ELSEVIER

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Functional Analysis 262 (2012) 3240–3308

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Free biholomorphic functions and operator model theory ☆

Gelu Popescu

Department of Mathematics, The University of Texas at San Antonio, San Antonio, TX 78249, USA

Received 29 April 2011; accepted 18 January 2012

Available online 31 January 2012

Communicated by D. Voiculescu

Dedicated to the memory of Mircea Reghis

Abstract

Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series in noncommutative indeterminates $Z_1, \dots, Z_n$ such that $f(0) = 0$ and the Jacobian $\det J_f(0) \neq 0$, and let $g = (g_1, \dots, g_n)$ be its inverse with respect to composition. We assume that f and g have nonzero radius of convergence and g is a bounded free holomorphic function on the open unit ball $[B(\mathcal{H})^n]_1$, where $B(\mathcal{H})$ is the algebra of bounded linear operators an a Hilbert space $\mathcal{H}$. In this paper, several results concerning the noncommutative multivariable operator theory on the unit ball $[B(\mathcal{H})^n]_1^-$ are extended to the noncommutative domain

$$\mathbb{B}_f(\mathcal{H}) := \{X \in B(\mathcal{H})^n\colon\ g(f(X)) = X \text{ and } \|f(X)\| \leqslant 1\}$$

for an appropriate evaluation $X \mapsto f(X)$. We develop an operator model theory and dilation theory for $\mathbb{B}_f(\mathcal{H})$, where the associated universal model is an n-tuple $(M_{Z_1}, \dots, M_{Z_n})$ of left multiplication operators acting on a Hilbert space $\mathbb{H}^2(f)$ of formal power series. All the results of this paper have commutative versions.

Keywords: Formal power series; Free holomorphic function; Inverse mapping theorem; Model theory; Invariant subspaces; Noncommutative Hardy space; Poisson transform; Characteristic function; Curvature invariant; Commutant lifting

☆ Research supported in part by an NSF grant.

E-mail address: gelu.popescu@utsa.edu.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2012.01.012

Contents

0. Introduction

In the last sixty years, the study of the unit ball of the algebra $B(\mathcal{H})$, of all bounded linear operators on a Hilbert space, has generated the celebrated Sz.-Nagy–Foiaş theory of contractions [37] and has had profound implications in mathematics and applied mathematics. In the last three decades, a *free analogue* of Sz.-Nagy–Foiaş theory on the unit ball of $B(\mathcal{H})^n$ has been pursued by the author and others (see [29,31], and the references therein). This theory has already had remarkable applications in complex interpolation on the unit ball of $\mathbb{C}^n$, multivariable prediction and entropy optimization, control theory, systems theory, scattering theory, and wavelet theory. On the other hand, it has been a source of inspiration for the development of several other areas of research such as tensor algebras over C^*-correspondences and free semigroup (resp. semigroupoid, graph) algebras (see [12–14]).

The present paper is an attempt to find large classes of noncommutative multivariable functions $g : \Omega \subset [B(\mathcal{H})^n]_1^- \to B(\mathcal{H})^n$ for which a reasonable operator model theory and dilation theory can be developed for the noncommutative domain $g(\Omega)$. In other words, we want to *transfer* the free analogue of Sz.-Nagy–Foiaş theory from the unit ball $[B(\mathcal{H})^n]_1$ to other noncommutative domains in $B(\mathcal{H})^n$, using appropriate maps.

In Section 1, we obtain inverse mapping theorems for formal power series in noncommutative indeterminates $Z_1, \ldots, Z_n$, and also for free holomorphic functions. More precisely, we show that an n-tuple $f = (f_1, \ldots, f_n)$ of formal power series with $f(0) = 0$ has an inverse $g = (g_1, \ldots, g_n)$ with respect to composition if and only if the Jacobian $\det J_f(0) \neq 0$. If, in addition, f and g have nonzero radius of convergence, we prove that there are open neighborhoods D and G of 0 in $B(\mathcal{H})^n$ such that $f|_D : D \to G$ and $g|_G : G \to D$ are free holomorphic functions inverses to each other.

Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series in indeterminates $Z_1, \ldots, Z_n$ such that $f(0) = 0$ and $\det J_f(0) \neq 0$, and assume that f and its inverse $g = (g_1, \ldots, g_n)$ have nonzero radius of convergence. By re-scaling, we can assume without loss of generality that g is a bounded free holomorphic function on the open unit ball

$$\left[B(\mathcal{H})^n\right]_1 := \left\{X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon\ X_1X_1^* + \cdots + X_nX_n^* < I\right\}.$$

We consider the noncommutative domain

$$\mathbb{B}_f(\mathcal{H}) := \left\{X \in B(\mathcal{H})^n \colon\ g\big(f(X)\big) = X \text{ and } \big\|f(X)\big\| \leqslant 1\right\}$$

for an appropriate evaluation $X \mapsto f(X)$ and using the functional calculus for row contractions to define $g(f(X))$. We remark that the domain above makes sense if we remove the condition $f(0) = 0$ and ask instead that f and g be n-tuples of noncommutative polynomials or certain free holomorphic functions. In this paper, several results concerning noncommutative multivariable operator theory on the unit ball $[B(\mathcal{H})^n]_1^-$ are extended to the noncommutative domain $\mathbb{B}_f(\mathcal{H})$.

In Section 2, we introduce three classes of n-tuples $f = (f_1, \ldots, f_n)$ for which an operator model theory and dilation theory for the domain $\mathbb{B}_f(\mathcal{H})$ will be developed. These classes consist of noncommutative polynomials, formal power series with $f(0) = 0$, and free holomorphic functions, respectively. When f belongs to any of these classes, we say that it has the *model property*. In this case, each domain $\mathbb{B}_f$ has a universal model $(M_{Z_1}, \ldots, M_{Z_n})$ of multiplication operators acting on a Hilbert space $\mathbb{H}^2(f)$ of formal power series.

