Computers and Mathematics with Applications 61 (2011) 1209–1216

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics

K. Moaddy [a], S. Momani [b,*], I. Hashim [a]

[a] School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM Bangi Selangor, Malaysia
[b] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

ARTICLE INFO

Article history:
Received 5 May 2010
Received in revised form 30 December 2010
Accepted 30 December 2010

Keywords:
Non-standard finite difference schemes
Fractional differential equations
Telegraph equation
Wave equation
Burgers equation

ABSTRACT

A non-standard finite difference scheme is developed to solve the linear partial differential equations with time- and space-fractional derivatives. The Grunwald–Letnikov method is used to approximate the fractional derivatives. Numerical illustrations that include the linear inhomogeneous time-fractional equation, linear space-fractional telegraph equation, linear inhomogeneous fractional Burgers equation and the fractional wave equation are investigated to show the pertinent features of the technique. Numerical results are presented graphically and reveal that the non-standard finite difference scheme is very effective and convenient for solving linear partial differential equations of fractional order.

1. Introduction

Many applications and models involving fractional derivatives show the importance and necessity of fractional calculus. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Fractional differentiation and integration operators are used to model problems in astrophysics and chemical physics [1–4], signal processing and system identification, control and robotics [5,6] and other areas of application.

Finding robust and stable numerical and analytical methods for solving the fractional differential equations has been an active research undertaking from several authors. Numerical and analytical methods have included Adomian decomposition method [7,8], variational iteration method [9,10], Adams–Bashforth–Moulton method [11–14]. The variational iteration method and the Adomian decomposition method have been extensively used to solve fractional partial differential equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

Erjaee [15] investigated the saddle and Hopf bifurcation points of predator–prey fractional differential equation systems with a constant rate harvesting using the non-standard finite difference method. Hussian et al. [16] used the non-standard discretization for solving fractional differential equations. Recently, Odibat and Momani [17] developed a semi-numerical method for solving linear partial differential equations of fractional order. This method is named as generalized differential transform method (GDTM) and is based on the two-dimensional differential transform method and generalized Taylor's formula. Very recently, many mathematicians and scientists worked on the problem of existence and uniqueness of solutions of fractional differential equations (cf. [18,19] and references cited therein).

This paper is devoted to develop a non-standard discretization scheme given by Mickens [20] to the Grunwald–Letnikov discretization process for the linear partial differential equations with time- and space-fractional derivatives. The non-standard finite difference scheme [21–25] has developed as an alternative method for solving a wide range of problems

* Corresponding author.
E-mail addresses: s.momani@ju.edu.jo, shahermm@yahoo.com (S. Momani).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.12.072

whose mathematical models involve algebraic, differential, biological models, and chaotic systems. The definition of Grunwald–Letnikov derivatives has been used in numerical analysis to discretize the fractional differential equations with Riemann–Liouville derivatives. The technique has many advantages over the classical techniques, and provides an efficient numerical solution.

The rest of the paper is organized as follows. In the next section we present mathematical preliminaries of the fractional calculus theory which are required for establishing our results and describe the non-standard finite difference scheme (NSFD) to solve the fractional partial differential equations. In Section 3, we present four examples to show the efficiency and simplicity of the scheme and we discuss numerical approximations to the solutions. In the last section we summarize the conclusions.

2. Preliminaries and notations

In this section we give some basic definitions and properties of the fractional calculus theory and non-standard discretization which are used further in this paper.

2.1. Grunwald–Letinkov approximation

We will begin with the single fractional differential equation (see [16])

$$D^{\alpha}x(t) = f(t, x(t)), \quad T \geq t \geq 0 \text{ and } x(t_0) = 0, \tag{1}$$

where $\alpha > 0$ and D^{α} denotes the fractional derivative, defined by

$$D^{\alpha}x(t) = J^{n-\alpha}D^{n}x(t), \tag{2}$$

where $n - 1 < \alpha \leq n$, $n \in N$ and J^n in the nth-order Riemann–Liouville integral operator defined as

$$J^{n}x(t) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1}x(\tau)\mathrm{d}\tau, \tag{3}$$

with $t > 0$.

