Available online at www.sciencedirect.com

SCIENCE DIRECT

Discrete Mathematics 299 (2005) 172–179

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

A characterization of pancyclic complements of line graphs☆

Jiping Liu

Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB., Canada T1K 3M4

Received 26 August 2003; received in revised form 23 February 2004; accepted 23 February 2004

Abstract

We characterize graphs G such that the complements of their line graphs are pancyclic.

Keywords: Line graphs; Complement graphs; Hamiltonian; Pancyclic; Bi-pancyclic

1. Introduction

By a graph G we mean a simple finite undirected graph. The vertex set of G is $V(G)$ and the edge set is $E(G)$. The *order*, *size*, and *maximum degree* of G are denoted by $p(G)(=|V(G)|)$, $q(G)(=|E(G)|)$ and $\Delta(G)$, respectively.

We use $N_G(u)$ to denote the set of vertices adjacent to u in G, and $E_G(u)$ to denote the set of edges incident with u in G. $d_G(u)$ is the degree of u in G.

The *line graph* of G, denoted by $L(G)$, has vertex set $E(G)$; two vertices e_1 and e_2 of $L(G)$ are adjacent if and only if they are incident as edges in G. The *complement graph* $\overline{G}$ of G has the same vertex set $V(G)$ as G; two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Thus $\overline{L(G)}$ is the complement of $L(G)$.

If w is an isolated vertex of G, then $L(G-w)=L(G)$ and $\overline{L(G-w)}=\overline{L(G)}$. Hence we can always assume that there are no isolated vertices in G when we consider $L(G)$ or $\overline{L(G)}$.

☆ Partially supported by NSERC of Canada.

E-mail address: liu@cs.uleth.ca.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.02.024

A graph is *pancyclic* if there is a cycle of length k for each $3 \leqslant k \leqslant p$. If a graph G is a bipartite graph, it does not contain any odd cycles. But if a bipartite graph $G = (X, Y)$ is *balanced*, that is, the two color classes X and Y have the same number of vertices, then we can expect cycles of length $2k$ for $2 \leqslant k \leqslant p/2$. Therefore, we define a bipartite graph to be *bi-pancyclic* if it contains cycles of length $2k$ for $2 \leqslant k \leqslant p/2$.

If G_1 and G_2 are vertex disjoint, then we use $G_1 + G_2$ to denote the disjoint union of the two graphs. We also use nG to denote the vertex disjoint union of n copies of G. Notation and terminology not mentioned above can be found in [3].

Nebeský [8] showed the following result.

Theorem 1.1. *If G is a graph of order $\geqslant 5$, then at least one of the following statements holds*:

(a) *G is connected and $L(G)$ is Hamiltonian*.
(b) *$\bar{G}$ is connected and $L(\overline{G})$ is Hamiltonian*.

This result was improved in [9] as follows.

Theorem 1.2. *If G is a graph of order $\geqslant 6$, then at least one of the following statements holds*:

(a) *G is connected and $L(G)$ is pancyclic*.
(b) *$\bar{G}$ is connected and $L(\overline{G})$ is pancyclic*.

Though at least one of $L(G)$ and $L(\overline{G})$ is Hamiltonian or pancyclic, it is NP-complete to decide which one is. In [7], we studied the Hamiltonian problem for the class of complements of line graphs, and obtained the following characterization.

Theorem 1.3. *The graph $\overline{L(G)}$ is Hamiltonian if and only if $\Delta(G) \leqslant q/2$, $d_G(u) + d_G(v) \leqslant q - 1$ for any two adjacent vertices u and v, and G is not one of 22 graphs in Fig. 1*.

This characterization provides a linear time recognition algorithm determining graphs G such that $\overline{L(G)}$ are Hamiltonian.

In this paper, we study the pancyclic property (which is stronger than the Hamiltonian property) of the class of complements of graphs. We characterize all graphs such that the complement graphs of their line graphs are pancyclic or bi-pancyclic. Again, this characterization provides a linear recognition algorithm determining graphs G such that $\overline{L(G)}$ are pancyclic.

2. The main results

Let $\mathscr{B} = \{2P_3, 2K_3, 2K_{1,3}, K_3 + K_{1,3}, G_1, G_2\}$ (see Fig. 2).

We have the following characterization of bi-pancyclic graphs $\overline{L(G)}$.

