Statistical Inference for the Burr Model Based on Progressively Censored Data

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Abstract—Maximum likelihood and Bayes estimates for the two parameters and the reliability function of the Burr Type XII distribution are obtained based on progressive Type II censored samples. An approximation based on the Laplace approximation method developed by Tierney and Kadane [1] and a bivariate prior density for the two unknown parameters, suggested by Al-Hussaini and Jaheen [2] are used for obtaining the Bayes estimates. These estimates are compared via Monte Carlo simulation study. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Burr Type XII distribution, Bayesian estimation, Progressive samples, Laplace approximation, Monte Carlo simulation.

1. INTRODUCTION

The two-parameter Burr Type XII distribution (which we shall simply denote by Burr $(c,k)$) was first introduced in the literature by Burr [3] and has gained special attention in the last two decades due to the potential of using it in practical situations. Its capacity to assume various shapes often permits a good fit when used to describe biological, clinical, or other experimental data. It has also been applied in areas of quality control, reliability studies, duration, and failure time modelling. The probability density function (PDF), cumulative distribution function (CDF), and reliability function (RF) of the Burr $(c,k)$ distribution are given, respectively, by

$$f(x) = c k x^{c-1} (1 + x^c)^{-(k+1)}, \quad x > 0, \quad c > 0, \quad k > 0,$$

$$F(x) = 1 - (1 + x^c)^{-k},$$

and

$$r(t) = (1 + t^c)^{-k}.$$

Inferences for the Burr $(c,k)$ model were discussed by many authors. Based on complete samples, Papadopoulos [4] obtained Bayesian estimation for the parameter $k$ and the reliability function $r(t)$ when the parameter $c$ is assumed to be known. Evans and Ragab [5] also obtained Bayes estimates of $k$ and the reliability function based on Type 2 censored samples. Al-Hussaini

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and Jaheen [2,6] used different techniques for obtaining Bayes estimates of the parameters $c$ and $k$, reliability and failure rate functions based on Type 2 censored samples. Al-Hussaini et al. [7] obtained the maximum likelihood, uniformly minimum variance unbiased, Bayes and empirical Bayes estimators for the parameter $k$ and reliability function when $c$ is known. Ali Mousa and Jaheen [8] obtained interval estimates of the parameter $k$ and reliability function when $c$ is known from a Bayesian approach based on Type 2 censored data. Ali Mousa [9] obtained empirical Bayes estimation of the parameter $k$ and the reliability function based on accelerated Type 2 censored data. Hossain and Nath [10] deal with an unweighted least squares estimation of the parameters and compared the results with the maximum likelihood and maximum product of spacing methods.

In this paper, maximum likelihood (ML) and Bayes estimates for the two parameters $c$, $k$ and the reliability function $r(t)$ of the Burr Type XII distribution are obtained based on progressive Type 2 censored samples. A brief account for the progressive censoring will be given in Section 2. When both of the two parameters $c$ and $k$ are unknown, a bivariate prior density, suggested by Al Hussaini and Jaheen [2], and the approximation due to Tierney and Kadane [1] are used for obtaining the Bayes estimates. These estimates are compared via Monte Carlo simulation study.

2. PROGRESSIVE TYPE II CENSORING

Suppose that $n$ independent items, with common continuous density $f(x)$, are put on a life test. Suppose, further, that a censoring scheme $(R_1, R_2, \ldots, R_m)$ is previously fixed such that immediately following the first failure, $X_1$, $R_1$ surviving items are removed from the experiment at random, and immediately following the second failure, $X_2$, $R_2$ surviving items are removed from the experiment at random; this process continues until, at the time of the $m^{th}$ observed failure $X_m$, the remaining $R_m$ items are removed from the test. The $m$ ordered observed failure times denoted by $X_{1;m:n}, X_{2;m:n}, \ldots, X_{m;m:n}$, are called progressive Type 2 right censored order statistics of size $m$ from a sample of size $n$ with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$. It is clear that $n = m + R_1 + R_2 + \cdots + R_m$. If the failure times of the $n$ items originally on the test are from a continuous population with CDF $F(x)$ and PDF $f(x)$, the joint probability density function for $X_1 = X_{1;m:n}, X_2 = X_{2;m:n}, \ldots, X_m = X_{m;m:n}$ is given by Balakrishnan and Sandhu [11],

$$f_{x_1,x_2,\ldots,x_m}(x_1,x_2,\ldots,x_m) = A \prod_{i=1}^{m} [f(x_i)|1 - F(x_i)]^{R_i},$$

where

$$A = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1).$$

Progressive Type 2 censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a compromise can be achieved between time consumption and the observation of some extreme values. For more details on the progressive censored samples, see [12-14].