In Section 3, we show that $T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is a pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$ if and only if there exists a Hilbert space $\mathcal{D}$ and a co-invariant subspace $\mathcal{M} \subseteq \mathbb{H}^2(f) \otimes \mathcal{D}$ under $M_{Z_1} \otimes I_\mathcal{D}, \ldots, M_{Z_n} \otimes I_\mathcal{D}$ such that the n-tuple $(T_1, \ldots, T_n)$ is unitarily equivalent to

$$\big(P_\mathcal{M}(M_{Z_1} \otimes I_\mathcal{D})|_\mathcal{M}, \ldots, P_\mathcal{M}(M_{Z_n} \otimes I_\mathcal{D})|_\mathcal{M}\big).$$

The C^*-algebra $C^*(M_{Z_1}, \ldots, M_{Z_n})$ turns out to be irreducible and

$$M_{Z_i}^* M_{Z_j} = \langle Z_j, Z_i \rangle_{\mathbb{H}^2(f)} I_{\mathbb{H}^2(f)}, \quad i, j \in \{1, \ldots, n\}.$$

If, in addition, f has radial approximation property, that is, there is $\delta \in (0, 1)$ such that rf has the model property for any $r \in (\delta, 1)$, we prove that, for any $T := (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$, there is a unique unital completely contractive linear map

$$\Psi_{f,T} : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{H})$$

such that

$$\Psi_{f,T}\big(M_{Z_\alpha} M_{Z_\beta}^*\big) = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+,$$

where $T_\alpha := T_{i_1} \cdots T_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k}$ is a word in the free semigroup $\mathbb{F}_n^+$ with generators $g_1, \ldots, g_n$. As a consequence we obtain a minimal dilation of T which is unique up to an isomorphism.

We define the domain algebra $\mathcal{A}(\mathbb{B}_f)$ as the norm-closure of all polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity. Under natural conditions on f, we use Paulsen's similarity result [16] to obtain a characterization for the completely bounded representations of $\mathcal{A}(\mathbb{B}_f)$. We also show that the set $M_{\mathcal{A}(\mathbb{B}_f)}$ of all characters of $\mathcal{A}(\mathbb{B}_f)$ is homeomorphic to $g(\overline{\mathbb{B}}_n)$, where $\overline{\mathbb{B}}_n$ is the closed unit ball of $\mathbb{C}^n$.

In Section 4, we provide a Beurling [5] type characterization of the invariant subspaces under the multiplication operators $M_{Z_1}, \ldots, M_{Z_n}$ associated with the noncommutative domain $\mathbb{B}_f$. More precisely, we show that if $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series with the model property, then a subspace $\mathcal{N} \subseteq \mathbb{H}^2(f) \otimes \mathcal{H}$ is invariant under each operator $M_{Z_1} \otimes I_\mathcal{H}, \ldots, M_{Z_n} \otimes I_\mathcal{H}$ if and only if there exists an inner multi-analytic operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ with respect to $M_{Z_1}, \ldots, M_{Z_n}$, i.e., Ψ is an isometry and $\Psi(M_{Z_i} \otimes I_\mathcal{E}) = (M_{Z_i} \otimes I_\mathcal{H})\Psi$ for any $i = 1, \ldots, n$, such that

$$\mathcal{N} = \Psi\big[\mathbb{H}^2(f) \otimes \mathcal{E}\big].$$

Using some of the results of this section and noncommutative Poisson transforms associated with the noncommutative domain $\mathbb{B}_f$, we provide a minimal dilation theorem for pure n-tuples of operators in $\mathbb{B}_f(\mathcal{H})$, which turns out to be unique up to an isomorphism.

In Section 5, we show that the eigenvectors for $M_{Z_1}^*, \ldots, M_{Z_n}^*$ are precisely the noncommutative Poisson kernels associated with the noncommutative domain $\mathbb{B}_f$ at the points in the set

$$\mathbb{B}_f^<(\mathbb{C}) := \{\lambda \in \mathbb{C}^n \colon g(f(\lambda)) = \lambda \text{ and } \|f(\lambda)\| < 1\},$$

that is, the formal power series

$$\Gamma_\lambda := \left(1 - \sum_{i=1}^n |f_i(\lambda)|^2\right)^{1/2} \sum_{\alpha \in \mathbb{F}_n^+} [\overline{f(\lambda)}]_\alpha f_\alpha, \quad \lambda \in \mathbb{B}_f^<(\mathbb{C}).$$

Moreover, they satisfy the equations $M_{Z_i}^* \Gamma_\lambda = \bar{\lambda}_i \Gamma_\lambda$, $i = 1, \ldots, n$. We define the noncommutative Hardy algebra $H^\infty(\mathbb{B}_f)$ to be the WOT-closure of all noncommutative polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity, and show that it coincides with the algebra of bounded left multipliers of $\mathbb{H}^2(f)$. The symmetric Hardy space $\mathbb{H}_s^2(f)$ associated with the noncommutative domain $\mathbb{B}_f$ is defined as the subspace $\mathbb{H}^2(f) \ominus \overline{J_c(1)}$, where J_c is the WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ generated by the commutators

$$M_{Z_i} M_{Z_j} - M_{Z_j} M_{Z_i}, \quad i, j = 1, \ldots, n.$$

We show that $\mathbb{H}_s^2(f) = \overline{\operatorname{span}}\{\Gamma_\lambda \colon \lambda \in \mathbb{B}_f^<(\mathbb{C})\}$ and can be identified with a Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$ of holomorphic functions on $\mathbb{B}_f^<(\mathbb{C})$, namely, the reproducing kernel Hilbert space with reproducing kernel $\Lambda_f : \mathbb{B}_f^<(\mathbb{C}) \times \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ defined by

$$\Lambda_f(\mu, \lambda) := \frac{1}{1 - \sum_{i=1}^n f_i(\mu)\overline{f_i(\lambda)}}, \quad \lambda, \mu \in \mathbb{B}_f^<(\mathbb{C}).$$

The algebra $P_{H_s^2(f)} H^\infty(\mathbb{B}_f)|_{\mathbb{H}_s^2(f)}$ coincides with the WOT-closed algebra generated by the operators $L_i := P_{\mathbb{H}_s^2(f)} M_{Z_i}|_{\mathbb{H}_s^2(f)}$, $i = 1, \ldots, n$, and can be identified with the algebra of all multipliers of the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$. Under this identification the operators $L_1, \ldots, L_n$ become the multiplication operators $M_{z_1}, \ldots, M_{z_n}$ by the coordinate functions. The n-tuple $(L_1, \ldots, L_n)$ turns out to be the universal model for the commutative n-tuples from $\mathbb{B}_f(\mathcal{H})$.