Theorem 2.1. *$\overline{L(G)}$ is bi-pancyclic if and only q is even and G is in $\mathscr{B}$ when $q = 4, 6$, and there exist two nonadjacent vertices of degree $q/2$ when $q \geqslant 8$.*

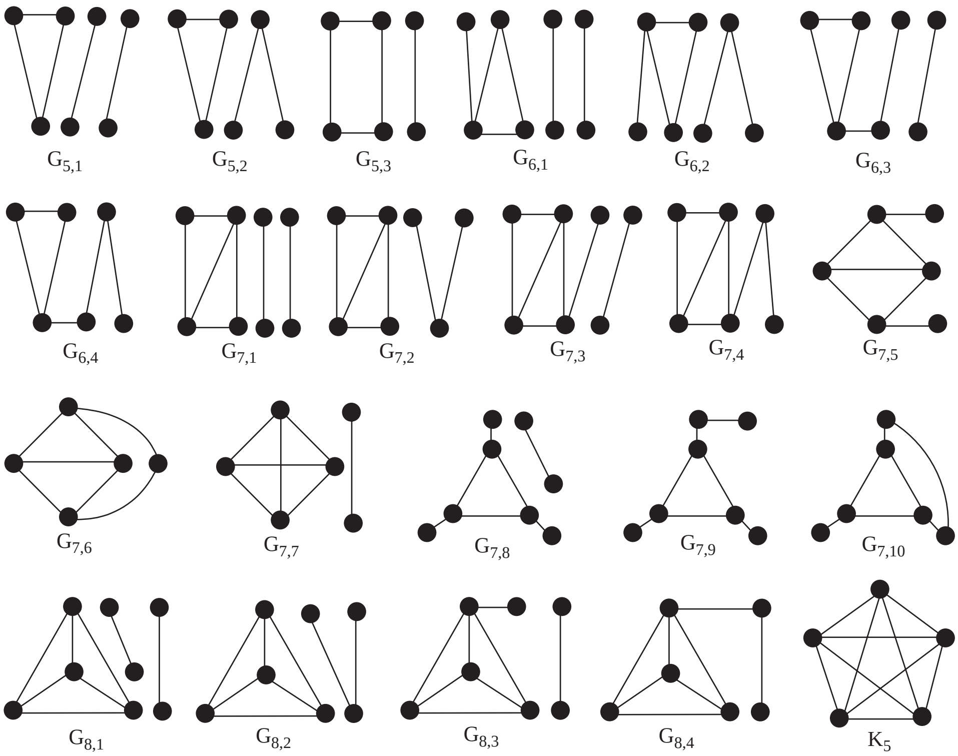

Fig. 1. $\mathscr{P}_4$.

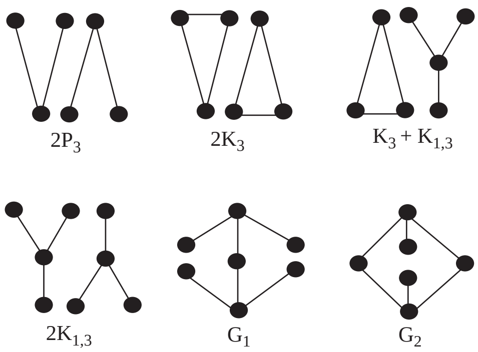

Fig. 2. $\mathscr{B}$.

Proof. If $\overline{L(G)}$ is bi-pancyclic, then $\overline{L(G)}$ must be a balanced bipartite graph. So the vertex set of $\overline{L(G)}$ can be partitioned into two independent sets of equal cardinality. If $q=4$, then $\overline{L(G)}$ is a 4-cycle. Therefore, $G=2P_3 \in \mathscr{B}$. If $q=6$, then corresponding to each independent set in $\overline{L(G)}$ is either a triangle or a 3-star in G. It is easy to verify that a triangle cannot

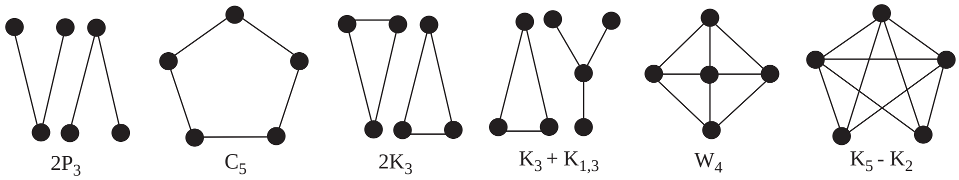

Fig. 3. $\mathscr{P}_3$.

intersect any triangle or star in G, and two stars can intersect in G at a maximum of two vertices of degree one. Therefore, $G \in \mathscr{B}$. Suppose now that $q \geqslant 8$. Then each independent set corresponds to a $q/2$-star in G, and the two $q/2$-stars share no common edges as $\overline{L(G)}$is a balanced bipartite graph and contains no isolated vertex. This implies that the two centers of the two stars are nonadjacent and have degree $q/2$.