When data are obtained by progressive censoring, inference problems for various distributions have been studied by many authors including Cohen [15,16], Mann [17], Gibbons and Vance [18], Cohen and Whitten [19], Cohen [20], Viveros and Balakrishnan [21], and Balakrishnan and Sandhu [11,22].

3. ESTIMATION BASED ON PROGRESSIVE SAMPLES

Suppose that $X_{1;m:n}, X_{2;m:n}, \ldots, X_{m;m:n}$ are $m$ progressive Type II right censored order statistics from a sample of size $n$ with progressive censoring scheme $(R_1, R_2, \ldots, \ldots).$
\( R_m \), drawn from a population whose PDF is as given by (1). Based on such a progressive Type II right censored sample, the likelihood function (LF), using (1), (2), and (4), takes the form
\[
\ell(c, k; x) = A(c)^m b(c; x) \exp(-kT^*),
\]
where \( x = (x_1, x_2, \ldots, x_m) \), \( x_i = x_{(i:n)} \), \( i = 1, 2, \ldots, m \),
\[
b(c; x) = \prod_{i=1}^{m} \frac{x_i^{c-1}}{1 + x_i^c}, \quad T^* = \sum_{i=1}^{m} (R_i + 1) \ln (1 + x_i^c),
\]
and \( A \) is as given by (5).

### 3.1. Maximum Likelihood Estimation

The log likelihood function is, by using (6), given by
\[
\ln \ell(c, k; x) \propto m \ln(ck) + \ln[b(c; x)] - kT^*.
\]
Assuming that \( c \) is known, the ML estimate of the parameter \( k \) and the reliability function \( r(t) \) are given by
\[
\hat{k}_{ML} = \frac{m}{T^*} \quad \text{and} \quad \hat{r}_{ML}(t) = (1 + t^c)^{-m/T^*},
\]
where \( T^* \) is as given by (7).

On the other hand, when the two parameters \( c \) and \( k \) are unknown, the likelihood equation for \( c \) can be written as
\[
\frac{m}{c} + \sum_{i=1}^{m} \ln x_i - \sum_{i=1}^{m} \frac[k(R_i + 1)]{1 + x_i^c} \ln x_i = 0.
\]
Substituting the value of \( k \) in (10) by its ML estimate given by (9) yields a nonlinear equation in \( c \), and by solving it we obtain the ML estimate of the parameter \( c \). Then, substituting the ML estimate of \( c \) in (9), we obtain the ML estimates of both the parameter \( k \) and the reliability function \( r(t) \).

### 3.2. Bayesian Estimation

When the parameter \( c \) is assumed to be known and \( k \) has a gamma conjugate prior density of the form
\[
\pi(k) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} k^\alpha e^{-\beta k}, \quad k > 0, \quad \alpha > -1, \quad \beta > 0,
\]
it follows, from (6) and (11), that the posterior density of \( k \) is given by
\[
q_t(k \mid x, c) = \frac{(T^* + \beta)^{m+\alpha+1}}{\Gamma(\alpha+1)} k^{m+\alpha} e^{-k(T^* + \beta)},
\]
where \( T^* \) is given by (7).

Under a squared error loss function, the Bayes estimate of the parameter \( k \) and the reliability function can be shown to be
\[
\hat{k}_B = \frac{m + \alpha + 1}{T^* + \beta} \quad \text{and} \quad \hat{r}_B(t) = \left[1 + \frac{\ln(1 + t^c)}{\beta + T^*}\right]^{-m+\alpha+1}.
\]
Clearly, when \( \alpha \to -1 \) and \( \beta \to 0 \), \( \hat{k}_B = \hat{k}_{ML} \), given by (9).
When both of the two parameters $c$ and $k$ are unknown, we use the bivariate prior density, suggested by Al-Hussaini and Jaheen [2], which is given by

$$g(c, k) = g_1(k \mid c)g_2(c),$$

(14)

where

$$g_1(k \mid c) = \frac{c^{\alpha+1}}{\Gamma(\alpha+1)\beta^{\alpha+1}}k^\alpha e^{-kc/\beta}, \quad k > 0, \quad \alpha > -1, \quad \beta > 0,$$