In Section 6, we define the characteristic function of an n-tuple $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ to be a certain multi-analytic operator $\Theta_{f,T} : \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ with respect to $M_{Z_1}, \ldots, M_{Z_n}$, and point out a natural connection with the characteristic function of a row contraction [19]. We present a model for pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ in terms of characteristic functions, and show that the characteristic function is a complete unitary invariant for pure n-tuples of operators in $\mathbb{B}_f(\mathcal{H})$.

Using ideas from [27], we introduce in Section 7 the curvature invariant of $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ by setting

$$\operatorname{curv}_f(T) := \lim_{m \to \infty} \frac{\operatorname{trace}[K_{f,T}^*(Q_{\leqslant m} \otimes I_{\mathcal{D}_{f,T}})K_{f,T}]}{\operatorname{trace}[K_{f,M_Z}^*(Q_{\leqslant m})K_{f,M_Z}]},$$

where $K_{f,T}$ is the noncommutative Poisson kernel associated with T, and $Q_{\leqslant m}$, $m = 0, 1, \ldots,$ is the orthogonal projection of $\mathbb{H}^2(f)$ on the linear span of the formal power series f_α, $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \leqslant m$. We show that the limit exists and provide an index type formula for the curvature in terms of the characteristic function. One of the main goals of this section is to show that the curvature is a complete numerical invariant for the finite rank submodules of the free Hilbert module $\mathbb{H}^2(f) \otimes \mathcal{K}$, where $\mathcal{K}$ is finite dimensional. Here, the Hilbert module structure of $\mathbb{H}^2(f)$ over $\mathbb{C}[Z_1, \ldots, Z_n]$ is defined by the universal model $(M_{Z_1}, \ldots, M_{Z_n})$ by setting

$$p \cdot h := p(M_{Z_1}, \ldots, M_{Z_n})h, \quad p \in \mathbb{C}[Z_1, \ldots, Z_n] \text{ and } h \in \mathbb{H}^2(f).$$

In our setting, the Hilbert module $\mathbb{H}^2(f)$ occupies the position of the rank-one free module in the algebraic theory [11].

In Section 8, we use the commutant lifting theorem for row contractions [18], to deduce an analogue for the pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$. As a consequence, and using the results from Section 5, we solve the Nevanlinna Pick interpolation problem for the noncommutative Hardy algebra $H^\infty(\mathbb{B}_f)$. We show that if $\lambda_1, \ldots, \lambda_m$ are m distinct points in $\mathbb{B}_f^<(\mathbb{C})$ and $A_1, \ldots, A_m \in B(\mathcal{K})$, then there exists $\Phi \in H^\infty(\mathbb{B}_f)\bar{\otimes} B(\mathcal{K})$ such that

$$\|\Phi\| \leqslant 1 \quad \text{and} \quad \Phi(\lambda_j) = A_j, \quad j = 1, \ldots, m,$$

if and only if the operator matrix

$$\left[\frac{I_{\mathcal{K}} - A_i A_j^*}{1 - \sum_{k=1}^n f_k(\lambda_i)\overline{f_k(\lambda_j)}}\right]_{m \times m}$$

is positive semidefinite.

We remark that, using the results from Section 5, we can provide commutative versions for all the results of the present paper. Moreover, a model theory and dilation theory for not necessarily pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ (resp. varieties in $\mathbb{B}_f(\mathcal{H})$) is developed in a sequel to the present paper.

1. Inverse mapping theorem for free holomorphic functions

Initiated in [30], the theory of free holomorphic (resp. pluriharmonic) functions on the unit ball of $B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, has been developed very recently (see [32–34]). Several classical results from complex analysis and hyperbolic geometry have free analogues in this noncommutative multivariable setting.

In this section, we obtain inverse mapping theorems for formal power series in noncommutative indeterminates and for free holomorphic functions. We recall [30] that a free holomorphic functions on the open operatorial n-ball of radius $\gamma > 0$ (or $\gamma = \infty$) is defined as a formal power series $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ in noncommutative indeterminates $Z_1, \ldots, Z_n$ with radius of convergence $r(f) \geqslant \gamma$, i.e., $\{a_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ are complex numbers with $r(f)^{-1} := \limsup_{k\to\infty}(\sum_{|\alpha|=k} |a_\alpha|^2)^{1/2k} \leqslant 1/\gamma$, where $\mathbb{F}_n^+$ is the free semigroup with n generators $g_1, \ldots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and

$|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$. If $(X_1, \ldots, X_n) \in B(\mathcal{H})^n$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$. A free holomorphic function f on the open ball

$$[B(\mathcal{H})^n]_\gamma := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n: \|X_1 X_n^* + \cdots + X_n X_n^*\|^{1/2} < \gamma\},$$

is the evaluation of f on the Hilbert space $\mathcal{H}$, that is, the mapping

$$[B(\mathcal{H})^n]_\gamma \ni (X_1, \ldots, X_n) \mapsto f(X_1, \ldots, X_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha \in B(\mathcal{H}),$$

where the convergence is in the operator norm. Due to the fact that a free holomorphic function is uniquely determined by its representation on an infinite dimensional Hilbert space, throughout this paper, we identify a free holomorphic function with its evaluation on a separable infinite dimensional Hilbert space.