To apply Mickens' scheme, we have chosen this Grunwald–Letnikov method approximation for the one-dimensional fractional derivative as follows (see [26]):

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{N}(-1)^{j}\binom{\alpha}{j}x(t-jh), \tag{4}$$

where $N = t/h$ and t denotes the integer part of t and h is the step size. Therefore, Eq. (4) is discretized as

$$\sum_{j=0}^{N} c^{\alpha}x(t_{n-j}) = f(t_n, x(t_n)), \quad n = 1, 2, 3, \ldots, \tag{5}$$

where $t_n = nh$ and c_j^{α} are the Grunwald–Letnikov coefficients defined as

$$c^{\alpha}{}_{j} = \left(1 - \frac{1+\alpha}{j}\right)c_{j-1}^{\alpha}, \quad \text{and} \quad c_0^{\alpha} = h^{-\alpha}, \quad j = 1, 2, 3, \ldots. \tag{6}$$

2.2. Non-standard discretization

In general, the non-standard finite difference rules, introduced by Mickens [27], do not lead to a unique discrete model for either ODEs or PDEs. Therefore, we give the basic rules of non-standard for ODEs which lead us to obtain the non-standard solution for such PDEs.

We consider the ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \tag{7}$$

the discrete derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y_{k+1} - y_k}{\phi(h, \lambda)}, \tag{8}$$

where ϕ is a function of the step size $h = \Delta t$. ϕ has the following properties:

$$\phi(h) = h + O(h^2), \tag{9}$$

where $h \to 0$.

Examples of functions $\phi_i(h, \lambda)$ that satisfy (9) are h, $\sin(h)$, $\sinh(h)$, $e^h - 1$.

The nonlinear terms can be in general being replaced by nonlocal discrete representations. For example,

$$y^2 \approx y_k y_{k+1},$$
$$y^3 \approx \left(\frac{y_{k+1} + y_{k-1}}{2}\right) y_k^2,$$

where $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$.

This way of constructing discrete derivatives can be easily extended to partial derivatives

$$\frac{\partial f(y,t)}{\partial y} = \frac{f_{m+1}^k - f_{m-1}^k}{\phi(\Delta y)}, \tag{10}$$

$$\frac{\partial^2 f(y,t)}{\partial t^2} = \frac{f_m^{k+1} - 2f_m^k + f_m^{k-1}}{\phi(\Delta x)}, \tag{11}$$

where

$$\phi(\Delta y) = \Delta y + O(\Delta y^2), \tag{12}$$

$$\phi(\Delta t) = \Delta t^2 + O(\Delta t^4). \tag{13}$$

Several examples discussing the discretization of the non-standard finite difference method for PDEs are given in [20].

3. Applications

In this section we apply the NSFD to obtain the numerical solution for the linear partial differential equations with time- and space-fractional derivatives.

3.1. Example 1

Consider the following linear inhomogeneous time-fractional equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{14}$$

where $f(x,t) = 2t^\alpha + 2x^2 + 2$, $t > 0$, $0 < \alpha \le 1$,

subject to the initial condition

$$u(x,0) = x^2, \tag{15}$$

and the boundary conditions

$$u(0,t) = 2t^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}, \qquad u(1,t) = 1 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}. \tag{16}$$

The exact solution of Eq. (14) introduced by Odibat and Momani [17] is as follows:

$$u(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}. \tag{17}$$

By applying Mickens scheme and using the Grunwald–Letnikov discretization method, the derivatives can be approximated as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{n=0}^{k+1} c_n u_m^{k+1-n}, \tag{18}$$

$$\frac{\partial u}{\partial x} = \frac{u_{m+1}^k - u_{m-1}^k}{\phi_1(\Delta x)}, \tag{19}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{m+1}^k - 2u_m^k + u_{m-1}^k}{\phi_2(\Delta x)}, \tag{20}$$

where $\Delta x = \Delta t = h$.

We chose the dominator functions as the following form

$$\phi_1(\Delta t) = 2(e^h - 1), \tag{21}$$

$$\phi_2(\Delta x) = 4\sin^2(h/2), \tag{22}$$

where the denominator functions $\phi_1(h)$ and $\phi_2(h)$ satisfy the following conditions

$$\phi_1(h) = h + O(h^2), \tag{23}$$

$$\phi_2(h) = h^2 + O(h^4), \tag{24}$$

and $c_0 = \frac{1}{2}\phi_1(h)^{-\alpha}$.