(15)

and

$$g_2(c) = \frac{1}{\Gamma(\delta)\gamma^\delta}c^{\delta-1}e^{-c/\gamma}, \quad c > 0, \quad \gamma > 0, \quad \delta > 0.$$

(16)

Hence, the bivariate prior density of $c$ and $k$ can be written as

$$g(c, k) = B_1c^{\alpha+\delta}k^\alpha \exp \left[ -c \left( \frac{1}{\gamma} + \frac{k}{\beta} \right) \right], \quad c > 0, \quad k > 0,$$

(17)

where

$$B_1^{-1} = \Gamma(\delta)\Gamma(\alpha+1)\gamma^\delta\beta^{\alpha+1}.$$

(18)

From (6) and (17), the joint posterior density function of $c$ and $k$ given the data is thus,

$$q_2(c, k \mid x) = \frac{\ell(c, k; x)g(c, k)}{\int \ell(c, k; x)g(c, k) \, dc \, dk}$$

(19)

$$= B_2c^{m+\alpha+\delta}k^{m+\alpha}b(c, x)e^{-c/\gamma} \left[ -k \left( \frac{T^* + \frac{c}{\beta}}{\gamma} \right) - \frac{c}{\gamma} \right],$$

where

$$B_2^{-1} = \Gamma(m + \alpha + 1) \int_0^\infty c^{m+\alpha+\delta}b(c, x)e^{-c/\gamma} \left[ -k \left( \frac{T^* + \frac{c}{\beta}}{\gamma} \right) - \frac{c}{\gamma} \right]^{-\alpha-\delta-1} \, dc.$$

(20)

Under a squared error loss function, the Bayes estimator $\hat{\phi}$ of a function $\phi(c, k)$ is its posterior mean given by

$$\hat{\phi} = E(\phi(c, k) \mid x) = \int c \int_0^\infty \phi(c, k)q_2(c, k \mid x) \, dc \, dk$$

$$= \frac{\int c \int_0^\infty \phi(c, k)\ell(c, k; x)g(c, k) \, dc \, dk}{\int \int_0^\infty \ell(c, k; x)g(c, k) \, dc \, dk},$$

(21)

where $\ell(c, k; x)$ and $g(c, k)$ are given, respectively, by (6) and (17).

The ratio of the two integrals given by (21) cannot, generally, be obtained in a closed form. Therefore, in such situations, we can use numerical integration technique, which can be computationally intensive, especially in high-dimensional parameter space. One can also use approximate methods such as the approximate form due to Lindley [23] or that of Tierney and Kadane [1]. We adopt here the Tierney and Kadane approximation since its error is of order $O(n^{-2})$, while the error in using Lindley's approximate form is of order $O(n^{-1})$. The regularity condition required for using Tierney-Kadane's form is that the posterior density should be unimodal. This condition can be shown to be held for the posterior density $q_2(c, k \mid x)$, given by (19), in a similar proof to that given by Al-Hussaini and Jaheen [6]. In the next section, we review this approximation (see [1] and [24]).
3.2.1. The approximation of Tierney and Kadane

Let $\ell(\theta; \mathbf{x})$ be the likelihood function of $\theta$ based on the $n$ observations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. These observations are not necessarily independent or identically distributed, and also both $x_i$ and $\theta$ may be multidimensional. Let $\pi(\theta)$ be the prior distribution defined on the parameter space $\Theta$. Let $q(\theta | \mathbf{x})$ denote the posterior distribution of $\theta$. The Bayes estimate of a function $\phi(\theta)$ is the posterior mean given by

$$
\hat{\phi} = E(\phi(\theta) | \mathbf{x}) = \int_{\Theta} \phi(\theta) q(\theta, \mathbf{x}) d\theta = \frac{\int_{\Theta} e^{nL^*(\theta)} d\theta}{\int_{\Theta} e^{nL(\theta)} d\theta},
$$

(22)

where

$$
L(\theta) = \frac{1}{n} \ln q(\theta | \mathbf{x}) \quad \text{and} \quad L^*(\theta) = L(\theta) + \frac{1}{n} \ln \phi(\theta).
$$

(23)