To prove the sufficiency, suppose G is a graph in $\mathscr{B}$ or there are two distinct vertices u and v such that $d(u) = d(v) = q/2$. It is easy to check that $\overline{L(G)}$is bi-pancyclic if $G \in \mathscr{B}$. Now suppose that $q \geqslant 8$ and let $N_G(u) = \{x_1, x_2, \ldots, x_{q/2}\}$ and $N_G(v) = \{y_1, y_2, \ldots, y_{q/2}\}$. Then G is the graph obtained by identifying some of the vertices in $N_G(u)$ and some vertices in $N_G(v)$. Without loss of generality, we assume that G is obtained by identifying the pairs of vertices $(x_1, y_1), \ldots, (x_k, y_k)$ for some k, or we assume that G is a disjoint union of the two stars. Let $K_{q/2,q/2}$ be a complete bipartite graph on the vertex set $N_G(u) \cup N_G(v)$ with the bipartition $(N_G(u), N_G(v))$. Let $M = \{x_1y_1, \ldots, x_{q/2}y_{q/2}\}$ and $H_q = K_{q/2,q/2} - M$. Then H_q is a spanning subgraph of $\overline{L(G)}$. Therefore, $\overline{L(G)}$is bi-pancyclic if H_q is. In the following, we are going to show that H_q is bi-pancyclic by induction on q.

If $q = 8$, it is easy to verify that H_8 contains cycles of lengths 4, 6 and 8, and hence H_8 is bi-pancyclic. Suppose that H_q is bi-pancyclic for $q \geqslant 8$. Then H_q contains cycles of lengths $4, 6, \ldots, q$. We need to show that H_{q+2} contains all cycles of lengths $4, 6, \ldots, q, q+2$. It is obvious that H_q is a subgraph of H_{q+2}, thus H_{q+2} contains cycles of lengths $4, 6, \ldots, q$ by the induction hypothesis. But it is well known that H_{q+2} is Hamiltonian (see Chvtal's result in Exercise 4.2.6 of Bondy [3]), and hence it contains a $(q+2)$-cycle. This proves that H_{q+2} contains all possible even length cycles. □

Let $\mathscr{P}_1 = \{G : \Delta(G) > q/2\}$, $\mathscr{P}_2 = \{G : \exists\, u \neq v \in V(G),\ d_G(u) + d_G(v) \geqslant q\}$, $\mathscr{P}_3 = \{2P_3, C_5, 2K_3, K_3 + K_{1,3}, W_4, K_5 - K_2\}$ (see Fig. 3) and let $\mathscr{P}_4$ be the set of graphs in Fig. 1.

The following theorem characterizes the pancyclic graphs $\overline{L(G)}$.

Theorem 2.2. *$\overline{L(G)}$is pancyclic if and only if G is not in $\bigcup_{i=1}^{4} \mathscr{P}_i$.*

Proof. If $\overline{L(G)}$is pancyclic, then $\overline{L(G)}$is Hamiltonian. By Theorem 1.3, we have $\Delta(G) \leqslant q/2$, $d_G(u) + d_G(v) \leqslant q - 1$ for any edge uv of G, and G is not a graph in Fig. 1. If there are two non-adjacent vertices u and v such that $d_G(u) = d_G(v) = q/2$, then $\overline{L(G)}$is bipartite and hence $\overline{L(G)}$is not pancyclic. It is also easy to verify that for $G \in \mathscr{P}_3$ the graph $\overline{L(G)}$is not pancyclic since there are no cycles of length 3 in $\overline{L(G)}$. This proves the necessity of the theorem.

We prove the sufficiency by induction on the number of edges in G.

Let G be a graph not in $\cup_{i=1}^{4}\mathscr{P}_i$.

If $q=3$, then $G=3K_2$. We have $\overline{L(G)}=K_3$, which is pancyclic.

If $q=4$, then G is either P_3+2K_2 or $4K_2$. We have $\overline{L(P_3+2K_2)}=K_4-K_2$ and $\overline{L(4K_2)}=K_4$, which are pancyclic.