A free holomorphic function f on $[B(\mathcal{H})^n]_\gamma$ is bounded if $\|f\|_\infty := \sup \|f(X)\| < \infty$, where the supremum is taken over all $X \in [B(\mathcal{H})^n]_\gamma$ and $\mathcal{H}$ is an infinite dimensional Hilbert space. Let $H^\infty_{\mathbf{ball}_\gamma}$ be the set of all bounded free holomorphic functions and let $A_{\mathbf{ball}_\gamma}$ be the set of all elements $f \in H^\infty_{\mathbf{ball}_\gamma}$ such that the mapping

$$[B(\mathcal{H})^n]_\gamma \ni (X_1, \ldots, X_n) \mapsto f(X_1, \ldots, X_n) \in B(\mathcal{H})$$

has a continuous extension to the closed ball $[B(\mathcal{H})^n]_\gamma^-$. We showed in [30] that $H^\infty_{\mathbf{ball}_\gamma}$ and $A_{\mathbf{ball}_\gamma}$ are Banach algebras under pointwise multiplication and the norm $\|\cdot\|_\infty$.

For each $i = 1, \ldots, n$, we define the free partial derivation $\frac{\partial}{\partial Z_i}$ on $\mathbb{C}[Z_1, \ldots, Z_n]$, the algebra of noncommutative polynomials with complex coefficients and indeterminats $Z_1, \ldots, Z_n$, as the unique linear operator on this algebra, satisfying the conditions

$$\frac{\partial I}{\partial Z_i} = 0, \qquad \frac{\partial Z_i}{\partial Z_i} = I, \qquad \frac{\partial Z_j}{\partial Z_i} = 0 \quad \text{if } i \neq j,$$

and

$$\frac{\partial(\varphi\psi)}{\partial Z_i} = \frac{\partial \varphi}{\partial Z_i}\psi + \varphi\frac{\partial \psi}{\partial Z_i}$$

for any $\varphi, \psi \in \mathbb{C}[Z_1, \ldots, Z_n]$ and $i, j = 1, \ldots n$. Note that if $\alpha = g_{i_1} \cdots g_{i_p}$, $|\alpha| = p$, and q of the $g_{i_1}, \ldots, g_{i_p}$ are equal to g_j, then $\frac{\partial Z_\alpha}{\partial Z_j}$ is the sum of the q words obtained by deleting each occurrence of Z_j in $Z_\alpha := Z_{i_1} \cdots Z_{i_p}$ and replacing it by the identity I. The directional derivative of Z_α at Z_i in the direction Y, denoted by $(\frac{\partial Z_\alpha}{\partial Z_i})[Y]$, is defined similarly by replacing each occurrence of Z_j in $Z_\alpha := Z_{i_1} \cdots Z_{i_p}$ by Y (see [10]). Note that $\frac{\partial Z_\alpha}{\partial Z_i} = (\frac{\partial Z_\alpha}{\partial Z_i})[I]$. These definitions extend to formal power series in the noncommuting indeterminates $Z_1, \ldots, Z_n$. If $F := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ is a power series, then the free partial derivative of F with respect to Z_i is the power series $\frac{\partial F}{\partial Z_i} := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha \frac{\partial Z_\alpha}{\partial Z_i}$.

We denote by $\mathbf{S}[Z_1, \dots, Z_n]$ the algebra of all formal power series in noncommuting indeterminates $Z_1, \dots, Z_n$ and complex coefficients. We remark that, for any power series $G \in \mathbf{S}[Z_1, \dots, Z_n]$,

$$\left(\frac{\partial F}{\partial Z_i}\right)[G] := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha \left(\frac{\partial Z_\alpha}{\partial Z_i}\right)[G]$$

is a power series in $\mathbf{S}[Z_1, \dots, Z_n]$. Indeed, it is enough to notice that all the monomials of degree $m \geqslant 1$ in $Z_1, \dots, Z_n$ occur in the sum $\sum_{k=0}^{m+1} \sum_{|\alpha|=k} (\frac{\partial Z_\alpha}{\partial Z_i})[G]$. Consequently, we can use the directional derivative of F at Z_i to define the mapping

$$\left(\frac{\partial F}{\partial Z_i}\right) : \mathbf{S}[Z_1, \dots, Z_n] \to \mathbf{S}[Z_1, \dots, Z_n], \qquad g \mapsto \left(\frac{\partial F}{\partial Z_i}\right)[G].$$

Let H be a formal power series in indeterminates $W_1, \dots, W_n$ and let $G = (G_1, \dots, G_n)$ be an n-tuple of formal power series in indeterminates $Z_1, \dots, Z_n$ with $G(0) = 0$. Then we have the following chain rule

$$\frac{\partial (H \circ G)}{\partial Z_i} = \sum_{k=1}^n \left\{ \left(\frac{\partial H}{\partial W_k}\right) \left[\frac{\partial G_k}{\partial Z_i}\right] \right\}_{W=G(Z)},$$

where $Z = (Z_1, \dots, Z_n)$ and $W = (W_1, \dots, W_n)$. Let $F := (F_1, \dots, F_n)$ be an n-tuple of formal power series in indeterminates $W_1, \dots, W_n$. We define the Jacobian matrix of F to be $J_F := [\frac{\partial F_i}{\partial W_j}]_{n \times n}$ with entries in $\mathbf{S}[W_1, \dots, W_n]$. Note that

$$J_{F \circ G} = \left[\sum_{k=1}^n \left\{ \left(\frac{\partial F_i}{\partial W_k}\right) \left[\frac{\partial G_k}{\partial Z_j}\right] \right\}_{W=G(Z)} \right]_{n \times n},$$

which, symbolically, can be written as

$$\big(J_F[\cdot]\big) \Diamond J_G = \left[\left(\frac{\partial F_i}{\partial W_k}\right)[\cdot] \right]_{n \times n} \Diamond \left[\frac{\partial G_k}{\partial Z_j}\right]_{n \times n},$$

which is the substitute for the matrix multiplication from the commutative case. In particular, we can easily deduce the following result.