Substituting Eqs. (18)–(20) into Eq. (14) yields

$$\sum_{n=0}^{k+1} c_n u_m^{k+1-n} + x_m \frac{u_{m+1}^k - u_{m-1}^k}{\phi_1(h)} + \frac{u_{m+1}^k - 2u_m^k + u_{m-1}^k}{\phi_2(h)} = f_m^k, \tag{25}$$

after doing some algebraic manipulation (25) gives

$$u_m^{k+1} = \frac{\gamma + \beta_1 u_m^k - u_{m+1}^k(\phi_1 + x_m\phi_2) - u_{m-1}^k(\phi_1 - x_m\phi_2)}{c_0\phi_1\phi_2}, \tag{26}$$

where $\gamma = \phi_1\phi_2\left(f_m^k - \sum_{n=2}^{k+1} c_n u_m^{k+1-n}\right)$ and $\beta_1 = 2\phi_1 - c_1\phi_1\phi_2$

where $x \to x_m = (\Delta x)m,\ t \to t_k = (\Delta t)k$.

3.2. Example 2

Consider the following space-fractional telegraph equation:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad t \ge 0,\ 0 < \alpha \le 2, \tag{27}$$

subject to the initial and boundary conditions

$$u(x, 0) = e^x, \quad 0 < x < 1, \tag{28}$$

$$u(0, t) = e^{-t}, \qquad \frac{\partial^2 u(0, t)}{\partial x} = e^{-t}, \quad t \ge 0. \tag{29}$$

When $\alpha = 1$, the exact solution for Eq. (27) introduced by Momani [8] using the Adomian decomposition method is as follows:

$$u(x, t) = e^{x-t}. \tag{30}$$

We define the derivatives as follows:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \sum_{n=0}^{m} c_n u_{m-n}^k, \tag{31}$$

$$\frac{\partial u}{\partial t} = \frac{u_m^{k+1} - u_m^{k-1}}{\phi_1(\Delta t)}, \tag{32}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_m^{k+1} - 2u_m^k + u_m^{k-1}}{\phi_2(\Delta t)}, \tag{33}$$

where $\Delta x = \Delta t = h$. Substituting Eqs. (31)–(33) into Eq. (27) gives

$$\sum_{n=0}^{m} c_n u_{m-n}^k = \frac{u_m^{k+1} - 2u_m^k + u_m^{k-1}}{\phi_2(h)} + \frac{u_m^{k+1} - u_m^{k-1}}{\phi_1(h)} + u, \tag{34}$$

for the u term in Eq. (34) we replace it by the following discrete linear

$$u = 2u - u \to 2\bar{u}_m^k - u_m^{k+1}, \tag{35}$$

where

$$\bar{u}_m^k = \frac{u_{m+1}^k + u_{m-1}^k}{2}. \tag{36}$$

Therefore, solving Eq. (34) for u_m^{k+1} yields

$$u_m^{k+1} = (-\beta_2 - (2\phi_1 + c_0\phi_1\phi_2)u_m^k + \phi_1\phi_2(u_{m+1}^k + u_{m-1}^k) - (\phi_2 - \phi_1)u_m^{k-1})/\phi_1\phi_2 - (\phi_1 + \phi_2), \tag{37}$$

where $\beta_2 = \phi_1\phi_2\sum_{n=1}^{m} c_n u_{m-n}^k$.

3.3. Example 3

Consider the following one-dimensional linear inhomogeneous fractional wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = g(x,t), \quad t > 0,\ 0 < \alpha \le 1, \tag{38}$$

where

$$g(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x + t\cos x, \quad x \in \Re, \tag{39}$$

subject to the initial and boundary conditions

$$u(x,0) = 0, \tag{40}$$

$$u(0,t) = 0, \qquad u(1,t) = 0.84147t. \tag{41}$$

The exact solution introduced by Odibat and Momani [10] is as follows:

$$u(x,t) = t\sin x. \tag{42}$$

Substituting (18) and (19) into Eq. (38) yields

$$\sum_{n=0}^{k} c_n u_m^{k-n} + \frac{u_{m+1}^k - u_{m-1}^k}{\phi_3(\Delta x)} = g_m^k, \tag{43}$$

where $\Delta x = \Delta t = h$.