If $q=5$, then G is one of the graphs in $\{5P_2, 3P_2+P_3, 2P_2+P_4, P_2+2P_3, P_2+P_5, P_3+P_4, P_6\}$. We have that $\overline{L(P_6)}$is a 5-cycle with a chord, which is pancyclic. If G is any of the remaining six graphs, then $\overline{L(P_6)}$is a spanning subgraph of $\overline{L(G)}$, thus, $\overline{L(G)}$is pancyclic.

Now we assume $q\geqslant 6$. If there is an edge uv such that $G-uv\notin\bigcup_{i=1}^{4}\mathscr{P}_i$, then $\overline{L(G-uv)}$is pancyclic by induction hypothesis. Thus $\overline{L(G-uv)}$contains cycles of length $3\leqslant k\leqslant q-1$. Notice that $\overline{L(G-uv)}$is a subgraph of $\overline{L(G)}$and $\overline{L(G)}$is Hamiltonian by Theorem 1.3. The Hamilton cycle is of length q. Therefore, $\overline{L(G)}$contains cycles of length $3\leqslant k\leqslant q$ and hence it is pancyclic. We may assume that $G-uv\in\bigcup_{i=1}^{4}\mathscr{P}_i$ for any edge $uv\in E(G)$.

Let uv be an edge of G satisfying the following conditions.

1. $d_G(u)=\Delta(G)$ and $d_G(u)+d_G(v)$ is maximum among all the edges;
2. subject to condition 1, the edge uv is contained in the maximum number of 3-cycles in G.

If $G-uv\in\mathscr{P}_1$, then there is a vertex x in $G-uv$ of degree larger than $(q-1)/2$. If x is u or v, then $d_G(u)\geqslant d_{G-uv}(x)+1>q/2$ by the choice of u and v. So $G\in\mathscr{P}_1$, which is a contradiction. If $x\neq u$, then $d_G(u)+d_G(x)>(q-1)/2+(q-1)/2=q-1$ implying that $G\in\mathscr{P}_2$, which again is a contradiction.

If $G-uv\in\mathscr{P}_2-\mathscr{P}_1$, then there are two vertices x and y in $G-uv$ such that $d_{G-uv}(x)+d_{G-uv}(y)\geqslant q-1$. This implies that $d_{G-uv}(x)=d_{G-uv}(y)=(q-1)/2$ since $G-uv\notin\mathscr{P}_1$. Therefore, q is odd. If x and y are not adjacent in $G-uv$, then $G-uv$ is the graph obtained by identifying some degree one vertices of two $(q-1)/2$ stars with centers x and y, respectively. If $u\notin\{x,y\}$, then $d_G(u)\leqslant 3$. In this case, G must be either $G_{7,5}$ or $G_{7,6}$ as shown in Fig. 1. Thus $G\in\mathscr{P}_4$, which is a contradiction. If $u\in\{x,y\}$, then by the selection of u, $d_G(u)=(q+1)/2>q/2$, which is also a contradiction since $G\notin\mathscr{P}_1$. Suppose now that xy is an edge of $G-uv$. If uv and xy are incident edges in G, then $d_G(x)+d_G(y)\geqslant q$, which is a contradiction as $G\notin\mathscr{P}_2$. So uv and xy are independent edges in G. Since $G\notin\mathscr{P}_1$, $d_G(u)\leqslant q/2$. This further implies that $d_G(u)=(q-1)/2$. Moreover, by the selection of v, $d_G(v)=(q-1)/2$ (otherwise, select x and y instead of u and v). Counting the number of edges incident with the four vertices x, y, u, v, we have $q\geqslant d_G(u)+d_G(v)+d_{G-uv}(x)+d_{G-uv}(y)-\varepsilon\geqslant 2(q-1)-\varepsilon$, where ε is the number of edges induced by the four vertices u, v, x and y. Hence $6\leqslant q\leqslant 2+\varepsilon\leqslant 8$. Recall that q is odd, hence $q=7$. If $\varepsilon=6$, then the four vertices u, v, x, y induce a K_4 and $d_G(u)=4>7/2=q/2$, which is a contradiction. Therefore, $\varepsilon=5$ and the four vertices induce a subgraph isomorphic to K_4-K_2 in G. Hence $G\in\{G_{7,1}, G_{7,2}, G_{7,3}, G_{7,4}, G_{7,5}, G_{7,6}\}\subset\mathscr{P}_4$ (see Fig. 1), which is a contradiction.

