mathfrak{t}, \mathfrak{t}^*)$ the orbits of the modified coadjoint action of $\mathbb{R}^d$ are the affine subspaces parallel to image$(\theta) \subset \mathfrak{t}^*$, and so the reduced spaces for this action are symplectomorphic to these affine subspaces. If θ is chosen so that the holonomy is closed (e.g., d is even and θ is invertible) then the same is true of the reduced spaces for the action of $\mathbb{T}^d$ on $T^*\mathbb{T}^d$ via the cylinder valued momentum map.

Example 3.8. Returning now to Example 2.15, the symplectic reduced spaces for the Heisenberg group with the symplectic structure $\Omega_{\mathrm{canon}} + \pi^* B_\Sigma$ on $T^* H$ are the orbits for the modified coadjoint action. Calculations show these to be the level sets of the Casimir function $f(\psi, \nu) = \frac{1}{2}\psi^2 - \omega^{-1}(\sigma, \nu)$, which are parabolic cylinders. Since the Hamiltonian holonomy $\mathcal{H}$ is closed, it follows from the results above that the same is true for reduction via the cylinder valued momentum map on $T^* G$.

Acknowledgements

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