Lemma 1.1. *Let $F := (F_1, \dots, F_n)$ and $G := (G_1, \dots, G_n)$ be formal power series in n-indeterminates and such that $G(0) = 0$. Then*

$$J_{F \circ G}(0) = J_F(0) J_G(0).$$

If F is an n-tuple of noncommutative polynomials, the condition $G(0) = 0$ is not necessary.

Theorem 1.2. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series in indeterminates $Z_1, \dots, Z_n$ and with the property that*

$$\det J_f(0) := \det\left[\frac{\partial f_i}{\partial Z_j}\bigg|_{Z=0}\right] \neq 0.$$

Then the set $\{f_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is linearly independent in the complex vector space of all formal power series in noncommuting indeterminates $Z_1, \dots, Z_n$ and complex coefficients.

Proof. First, we consider the case when $f(0) = 0$. Let $A := J_f(0)^t$, where t stands for the transpose, and let $f = G = [G_1, \dots G_n]$ be an n-tuple of power series in noncommuting indeterminates $Z_1, \dots, Z_n$, of the form $G = [Z_1, \dots, Z_n]A + [Q_1, \dots, Q_n]$, where $Q_1, \dots, Q_n$ are noncommutative power series containing only monomials of degree greater than or equal to 2. In what follows, we prove that the composition map $C_G : \mathbf{S}[Z_1, \dots, Z_n] \to \mathbf{S}[Z_1, \dots, Z_n]$ defined by $C_G\Psi := \Psi \circ G$ is an injective homomorphism. Let F be a formal power series such that $F \circ G = 0$. Since $A \in M_{n\times n}$ there is a unitary matrix $U \in M_{n\times n}$ such that $U^{-1}AU$ is an upper triangular matrix. Setting $\Phi_U := [Z_1, \dots, Z_n]U$, the equation $F \circ G = 0$ is equivalent to $F' \circ G' = 0$, where $F' := \Phi_U \circ F \circ \Phi_{U^{-1}}$ and

$$G' := \Phi_U \circ G \circ \Phi_{U^{-1}} = [Z_1, \dots, Z_n]U^{-1}AU + U^{-1}[Q_1, \dots, Q_n]U.$$

Therefore, we can assume that $A = [a_{ij}] \in M_{n\times n}$ is an invertible upper triangular matrix and, therefore $a_{ii} \neq 0$ for any $i = 1, \dots, n$. We introduce a total order $\leqslant$ on the free semigroup $\mathbb{F}_n^+$ as follows. If $\alpha, \beta \in \mathbb{F}_n^+$ with $|\alpha| < |\beta|$ we say that $\alpha < \beta$. If $\alpha, \beta \in \mathbb{F}_n^+$ are such that $|\alpha| = |\beta|$, then $\alpha = g_{i_1} \cdots g_{i_k}$ and $\beta = g_{j_1} \cdots g_{j_k}$ for some $i_1, \dots, i_k, j_1, \dots, j_k \in \{1, \dots, k\}$. We say that $\alpha < \beta$ if either $i_1 < j_1$ or there exists $p \in \{2, \dots, k\}$ such that $i_1 = j_1, \dots, i_{p-1} = j_{p-1}$ and $i_p < j_p$. The relation $\leqslant$ is a total order on $\mathbb{F}_n^+$. According to the hypothesis and due to the fact that A is an upper triangular matrix, we have

$$G_j = \sum_{i=1}^{j} a_{ij} Z_i + Q_j, \quad j = 1, \dots, n. \tag{1.1}$$

Consequently, if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \dots i_k \in \{1, \dots, n\}$, then

$$G_\alpha := G_{i_1} \cdots G_{i_k} = L^{<\alpha} + a_{i_1 i_1} \cdots a_{i_k i_k} Z_\alpha + \Psi^{(\alpha)}, \tag{1.2}$$

where $L^{<\alpha}$ is a power series containing only monomials Z_β such that $|\beta| = |\alpha|$ and $\beta < \alpha$, and $\Psi^{(\alpha)}$ is a power series containing only monomials Z_γ with $|\gamma| \geqslant |\alpha| + 1$.

Now, assume that F has the representation $F = \sum_{p=0}^\infty \sum_{|\alpha|=p} c_\alpha Z_\alpha$, $c_\alpha \in \mathbb{C}$, and satisfies the equation $F \circ G = 0$. We will show by induction over p, that $\sum_{|\alpha|=p} c_\alpha Z_\alpha = 0$ for any $p = 0, 1, \dots$. Note that the above-mentioned equation is equivalent to

$$\sum_{p=0}^{\infty} \sum_{|\alpha|=p} c_\alpha G_\alpha = 0. \tag{1.3}$$

Due to relation (1.1), we have $c_0 = 0$. Assume that $c_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| < k$. According to Eqs. (1.2) and (1.3), we have

$$\sum_{|\alpha|=k} c_\alpha \big(L^{<\alpha} + d_A(\alpha) Z_\alpha + \Psi^{(\alpha)}\big) + \sum_{p=k+1}^{\infty} \sum_{|\alpha|=p} c_\alpha G_\alpha = 0,$$

where $d_A(\alpha) := a_{i_1 i_1} \cdots a_{i_k i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $i_1, \dots, i_k \in \{1, \dots, n\}$. Since $\Psi^{(\alpha)}$ is a power series containing only monomials Z_γ with $|\gamma| \geqslant |\alpha| + 1$, and the power series G_α, $|\alpha| \geqslant k + 1$, contains only monomials Z_σ with $|\sigma| \geqslant k + 1$, we deduce that

$$\sum_{|\alpha|=k} c_\alpha \big(L^{<\alpha} + d_A(\alpha) Z_\alpha\big) = 0. \tag{1.4}$$