The denominator function $\phi_3(h)$ satisfies the following condition:

$$\phi_3(h) = h + O(h^2), \tag{44}$$

and in this case we chose $\phi_3(h) = 2\sin(h)$.

Solving Eq. (43) for u_{m+1}^k gives

$$u_{m+1}^k = 2\left(g_m^k - c_0 u_m^k + \sum_{n=1}^{k} c_n u_m^{k-n}\right)\phi_3(h) + u_{m-1}^k, \tag{45}$$

where $c_0 = \frac{1}{2}\phi_3(h)^{-\alpha}$.

3.4. Example 4

In this example we consider the one-dimensional linear inhomogeneous fractional Burgers equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = q(x,t), \quad t > 0,\ 0 < \alpha \le 1, \tag{46}$$

where

$$q(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad x \in \Re, \tag{47}$$

subject to the initial and boundary conditions

$$u(x,0) = x^2, \tag{48}$$

$$u(0,t) = t^2, \qquad u(1,t) = 1 + t^2. \tag{49}$$

The exact solution introduced by Odibat and Momani [10]:

$$u(x,t) = x^2 + t^2. \tag{50}$$

Substituting Eqs. (18)–(20) into Eq. (46) yields

$$\sum_{n=0}^{k} c_n u_m^{k-n} + \frac{u_{m+1}^k - u_{m-1}^k}{\phi_3(\Delta x)} + \frac{u_{i+1}^j - 2u_m^k + u_{m-1}^k}{\phi_2(\Delta x)} = q_m^k, \tag{51}$$

where $\Delta x = \Delta t = h$, $c_0 = \frac{1}{2}\phi_3(h)^{-\alpha}$ and the denominator function is $\phi_3(h) = 2\sin h$, and satisfies (44).

Solving Eq. (51) for u_{m+1}^k gives

$$u_{m+1}^k = \frac{\beta_3 + 2u_m^k(\phi_3 + \phi_2\phi_3 c_0) - u_{m-1}^k(\phi_2 + \phi_3)}{\phi_3 - \phi_2}, \tag{52}$$

where $\beta_3 = \phi_2\phi_3\left(\sum_{n=1}^{k} c_n u_m^{k-n} - q_m^k\right)$.

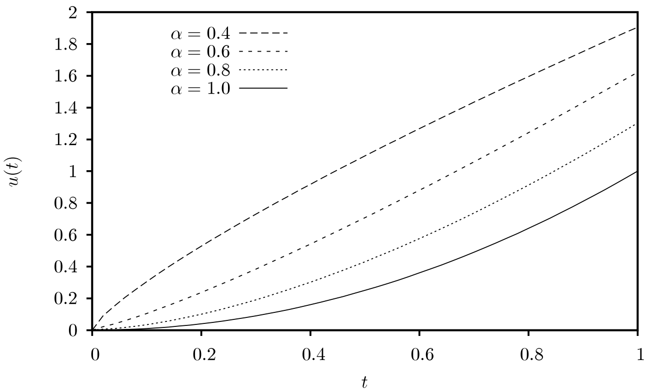

Fig. 1. Different solutions of the linear inhomogeneous time-fractional equation in Example 1 using the non-standard finite difference scheme where $x = 1/50$ with time step $h = 1/50$.

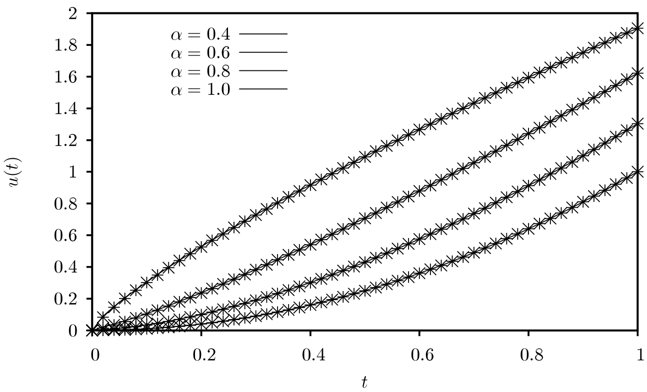

Fig. 2. Comparison of the results of the non-standard finite difference scheme and the exact solution considering $x = 1/50$ for Example 1.

