JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 61, 113–115 (1977)

Hurwitz' Theorem Implies Rouché's Theorem[1]

Alexander Abian

Department of Mathematics, Iowa State University, Ames, Iowa 50011

Submitted by Masanao Aoki

It is well known that Hurwitz's theorem is easily proved from Rouché's theorem. We show that conversely, Rouché's theorem is readily proved from Hurwitz' theorem. Since Hurwitz' theorem is easily proved from the formula giving the number of roots of an analytic function, our result thus gives also a simple proof of Rouché's theorem.

We recall that Hurwitz' theorem runs as follows. *If a sequence* $(f_n(z))_{n\in\omega}$ *of functions* $f_n(z)$ *continuous on a compact set F and analytic in the interior of F, is uniformly convergent on F, and if the function* $f(z) = \lim_n f_n(z)$ *vanishes nowhere on the boundary B of F, then, beginning from a certain value of n, all the functions* $f_n(z)$ *have in the interior of F the same number of roots (counting each root as many times as its multiplicity indicates) as the function* $f(z)$.

Based on the above, we prove:

THEOREM (Rouché). *If* $g(z)$ *and* $f(z)$ *are functions continuous on a compact set F and analytic in the interior of F, and if* $|g(z)| < |f(z)|$ *on the boundary B of F, then the function* $g(z) + f(z)$ *has in the interior of F the same number of roots (counting each root as many times as its multiplicity indicates) as the function* $f(z)$.

Proof. Let us consider the sequence $(f_{0,n}(z))_{n\in\omega}$ of functions $f_{0,n}(z)$, where

$$f_{0,n}(z) = f(z) + g(z) \cdot (n/(1 + n)).$$

Since $|g(z)| < |f(z)|$ on B, we see that $\lim_n f_{0,n}(z) = g(z) + f(z)$ vanishes nowhere on B. Moreover, in view of the remaining hypotheses of the theorem, the conclusion of Hurwitz' theorem applies and hence, for some $n_0 > 0$ it is the case that the function $f_0(z)$ given by

$$f_0(z) = f(z) + g(z) \cdot (n_0/(1 + n_0)) \tag{1}$$

has in the interior of F the same number of roots as the function $g(z) + f(z)$. Clearly, $f_0(z)$ vanishes nowhere on B.

Next, let us consider the sequence $(f_{1,n}(z))_{n\in\omega}$ of functions $f_{1,n}(z)$, where

$$f_{1,n}(z) = f(z) + g(z) \cdot (n_0/(1 + n_0)) (n/(1 + n)).$$

[1] This work was partly supported by Iowa State University SHRI.

ISSN 0022-247X

Again, we see that $\lim_n f_{1,n}(z) = f_0(z)$ vanishes nowhere on B, and again, in view of the hypotheses of the theorem, the conclusion of Hurwitz' theorem applies. Hence, for some $n_1 > 0$ we see that the function $f_1(z)$ given by

$$f_1(z) = f(z) + g(z) \cdot (n_0/(1 + n_0))\,(n_1/(1 + n_1)) \tag{2}$$

has in the interior of F the same number of roots as the function $f_0(z)$ and therefore as well as the function $g(z) + f(z)$. Clearly, $f_1(z)$ vanishes nowhere on B.

Continuing in the above manner, we obtain a sequence of functions $f_0(z)$, $f_1(z), f_2(z),\ldots$, each having in the interior of F the same number of roots as the function $g(z) + f(z)$. Also, it is clear that these functions form a sequence $(f_n(z))_{n\in\omega}$ of functions $f_n(z)$ continuous on the compact set F and analytic in the interior of F.

Based on (1) and (2), we consider the infinite product

$$\left(\frac{n_0}{1 + n_0}\right)\left(\frac{n_1}{1 + n_1}\right)\left(\frac{n_2}{1 + n_2}\right)\cdots\left(\frac{n_k}{1 + n_k}\right)\cdots \tag{3}$$

of positive real numbers. Since the partial products of (3) form a strictly decreasing sequence of positive real numbers, we see that the infinite product (3) converges, say, to r_ω, where

$$r_\omega = \lim \left(\frac{n_0}{1 + n_0}\right)\left(\frac{n_1}{1 + n_1}\right)\left(\frac{n_2}{1 + n_2}\right)\cdots\left(\frac{n_k}{1 + n_k}\right)\cdots \qquad \text{with} \qquad 0 \leqslant r_\omega < 1.$$

Hence, the sequence $(f_n(z))_{n\in\omega}$ converges uniformly on F, say to the function $f_\omega(z)$, where

$$f_\omega(z) = f(z) + g(z) \cdot r_\omega\,. \tag{4}$$

Moreover, since $0 \leqslant r_\omega < 1$ and since $|g(z)| < |f(z)|$ on the boundary B of F, we see that $f_\omega(z)$ vanishes nowhere on B. But then $f_\omega(z)$ has in the interior of F the same number of roots as $g(z) + f(z)$. This follows from Hurwitz' theorem and the fact that (as stated above) in the interior of F, for every $n \in \omega$ the function $f_n(z)$ has the same number of roots as the function $g(z) + f(z)$.

Now, if in (4) it is the case that $r_\omega = 0$ then $f_\omega(z) = f(z)$ and, in view of the above considerations, the proof of the theorem is complete. If, however, $r_\omega \neq 0$, then we continue the process and we obtain

$$f_{\omega+1} = f(z) + g(z) \cdot r_\omega \cdot (m_1/(1 + m_1))$$

analogous to $f_1(z)$, as given in (2); and we obtain

$$f_{\omega+\omega} = f(z) + g(z) \cdot r_\omega \cdot r_{\omega+\omega}$$

analogous to $f_\omega(z)$, as given in (4). Moreover,

$$0 < r_\omega \cdot r_{\omega+\omega} < r_\omega\,.$$

Since there is no nondenumerable strictly decreasing (or increasing) sequence of real numbers indexed by ordinal numbers (or indexed by any well-ordered set, because otherwise there would be nondenumerably many rational numbers) we see that the above process must terminate at a denumerable limit ordinal λ, with $r_\lambda = 0$. But then $f_\lambda = f(z)$ and the proof of the theorem is complete, just as in the case of $r_\omega = 0$ mentioned above.

Reference

1. S. Saks and A. Zygmund, "Analytic Functions," pp. 157–159, Elsevier, New York, 1971.