Note that $G-uv$ has size at least 5. Therefore $G-uv\neq 2P_3$. Now we only need to check the cases when $G-uv\in(\mathscr{P}_3-\{2P_3\})\cup\mathscr{P}_4$. We have either $d_G(u)=\Delta(G-uv)$ or $d_G(u)=\Delta(G-uv)+1$ since $d_G(u)=\Delta(G)$.

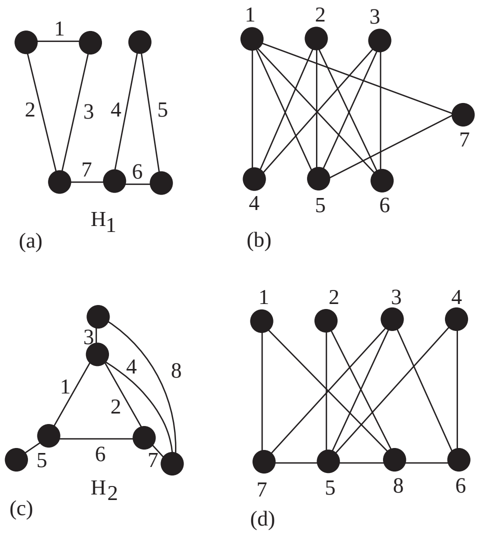

Fig. 4.

Suppose first that $d_G(u) = \Delta(G - uv)$. From the selection criterion 1 of u and v ($d_G(u) = \Delta(G)$ and $d_G(u) + d_G(v)$ is the maximum), $G - uv$ can be only one of the graphs in $A = \{K_3 + K_{1,3}, K_5 - K_2, G_{5,2}, G_{5,2}, G_{6,2}, G_{6,3}, G_{6,4}, G_{7,1}, G_{7,2}, G_{7,4}\}$. By considering further the selection criterion 2 of u and v (the edge uv is contained in the maximum number of triangles), we can reduce the set A to $\{K_5 - K_2, G_{5,2}, G_{7,1}, G_{7,2}\}$. We have $(K_5 - K_2) + uv = K_5 \in \mathscr{P}_4$, $G_{5,2} + uv = 2K_3 \in \mathscr{P}_3$, $G_{7,1} + uv = G_{8,1} \in \mathscr{P}_4$, and $G_{7,2} + uv = G_{8,2} \in \mathscr{P}_4$. In any case, we obtain a contradiction.

Now we consider the case when $d_G(u) = \Delta(G - uv) + 1$. From the selection criterion 1 of u and v ($d_G(u) = \Delta(G)$ and $d_G(u) + d_G(v)$ is the maximum), $G - uv$ can be only one of the graphs in $A = \{C_5, 2K_3, K_3 + K_{1,3}, G_{5,1}, G_{5,2}, G_{5,3}, G_{6,1}, G_{6,2}, G_{6,3}, G_{6,4}, G_{7,2}, G_{7,3}, G_{7,4}, G_{7,5}, G_{7,6}, G_{7,9}, G_{7,10}, G_{8,2}, G_{8,4}\}$. When considering the selection criterion 2 of u and v (uv is contained in the maximum of triangles), A can be further reduced to $\{C_5, 2K_3, K_3 + K_{1,3}, G_{5,2}, G_{5,3}, G_{6,2}, G_{6,3}, G_{6,4}, G_{7,3}, G_{7,4}, G_{7,5}, G_{7,6}, G_{7,10}\}$.

If $G - uv \in \{C_5, K_3 + K_{1,3}, G_{5,2}, G_{5,3}, G_{6,2}, G_{6,4}, G_{7,5}, G_{7,6}\}$, then $d_G(u) + d_G(v) \geqslant q$. Thus $G \in \mathscr{P}_2$, which is a contradiction.

If $G - uv = G_{6,3}$, then $d_G(u) = 4 > 7/3$ and $G \in \mathscr{P}_1$, which is a contradiction. We note that $G_{7,3} + uv = G_{8,3} \in \mathscr{P}_4$, $G_{7,4} + uv = G_{8,4} \in \mathscr{P}_4$.