4. Results and discussion

Four fractional partial differential equations with boundary conditions were numerically solved using the non-standard finite difference method based on Grunwald–Letnikov method as coded in the computer algebra package Maple. In Maple, the number of variable digits controlling the number of significant digits is set to 18 in all the calculations done in this paper. In this present work, we fix the benchmark time step size $h = 0.02$.

Note that this non-standard finite difference scheme has the following features:

(i) The discrete model is explicit.
(ii) The denominator functions ϕ_1, ϕ_2 and ψ for the discrete first- and second-derivatives have a non-standard form.
(iii) A central difference scheme replaces the first and second order space derivative.
(iv) For the linear terms involving the dependent variable may require "nonlocal" discretizations.

Fig. 1 shows the different solutions of the linear inhomogeneous time-fractional equation in Example 1 using the non-standard finite difference scheme with different fractional derivatives $\alpha = 0.4, 0.6, 0.8$ and 1. Fig. 2 shows the numerical solutions and the exact solutions for different values of α when $x = 1/50$. From the numerical results in Figs. 1 and 2, it is to conclude that the numerical solutions obtained using the non-standard finite difference scheme are in good agreement with the exact solutions and the approximate solutions obtained using the generalized differential transform method [17] for all values of α and t.

Figs. 3–5 show the approximate solutions for Examples 2, 3 and 4 obtained for different values of α using the non-standard finite difference scheme, respectively. From the graphical results in these figures, it is clear that the approximate solutions are in good agreement with the exact solutions and the solution continuously depends on the time-fractional derivative.

5. Conclusions

Numerical solutions of the linear partial differential equations of fractional order are derived using the non-standard finite difference scheme. The results of this method are in good agreement with those obtained by using the already existing ones. It may be concluded that the non-standard method is a very powerful and efficient technique for solving the model. The basic idea described in this paper is expected to be further employed to solve other similar linear and nonlinear problems in fractional calculus.

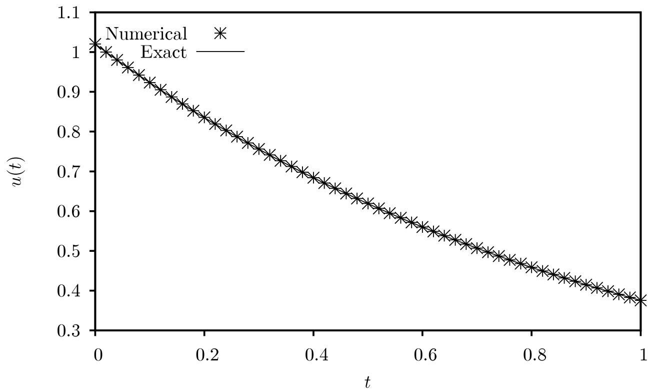

Fig. 3. Comparison of the results of the non-standard finite difference scheme and the exact solution in Example 2 considering $x = 1/50$ and $\alpha = 1$.

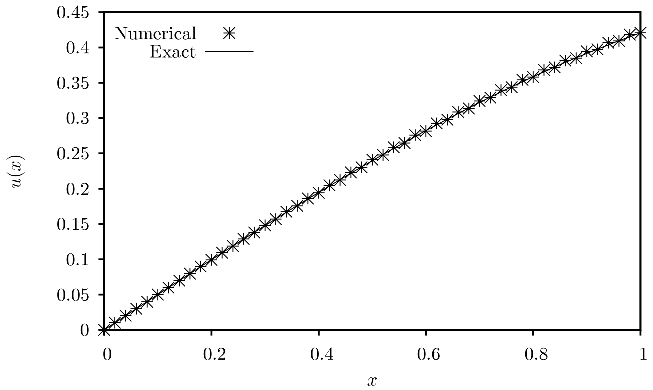

Fig. 4. Comparison of the results of the non-standard finite difference scheme and the exact solution in Example 3 considering $\alpha = 0.9$ and $t = 1/2$.