We arrange the elements of the set $\{\alpha \in \mathbb{F}_n^+ : |\alpha| = k\}$ increasingly with respect to the total order, i.e., $\beta_1 < \beta_2 < \cdots < \beta_{n^k}$. Note that $\beta_1 = g_1^k$ and $\beta_{n^k} = g_n^k$. Relation (1.4) becomes

$$\sum_{j=1}^{n^k} \big(c_{\beta_j} L^{<\beta_j} + c_{\beta_j} d(\beta_j) Z_{\beta_j}\big) = 0. \tag{1.5}$$

Taking into account that $L^{<\alpha}$ is a power series containing only monomials Z_β such that $|\beta| = |\alpha|$ and $\beta < \alpha$, one can see that the monomial $Z_{\beta_{n^k}}$ occurs just once in (1.5). Consequently, we must have $c_{\beta_{n^k}} d(\beta_{n^k}) = 0$. Since $0 \neq a_{nn}^k = d(\beta_{n^k})$, we must have $c_{\beta_{n^k}} = 0$. Then, Eq. (1.5) becomes

$$\sum_{j=1}^{n^k - 1} \big(c_{\beta_j} \Psi^{<\beta_j} + c_{\beta_j} d(\beta_j) Z_{\beta_j}\big) = 0.$$

Continuing the process, we deduce that $c_{\beta_j} = 0$ for $j = 1, \dots, n^k$. Therefore $c_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = k$, which completes our induction. This shows that $F = 0$.

Now, we consider the case when $f(0) \neq 0$. Then $f_i = f_i(0) I + G_i$, $i = 1, \dots, n$, for some n-tuple $G = [G_1, \dots G_n]$ of formal power series in $\mathbf{S}[Z_1, \dots, Z_n]$ with $G(0) = 0$. According to the first part of the proof, the set $\{G_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is linearly independent in $\mathbf{S}[Z_1, \dots, Z_n]$. Consequently, setting $\mathcal{M}_k := \operatorname{span}\{G_\alpha\}_{|\alpha| \leqslant k}$, $k \geqslant 0$, we have $\dim \mathcal{M}_k = 1 + n + n^2 + \cdots + n^k$. Now, assume that $\{f_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is not linearly independent in $\mathbf{S}[Z_1, \dots, Z_n]$. Then there exists $m \geqslant 1$ such that $\{f_\alpha\}_{|\alpha| \leqslant m}$ is not linearly independent. This shows that the space $\mathcal{N}_m := \operatorname{span}\{f_\alpha\}_{|\alpha| \leqslant m}$ has $\dim \mathcal{N}_m < 1 + n + n^2 + \cdots + n^m = \dim \mathcal{M}_m$. On the other hand, note that for each $\alpha \in \mathbb{F}_n^+$, f_α is a linear combination of G_β with $\beta \in \mathbb{F}_n^+$, $|\beta| \leqslant |\alpha|$, and each G_α is a linear combination of f_β with $\beta \in \mathbb{F}_n^+$, $|\beta| \leqslant |\alpha|$. Consequently, $\mathcal{N}_m = \mathcal{M}_m$ and, therefore, $\dim \mathcal{N}_m = \dim \mathcal{M}_m$, which is in contradiction with the strict inequality above. The proof is complete. $\square$

Now we prove an inverse mapping theorem for formal power series in noncommuting indeterminates.

Theorem 1.3. *Let $F = (F_1, \dots, F_n)$ be an n-tuple of formal power series in indeterminates $Z_1, \dots, Z_n$. Then the following statements are equivalent.*

(i) *There is an n-tuple of formal power series $G = (G_1, \ldots, G_n)$ such that*

$$G(0) = 0 \quad \textit{and} \quad F \circ G = id.$$

(ii) $F(0) = 0$ *and the Jacobian* $\det J_F(0) \neq 0$.

In this case, G is unique and $G \circ F = id$.

Proof. Assume that item (i) holds. For each $i = 1, \ldots, n$, let

$$F_i := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha^{(i)} Z_\alpha \quad \text{and} \quad G_i := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} b_\alpha^{(i)} Z_\alpha$$

be such that $G_i(0) = 0$ and $F \circ G = id$. Hence, we deduce that

$$a_g^{(i)} + \sum_{j=1}^{n} a_{g_j}^{(i)} G_j + \sum_{|\alpha| \geqslant 2} a_\alpha^{(i)} G_\alpha = Z_i, \quad i = 1, \ldots, n.$$

Since $G_i(0) = 0$, if $|\alpha| \geqslant 2$, then each monomial in G_α has degree $\geqslant 2$. Consequently, we have $a_g^{(i)} = 0$ for $i = 1, \ldots, n$, i.e., $F(0) = 0$, and $\sum_{j=1}^{n} a_{g_j}^{(i)} b_{g_p}^{(j)} = \delta_{ip}$ for any $i, p \in \{1, \ldots, n\}$. The latter condition is equivalent to $J_F(0) J_G(0) = I_n$, which implies $\det J_F(0) \neq 0$ and $\det J_G(0) \neq 0$. Therefore, (ii) holds. Now, we prove the implication (ii) $\Longrightarrow$ (i). Assume that condition (ii) is satisfied and let $F_i := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha^{(i)} Z_\alpha$. We need to find and n-tuple $G = (G_1, \ldots, G_n)$ with $G_i := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} b_\alpha^{(i)} Z_\alpha$ such that $G(0) = 0$ and $F \circ G = id$. Therefore, we should have

$$\sum_{|\alpha| \geqslant 1} a_\alpha^{(i)} G_\alpha = Z_i, \quad i = 1, \ldots, n. \tag{1.6}$$