If $G - uv = 2K_3$, then G is the graph H_1 as shown in Fig. 4 (uv is the edge with label 7), and $\overline{L(H_1)}$is pancyclic. Finally if $G - uv = G_{7,10}$, then G is the graph H_2 as shown in Fig. 4(c) (uv is the edge with label 4). $\overline{L(H_2)}$is the graph as shown in Fig. 4(b). It is easy to check that $\overline{L(H_2)}$is pancyclic.

This completes the proof. □

It is well known that both the Hamiltonian problem and Hamilton-path problem are NP-complete for general graphs. For any graph G, let G' be the graph obtained by adding to

G a new vertex v and edges vu for all $u \in V(G)$. It is easy to see that G has a Hamilton path iff G' is pancyclic. Moreover, this transformation is linear. This proves the following results, which may be not new.

Theorem 2.3. *The problem that decides whether G is pancyclic on a given graph G is NP-complete.*

For the class of complements of line graphs, our characterization provides the following linear time algorithm recognizing pancyclic complements of line graphs.

Theorem 2.4. *The problem that decides whether $\overline{L(G)}$ is pancyclic or bi-pancyclic on a given graph G is linear-time solvable.*

Proof. It takes time $O(p)$ to determine whether G contains two nonadjacent vertices of degree $q/2$ or whether $G \in \mathscr{P}_1$; it takes time $O(p+q)$ to verify whether $G \in \mathscr{P}_2$; it takes constant time to check whether a given graph G is in $\mathscr{B}$, $\mathscr{P}_3$ or $\mathscr{P}_4$. The input of G takes time $O(p+q)$. Now the theorem follows from Theorems 2.1 and 2.2. □

The concept of pancyclic graph was first introduced by Bondy. He also made the following "meta-conjecture": *every condition which implies that a graph is Hamiltonian also implies that it is pancyclic, with the possible exception of a simple family of exceptional graphs.* Although this meta-conjecture is sometimes false, it turns out to be true for the family of complements of line graphs.

Theorem 2.5. *$\overline{L(G)}$ is pancyclic or bi-pancyclic if and only if $\overline{L(G)}$ is Hamiltonian and $G \notin \{C_5, W_4, K_5 - K_2, K_{2,3}\}$.*

Proof. If $\overline{L(G)}$ is pancyclic or bi-pancyclic, then $G \notin \{C_5, W_4, K_5 - K_2, K_{2,3}\}$, and $\overline{L(G)}$ is Hamiltonian.

Conversely, if $\overline{L(G)}$ is Hamiltonian and $G \notin \{C_5, W_4, K_5 - K_2, K_{2,3}\}$, then $G \notin \mathscr{P}_1 \cup \mathscr{P}_4$. If $\overline{L(G)}$ is not a balanced bipartite graph, then $G \notin \mathscr{P}_3$. Note that $d_G(u) + d_G(v) \leqslant q - 1$ for any edge uv in G by Theorem 1.3, so if $G \in \mathscr{P}_2$, then there are nonadjacent vertices u and v such that $d_G(u) + d_G(v) \leqslant q$. This implies that $d_G(u) = d_G(v) = q/2$ and thus $\overline{L(G)}$ is a balanced bipartite graph. This is a contradiction. Therefore, $G \notin \mathscr{P}_2$. By Theorem 2.2, $\overline{L(G)}$ is pancyclic.

If G is a balanced bipartite graph, then there are two nonadjacent vertices of degree $q/2$ for $q \geqslant 8$ or $G \in \mathscr{B} \cup \{K_{2,3}\}$ when $q \leqslant 6$. But $G \neq K_{2,3}$, thus $\overline{L(G)}$ is bi-pancyclic. □

Combining Theorems 1.3 and 2.5 we have the following characterization.

Corollary 2.6. *$\overline{L(G)}$ is pancyclic or bi-pancyclic if and only if $\Delta(G) \leqslant q/2$, $d_G(u) + d_G(v) \leqslant q - 1$ for any two adjacent vertices u and v, and $G \notin \mathscr{P}_4 \cup \{C_5, W_4, K_5 - K_2, K_{2,3}\}$.*

Acknowledgements

The author would like to thank Dr. Siming Zhan for many helpful discussions, and the referees for their valuable comments.

References

[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[7] J. Liu, A characterization of Hamiltonian complement graphs of line graphs, submitted for publication.
[8] L. Nebeský, On hamiltonian line graphs, Comment. Math. Univ. Carolinae 14 (1973) 107–112.
[9] L. Nebeský, On pancyclic line graphs, Czchoslovak Math. J. 28 (103) (1978) 650–655.