Following [1], equation (22) can be approximated in the form

$$
\hat{\phi}_{BT} = \left[ \frac{\det \Sigma^*}{\det \Sigma} \right]^{1/2} \operatorname{exp} \left[ n \left\{ L^* (\hat{\theta}^*) - L (\hat{\theta}) \right\} \right] = \left[ \frac{\det \Sigma^*}{\det \Sigma} \right]^{1/2} \frac{\phi (\hat{\theta}^*) q (\hat{\theta}^* | \mathbf{x})}{q (\hat{\theta} | \mathbf{x})},
$$

(24)

where $\hat{\theta}^*$ and $\hat{\theta}$ maximize $L^*(\theta)$ and $L(\theta)$, respectively, and $\Sigma^*$ and $\Sigma$ are the negatives of the inverses of the matrices of second derivatives of $L^*(\theta)$ and $L(\theta)$, at $\hat{\theta}^*$ and $\hat{\theta}$, respectively.

We now apply this approximation to obtain the Bayes estimators of the Burr parameters $c$ and $k$ and the reliability function $r(t)$, given by (3). From (19) and (23), the functions $L$ and $L^*$ are, respectively, given by

$$
L(c, k) = \frac{1}{n} \left[ (m + \alpha + \beta) \ln c + (m + \alpha) \ln k + \ln b(c, \mathbf{x}) - k \left( T^* + \frac{c}{\beta} \right) - \frac{c}{\gamma} \right]
$$

(25)

and

$$
L^*(c, k) = L(c, k) + \frac{1}{n} \ln \phi(c, k).
$$

(26)

Differentiating (25) with respect to $c$ and $k$, the first derivatives of $L(c, k)$ are

$$
L_1 = \frac{\partial L}{\partial c} = \frac{1}{n} \left[ \frac{m + \alpha + \delta}{c} + \sum_{i=1}^{m} \ln x_i - \sum_{i=1}^{m} x_i^c a_i(c) - \frac{1}{\gamma} - k \left( \frac{1}{\beta} + \sum_{i=1}^{m} (R_i + 1)x_i^c a_i(c) \right) \right],
$$

(27)

and

$$
L_2 = \frac{\partial L}{\partial k} = \frac{1}{n} \left[ \frac{m + \alpha}{k} - \left( T^* + \frac{c}{\beta} \right) \right],
$$

(28)

where $T^*$ is given by (7) and

$$
a_i(c) = \frac{\ln x_i}{1 + x_i^c}, \quad i = 1, 2, \ldots, m.
$$

(29)

The unique posterior mode, $\left( \hat{c}_D, \hat{k}_D \right)$, is obtained by equating (27) and (28) to zero and then solving the resulting nonlinear equations in $c$ and $k$.

The second derivatives of $L(c, k)$ with respect to $c$ and $k$ are

$$
L_{11} = \frac{\partial^2 L}{\partial c^2} = -\frac{1}{n} \left[ \frac{m + \alpha + \delta}{c^2} + \sum_{i=1}^{m} x_i^c a_i^2(c) + k \sum_{i=1}^{m} (R_i + 1)x_i^c a_i^2(c) \right],
$$

$$
L_{12} = \frac{\partial^2 L}{\partial c \partial k} = -\frac{1}{n} \left[ \frac{1}{\beta} + \sum_{i=1}^{m} (R_i + 1)x_i^c a_i(c) \right] = L_{21},
$$

(30)

$$
L_{22} = \frac{\partial^2 L}{\partial k^2} = -\frac{1}{n} \left( \frac{m + \alpha}{k^2} \right),
$$

(31)
From (30), one can see that
\[
\det \Sigma = \frac{1}{L_{11}L_{22} - L_{12}^2},
\]  
(31)
evaluated at the posterior mode \((\hat{c}_D, \hat{k}_D)\). Similar derivations are needed to determine the mode of \(L^*(c, k)\) and \(\det \Sigma^*\) in the following cases.