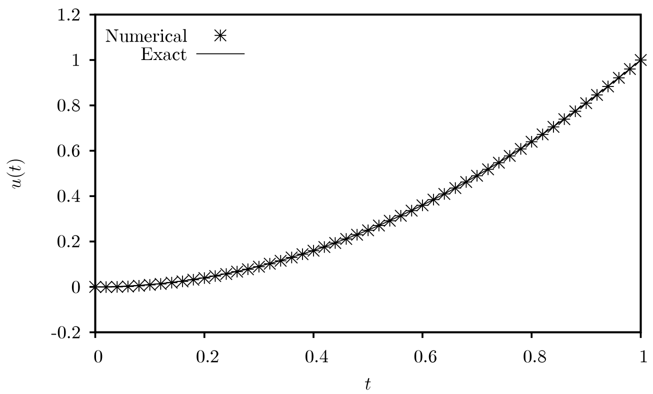

Fig. 5. Comparison of the results of the non-standard finite difference scheme and the exact solution in Example 4 considering $\alpha = 0.7$ and $x = 1/50$.

Acknowledgements

The authors gratefully acknowledge the financial support received from the ZAMALAH scheme of Universiti Kebangsaan Malaysia and the Research Grant UKM-ST-07- FRGS-0028-2009.

References

[1] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[2] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.
[3] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[4] C.P. Li, W.H. Deng, Remarks on fractional derivatives, Appl. Math. Comput. 187 (2007) 777–784.
[5] G.M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, Phys. Rep. 371 (2002) 461–580.

[6] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific, Singapore, 2009.
[7] S. Momani, Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, Appl. Math. Comput. 165 (2) (2005) 459–472.
[8] S. Momani, Analytic and approximate solutions of the space- and time-fractional telegraph equations, Appl. Math. Comput. 170 (2005) 1126–1134.
[9] S. Momani, Z. Odibat, A. Alawneh, Variational iteration method for solving the space- and time-fractional KdV equation, Numer. Methods Partial Differential Equations J. 24 (1) (2008) 262–271.
[10] Z. Odibat, S. Momani, The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics, Comput. Math. Appl. 58 (2009) 2199–2208.
[11] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.
[12] K. Diethelm, N.J. Ford, A.D. Freed, A predictor–corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynam. 29 (2002) 3–22.
[13] C.P. Li, G.J. Peng, Chaos in Chen's system with a fractional order, Chaos Solitons Fractals 22 (2004) 443–450.
[14] K. Diethelm, N.J. Ford, A.D. Freed, Detailed error analysis for a fractional Adams method, Numer. Algorithms 26 (2004) 31–52.
[15] G.H. Erjaee, Numerical bifurcation of predator–prey fractional differential equations with a constant rate harvesting, J. Phys. Conf. Ser. 96 (2007) 012045.
[16] G. Hussian, M. Alnasr, S. Momani, Non-standard discretization for fractional differential equations, in: Proceeding of 8th Seminar of Differential Equations and Dynamical Systems in, Isfahan, Iran.
[17] Z. Odibat, S. Momani, A generalized differential transform method for linear partial differential equations of fractional order, Appl. Math. Lett. 21 (2008) 194–199.
[18] D. Baleanu, O.G. Mustafa, R.P. Agarwal, On the solution set for a class of sequential fractional differential equations, J. Phys. A: Math. 43 (38) (2010) 385209.
[19] D. Baleanu, O.G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Comput. Math. Appl. 59 (5) (2010) 1835–1841.
[20] R.E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore, 1994.
[21] R.E. Mickens, Exact solutions to a finite difference model of a nonlinear reactionadvection equation: implications for numerical analysis, Numer. Methods Partial Differential Equations Fractals 5 (1989) 313–325.
[22] R.E. Mickens, A. Smith, Finite difference models of ordinary differential equations: in uence of denominator models, J. Franklin Inst. Fractals 327 (1990) 143–145.
[23] R.E. Mickens, Nonstandard finite difference schemes for reaction–diffusion equations, Numer. Methods Partial Differential Equations Fractals 15 (1999) 201–214.
[24] R.E. Mickens (Ed.), Advances in the Applications of Nonstandard Finite Difference Schemes, World Scientific, Singapore, 2005.
[25] in: R.E. Mickens (Ed.), Applications of Nonstandard Finite Difference Schemes, Singapore, 2000.
[26] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection dispersion flow equations, J. Comput. Appl. Math. 172 (2003) 65–77.
[27] R.E. Mickens, A nonstandard finite difference scheme for a fisher PDE having nonlinear diffusion, Comput. Math. Appl. 45 (2003) 429–436.