We denote by $\operatorname{Coef}_{Z_\alpha}(H)$ the coefficient of the monomial Z_α, $\alpha \in \mathbb{F}_n^+$, in the formal power series H. Due to relation (1.6), we have

$$\delta_{ip} = \operatorname{Coef}_{Z_p}(Z_i) = \sum_{j=1} a_{g_j}^{(i)} \operatorname{Coef}_{Z_p}(G_j) = \sum_{j=1} a_{g_j}^{(i)} b_{g_p}^{(j)}$$

for any $i, p \in \{1, \ldots, n\}$. Hence, we deduce that $J_F(0) J_G(0) = I_n$, where $J_F(0) = [a_{g_j}^{(i)}]_{i,j=1,\ldots,n}$ and $J_G(0) = [b_{g_j}^{(i)}]_{i,j=1,\ldots,n}$. This implies that $J_G(0)$ is the inverse of $J_F(0)$ and, therefore, the coefficients $\{b_\alpha\}_{|\alpha|=1, i=1,\ldots,n}$ are uniquely determined and $\det J_G(0) \neq 0$. Now, we prove by induction over m that the coefficients $\{b_\alpha^{(i)}\}_{|\alpha| \leqslant m,\, i=1,\ldots,n}$ are uniquely determined by condition (1.6). Assume that the coefficients $\{b_\alpha^{(i)}\}_{|\alpha| \leqslant m-1,\, i=1,\ldots,n}$, $m \geqslant 2$, are uniquely determined by (1.6). Let $\beta = g_{p_1} \cdots g_{p_m} \in \mathbb{F}_n^+$ with $p_1, \ldots, p_m \in \{1, \ldots, n\}$ and $m \geqslant 2$. Note that condition (1.6) implies

$$
\begin{aligned}
&\operatorname{Coef}_{Z_\beta}\left(\sum_{|\alpha|\geqslant 1} a_\alpha^{(i)} G_\alpha\right)\\
&\quad=\operatorname{Coef}_{Z_\beta}\left(\sum_{j_1=1}^n a_{g_{j_1}}^{(i)} G_{j_1}+\sum_{j_1,j_2=1}^n a_{g_{j_1}g_{j_2}}^{(i)} G_{j_1}G_{j_2}+\cdots+\sum_{j_1,\dots,j_m=1}^n a_{g_{j_1}\cdots g_{j_m}}^{(i)} G_{j_1}\cdots G_{j_m}\right)\\
&\quad=\sum_{j_1=1}^n a_{g_{j_1}}^{(i)} b_\beta^{(j_1)}+\sum_{j_1,j_2=1}^n a_{g_{j_1}g_{j_2}}^{(i)}\left(\sum_{\sigma_1\sigma_2=\beta,\sigma_1,\sigma_2\in\mathbb{F}_n^+\backslash\{g_0\}} b_{\sigma_1}^{(j_1)} b_{\sigma_2}^{(j_2)}\right)\\
&\qquad+\cdots+\sum_{j_1,\dots,j_m=1}^n a_{g_{j_1}\cdots g_{j_m}}^{(i)} b_{g_{p_1}}^{(j_1)}\cdots b_{g_{p_m}}^{(j_1)}=0
\end{aligned}
$$

for each $i=1,\dots,n$. We consider the matrices

$$
J_F(0)=\left[a_{g_{j_1}}^{(i)}\right]_{i,j_1=1,\dots,n},\qquad A_{n\times n^k}:=\left[a_{g_{j_1}g_{j_2}\cdots g_{j_k}}^{(i)}\right]_{i,j_1,\dots,j_k=1,\dots,n}
$$

for $2\leqslant k\leqslant m$, and the column matrices

$$
B_{n\times 1}^{(\beta)}:=\left[b_\beta^{(i)}\right]_{i=1,\dots,n},\qquad B_{n^k\times 1}^{(\beta)}:=\left[\sum_{\sigma_1\cdots\sigma_k=\beta,\sigma_1,\dots,\sigma_k\in\mathbb{F}_n^+\backslash\{g_0\}} b_{\sigma_1}^{(j_1)}\cdots b_{\sigma_k}^{(j_k)}\right]_{j_1,\dots,j_k=1,\dots,n}
$$

for $2\leqslant k\leqslant m$. The equality above is equivalent to

$$
J_F(0)B_{n\times 1}^{(\beta)}+A_{n\times n^2}B_{n^2\times 1}^{(\beta)}+\cdots+A_{n\times n^m}B_{n^m\times 1}^{(\beta)}=0_{n\times 1},
$$

where $0_{n\times 1}$ is the column zero matrix. Since the entries of the matrices $B_{n^2\times 1}^{(\beta)},\dots,B_{n^m\times 1}^{(\beta)}$ contain only coefficients $b_\omega^{(j)}$, where $|\omega|\leqslant m-1$ and $j=1,\dots,n$, the relation

$$
B_{n\times 1}^{(\beta)}=-J_F(0)^{-1}A_{n\times n^2}B_{n^2\times 1}^{(\beta)}-\cdots-J_F(0)^{-1}A_{n\times n^m}B_{n^m\times 1}^{(\beta)}
$$

shows that the coefficients $\{b_\beta^{(i)}\}_{|\beta|=m,i=1,\dots,n}$ are uniquely determined. This completes our proof by induction. Therefore, (i) holds. Since $G(0)=0$ and $\det J_G(0)\neq 0$, the result we proved above implies the existence of an n-tuple of formal power series $H=(H_1,\dots,H_n)$ such that $H(0)=0$, $\det J_H(0)\neq 0$, and $G\circ H=id$. Using (i), we deduce that

$$
H=id\circ H=(F\circ G)\circ H=F\circ(G\circ H)=F\circ id=F
$$

and $G\circ F=id$. The uniqueness of G is now obvious. The proof is complete. $\square$