(i) When \(\phi(c, k) = c\), equation (26) becomes
\[
_cL^*(c, k) = L(c, k) + \frac{1}{n} \ln c.
\]  
(32)
Thus, the first derivatives of \(_cL^*(c, k)\) with respect to \(c\) and \(k\) are
\[
_cL^*_1 = L_1 + \frac{1}{n c} \quad \text{and} \quad _cL^*_2 = L_2,
\]  
(33)where \(L_1\) and \(L_2\) are given, respectively, by (27) and (28). The second derivatives are
\[
_cL^*_1 = L_{11} - \frac{1}{n c^2}, \quad _cL^*_2 = L_{12}, \quad \text{and} \quad _cL^*_2 = L_{22},
\]  
(34)where \(L_{11}, L_{12}, \) and \(L_{22}\) are the second derivatives given by (30). Using (33), we may obtain the mode \((\hat{c}^*_1, \hat{k}^*_1)\) of \(_cL^*(c, k)\), and from (34), we get
\[
\det \Sigma^*_c = \frac{1}{_cL_{11}^*L_{22}^* - (cL_{12}^*)^2},
\]  
(35)evaluated at the mode \((\hat{c}^*_1, \hat{k}^*_1)\). Substituting from (31) and (35) in (24), the Bayes estimator of \(c\) takes the form
\[
\hat{c}_{BT} = \left( \frac{L_{11}L_{22} - (L_{12})^2}{_cL_{11}^*L_{22}^* - (cL_{12}^*)^2} \right)^{1/2} \frac{\hat{c}_1^* q_2 \left( \hat{c}^*_1, \hat{k}^*_1 \mid \hat{c}_D, \hat{k}_D \right)}{q_2 \left( \hat{c}_D, \hat{k}_D \mid \hat{c}, \hat{k} \right)},
\]  
(36)where \(q_2(c, k \mid x)\) is the posterior density given by (19) evaluated at the modes \((\hat{c}^*_1, \hat{k}^*_1)\) and \((\hat{c}_D, \hat{k}_D)\) of the functions \(_cL^*(c, k)\) and \(L(c, k)\), respectively.

(ii) When \(\phi(c, k) = k\), we have from (26),
\[
kL^*(c, k) = L(c, k) + \frac{1}{n} \ln k.
\]  
(37)The first and second derivatives of \(kL^*(c, k)\) with respect to \(c\) and \(k\) are
\[
kL^*_1 = L_1 \quad \text{and} \quad kL^*_2 = L_2 + \frac{1}{nk},
\]  
(38)and
\[
kL^*_{11} = L_{11}, \quad kL^*_{12} = L_{12}, \quad \text{and} \quad kL^*_{22} = L_{22} - \frac{1}{nk^2},
\]  
(39)where \(L_1, L_2, L_{11}, L_{12}, \) and \(L_{22}\) are the first and second derivatives of \(L(c, k)\) given, respectively, by (27), (28), and (30). Using (39), we have
\[
\det \Sigma^*_k = \frac{1}{kL_{11}^*kL_{22}^* - (kL_{12}^*)^2},
\]  
(40)evaluated at the mode \((\hat{c}^*_2, \hat{k}^*_2)\) of \(kL^*(c, k)\), which may be determined by using (38). The Bayes estimator of the parameter \(k\) is, by using (24), (31), and (40), given by
\[
\hat{k}_{BT} = \left( \frac{L_{11}L_{22} - (L_{12})^2}{kL_{11}^*kL_{22}^* - (kL_{12}^*)^2} \right)^{1/2} \frac{\hat{k}_2^* q_2 \left( \hat{c}_2^*, \hat{k}_2^* \mid \hat{c}_D, \hat{k}_D \right)}{q_2 \left( \hat{c}_D, \hat{k}_D \mid \hat{c}, \hat{k} \right)},
\]  
(41)
where \( q_2(c, k | x) \) is the posterior density given by (19) evaluated at the modes \((\hat{c}_2, \hat{k}_2)\) and \((\hat{c}_D, \hat{k}_D)\) of the functions \( rL^*(c, k) \) and \( L(c, k) \), respectively.

(iii) When \( \phi(c, k) = r(t) = (1 + t^c)^{-k} \), equation (26) takes the form

\[
L^*(c, k) = L(c, k) - \frac{k}{n} \ln(1 + t^c).
\]  

The first and second derivatives of \( rL^*(c, k) \) with respect to \( c \) and \( k \) are

\[
rL_1^* = L_1 - \frac{k}{n} \frac{t^c \ln t}{1 + t^c} \quad \text{and} \quad rL_2^* = L_2 - \frac{1}{n} \ln(1 + t^c),
\]

and

\[
rL_{11}^* = L_{11} - \frac{k t^c}{n} \left( \frac{\ln t}{1 + t^c} \right)^2, \quad rL_{12}^* = L_{12} - \frac{t^c}{n} \left( \frac{\ln t}{1 + t^c} \right), \quad \text{and} \quad rL_{22}^* = L_{22}.
\]

where \( L_1, L_2, L_{11}, L_{12}, \) and \( L_{22} \) are the first and second derivatives of \( L(c, k) \) given, respectively, by (27), (28), and (30). Using (44), we have

\[
det \Sigma_r^* = \frac{1}{rL_{11}^* rL_{22}^* - (rL_{12}^*)^2},
\]

evaluated at the mode \((\hat{c}_2, \hat{k}_2)\) of \( rL^*(c, k) \), which may be determined by using (43). The Bayes estimator of \( r(t) \) is, by using (24), (31), and (45), given by

\[
\hat{r}_B(t) = \left( \frac{L_{11} L_{22} - (L_{12})^2}{rL_{11}^* rL_{22}^* - (rL_{12}^*)^2} \right)^{1/2} \left( 1 + t^{c_2} \right)^{-k_2} \frac{q_2(\hat{c}_2, \hat{k}_2 | x)}{q_2(\hat{c}_D, \hat{k}_D | x)},
\]

where \( q_2(c, k | x) \) is the posterior density given by (19) evaluated at the modes \((\hat{c}_2, \hat{k}_2)\) and \((\hat{c}_D, \hat{k}_D)\) of the functions \( rL^*(c, k) \) and \( L(c, k) \), respectively.

4. SIMULATION STUDY AND COMPARISONS

In the following, a simulation algorithm used to generate progressive Type II censored samples from the Burr Type XII distribution is introduced, and the Bayes estimates of the Burr parameters \( c \) and \( k \) and the reliability function \( r(t) \) are compared with their corresponding ML estimates via Monte Carlo simulation study.

4.1. Simulation Algorithm

Applying the algorithms of Balakrishnan and Sandhu [11] and Aggarwala and Balakrishnan [13], the following steps are used to generate a progressive Type II censored sample from the Burr \((c, k)\) distribution.

1. Generate \( m \) independent \( U(0, 1) \) random variables \( W_1, W_2, \ldots, W_m \).
2. For given values of the progressive censoring scheme \( R_1, R_2, \ldots, R_m \), set

\[
V_i = W_i^{1/(1 + R_1 + R_2 + \ldots + R_{i-1} + R_{i+1} + \ldots + R_m)}, \quad i = 1, 2, \ldots, m.
\]

3. Set \( U_i = 1 - (V_m V_{m-1} \ldots V_{m-i+1}), \) \( i = 1, 2, \ldots, m; \) then \( U_1, U_2, \ldots, U_m \) is a progressive Type II censored sample of size \( m \) from \( U(0, 1) \).
4. Thus, for given values of parameters \( c \) and \( k \), \( X_i = [(1 - U_i)^{-1/k} - 1]^{1/c}, \) \( i = 1, 2, \ldots, m, \) is the required progressive Type II censored sample of size \( m \) from Burr \((c, k)\) distribution.
4.2. Comparison of the Estimates

The ML and Bayes estimates of the two parameters \( c \) and \( k \) and the reliability function \( r(t) \) are compared via Monte Carlo simulation study according to the following steps.

1. For given values of the prior parameters \( (\gamma, \delta) \), generate \( c \) from the gamma distribution whose PDF is given by (16).

2. For a given vector \( (\alpha, \beta, c) \), \( c \) is the value obtained in Step (1), generate \( k \) from the conditional gamma distribution with PDF given by (15).

3. Using \( (c, k) \), obtained in (1) and (2), generate a progressive Type II censored sample of size \( m \) with given values of \( R_i \), \( i = 1, 2, \ldots, m \), from the Burr Type XII distribution whose PDF is given by (1) according to the above simulation algorithm.

4. The ML estimate of the parameter \( c \) is computed by solving the nonlinear equation (10) using the "ZSPOW" routine from the IMSL [25]. Substituting this estimate in equation (9) would yield the ML estimates of both the parameter \( k \) and the reliability function \( r(t) \), for some given value of \( t \). The corresponding Bayes estimates given, respectively, by (36), (41), and (51) are also computed. Then, the squared deviations \( (\hat{c} - c)^2 \), \( (\hat{k} - k)^2 \), and \( (\hat{r}(t) - r(t))^2 \) are obtained, where \( \hat{\cdot} \) stands for either ML or Bayes estimate.

For 1000 repetitions, the estimated risks (ER) of the different estimators are computed as the average of their squared deviations. Table 2 displays the estimated risks of the ML and Bayes estimators of \( c \), \( k \), and \( r(0.5) \). Three different cases of \( m \) and the censoring scheme, \( R_i \), \( i = 1, 2, \ldots, m \), as shown in Table 1, with two different vectors of the prior parameters \( (\alpha, \beta, \gamma, \delta) \) are considered in Table 2. For the sake of comparison, the values of \( m \) and \( R_i \)'s are chosen such that \( \sum R_i = n - m = 10 \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( m )</th>
<th>( R_i, i = 1, 2, \ldots, m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>0 2 0 0 0 1 2 0 0 0 1 0 0 0 3</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0 1 0 0 0 0 2 0 0 0 2 0 0 0 1 0 0 1</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>1 0 2 0 0 1 0 2 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1</td>
</tr>
</tbody>
</table>

Table 1. The sample size \( m \) and censoring scheme \( R_i, i = 1, 2, \ldots, m \).

<table>
<thead>
<tr>
<th>( (\alpha, \beta, \gamma, \delta) )</th>
<th>Case</th>
<th>( \text{ER}(\hat{c}_{ML}) )</th>
<th>( \text{ER}(\hat{c}_{BT}) )</th>
<th>( \text{ER}(\hat{k}_{ML}) )</th>
<th>( \text{ER}(\hat{k}_{BT}) )</th>
<th>( \text{ER}(r_{ML}(t)) )</th>
<th>( \text{ER}(r_{BT}(t)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (5, 0.5, 0.5, 6) )</td>
<td>1</td>
<td>1.6330</td>
<td>0.3737</td>
<td>1.7079</td>
<td>0.1115</td>
<td>0.00443</td>
<td>0.00323</td>
</tr>
<tr>
<td>2</td>
<td>1.6630</td>
<td>0.3314</td>
<td>0.2399</td>
<td>0.0962</td>
<td>0.00295</td>
<td>0.00268</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.5358</td>
<td>0.2609</td>
<td>0.1106</td>
<td>0.0582</td>
<td>0.00193</td>
<td>0.00172</td>
<td></td>
</tr>
<tr>
<td>( (6, 1, 0.6, 8) )</td>
<td>1</td>
<td>2.1465</td>
<td>0.5851</td>
<td>1.7674</td>
<td>0.1778</td>
<td>0.00202</td>
<td>0.00204</td>
</tr>
<tr>
<td>2</td>
<td>1.3394</td>
<td>0.6390</td>
<td>1.2073</td>
<td>0.1330</td>
<td>0.00184</td>
<td>0.00149</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.8026</td>
<td>0.4212</td>
<td>0.3314</td>
<td>0.0889</td>
<td>0.00122</td>
<td>0.00105</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Estimated risks (ER) of the estimates of \( c, k, \) and \( r(t) \).

5. CONCLUDING REMARKS

(1) The progressive Type II censored scheme is a general one. When \( R_i = 0, i = 1, 2, \ldots, m \), it reduces to the general order statistics case. Further, when \( R_i = 0, i = 1, 2, \ldots, m - 1 \), and \( R_m = n - m \), it reduces to a Type 2 censored sample.

(2) It can be seen from Table 2 that the Bayes estimates are better than their corresponding ML estimates, for the considered cases of the sample size \( m \) and the prior parameters \( \alpha, \beta, \gamma, \) and \( \delta \). This is not surprising since the parameters \( c \) and \( k \) are assumed to be random
variables rather than unknown constants. However, more investigations are needed to see
the robustness of the choice of the prior in this situation.

(3) It has been verified by Tierney and Kadane [1] that the error in using their approximation
is of order $O(m^{-2})$. So, for relatively small sample size $m$, Tierney and Kadane approxima-
tion works very well as compared with the ML estimates. Obviously, both of the Bayes
and ML estimates become better when the progressive sample size $m$ becomes large.

(4) Different values of the sample size $m$ and the censoring scheme $R_s$, rather than those
appearing in the above two tables, have been considered but did not change the previous
conclusion.

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