The n-tuple $G=(G_1,\dots,G_n)$ of Theorem 1.3 is called the inverse of $F=(F_1,\dots,F_n)$ with respect to the composition of power series. We remark that under the conditions of Theorem 1.3, the composition map $C_F:\mathbf{S}[Z_1,\dots,Z_n]\to\mathbf{S}[Z_1,\dots,Z_n]$ defined by $C_F\Lambda:=\Lambda\circ F$ is an algebra isomorphism.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k \geqslant 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. We denote $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and $e_{g_0} := 1$. Note that $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is an orthonormal basis for $F^2(H_n)$. Define the left (resp. right) creation operators S_i (resp. R_i), $i = 1, \ldots, n$, acting on $F^2(H_n)$ by setting

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

(resp. $R_i \varphi := \varphi \otimes e_i$). Note that $S_i R_j = R_j S_i$ for $i, j \in \{1, \ldots, n\}$. The noncommutative disk algebra $\mathcal{A}_n$ (resp. $\mathcal{R}_n$) is the norm-closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra F_n^∞ (resp. R_n^∞) is the weakly closed version of $\mathcal{A}_n$ (resp. $\mathcal{R}_n$). These algebras were introduced in [21] in connection with a noncommutative version of the classical von Neumann inequality [38] and studied in [20, 23,8].

Let $\Omega \subset B(\mathcal{H})^n$ be a set containing a ball $[B(\mathcal{H})^n]_r$ for some $r > 0$. We say that $f : \Omega \to B(\mathcal{H})$ is a free holomorphic function on Ω if there are some complex numbers a_α, $\alpha \in \mathbb{F}_n^+$, such that

$$f(X) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha, \quad X = (X_1, \ldots, X_n) \in \Omega,$$

where the convergence is in the operator norm. As in [30], one can show that any free holomorphic function on Ω has a unique representation as above.

If $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series, we define the radius of convergence of f by setting $r(f) = \min_{i=1,\ldots,n} r(f_i)$. According to [30], f_i is a free holomorphic function on the open ball $[B(\mathcal{H})^n]_{r(f)}$ for any $i = 1, \ldots, n$. The next result can be viewed as an inverse function theorem for free holomorphic functions.

Theorem 1.4. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with nonzero radius of convergence such that $f(0) = 0$ and $\det J_f(0) \neq 0$. Let $g = (g_1, \ldots, g_n)$ be the inverse power series of f with respect to composition.*

If g has a non-zero radius of convergence, then there are open neighborhoods D and G of 0 in $B(\mathcal{H})^n$ such that $f|_D : D \to G$ is a bijective free holomorphic function whose inverse is a free holomorphic function on G which coincides with $g|_G : G \to D$.

Proof. First, note that according to Theorem 1.3, since $f(0) = 0$ and $\det J_f(0) \neq 0$, there is an n-tuple $g = (g_1, \ldots, g_n)$ of formal power series such that $g(0) = 0$ and $g \circ f = f \circ g = id$. Assume that f and g have nonzero radius of convergence $r(f) > 0$ and $r(g) > 0$, respectively. Fix $\epsilon_0 > 0$ such that $\epsilon_0 < r(g)$. Since $r(f) > 0$ and $f(0) = 0$, the Schwartz lemma for free holomorphic functions (see [30]) implies that there is $\delta_0 \in (0, r(f))$ such that $\|f(Y)\| < r(g) - \epsilon_0$ for any $Y \in [B(\mathcal{H})^n]_{\delta_0}^-$. On the other hand, using Theorem 1.2 from [33], the composition $Y \mapsto g(f(Y))$ is a free holomorphic function on $[B(\mathcal{H})^n]^-_{\delta_0}$. Due to the uniqueness theorem for free holomorphic functions and the fact that $g \circ f = id$ as formal power series, we deduce that $g(f(Y)) = Y$ for any $Y \in [B(\mathcal{H})^n]^-_{\delta_0}$. Hence, $f|_{[B(\mathcal{H})^n]^-_{\delta_0}}$ is a one-to-one free holomorphic function.

Now, fix $c_0 \in (0, \delta_0)$. Since $r(g) > 0$ and $g(0) = 0$, using again the Schwartz lemma for free holomorphic functions, we find $\gamma \in (0, r(g))$ such that $\|g(X)\| < \delta_0 - c_0$ for any $X \in [B(\mathcal{H})^n]^-_\gamma$. As above, the composition $X \mapsto f(g(X))$ is a free holomorphic function on $[B(\mathcal{H})^n]^-_\gamma$. Due to the uniqueness theorem for free holomorphic functions and that $f \circ g = id$ as formal power series, we deduce that $f(g(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$. Consequently, $g|_{[B(\mathcal{H})^n]_\gamma}$ is a one-to-one free holomorphic function.

Set $G := [B(\mathcal{H})^n]_\gamma$ and $D := g([B(\mathcal{H})^n]_\gamma)$. Note that g and f are free holomorphic (and, therefore, continuous) on $[B(\mathcal{H})^n]_{r(g)} \supset G$ and $[B(\mathcal{H})^n]_{\delta_0} \supset [B(\mathcal{H})^n]_{\delta_0 - c_0} \supset D$, respectively. Due to the fact that $f|_{[B(\mathcal{H})^n]_{\delta_0}} : [B(\mathcal{H})^n]_{\delta_0} \to B(\mathcal{H})^n$ is a one-to-one continuous function and $f(g(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$, we deduce that the pre-image $((f|_{[B(\mathcal{H})^n]_{\delta_0}})^{-1}([B(\mathcal{H})^n]_\gamma)$ is an open set in $[B(\mathcal{H})^n]_{\delta_0}$ which coincides with

$$\big[B(\mathcal{H})^n\big]_{\delta_0} \cap g\big(\big[B(\mathcal{H})^n\big]_\gamma\big) = \big[B(\mathcal{H})^n\big]_{\delta_0 - c_0} \cap g\big(\big[B(\mathcal{H})^n\big]_\gamma\big) = g\big(\big[B(\mathcal{H})^n\big]_\gamma\big) = D.$$

Consequently, since $D \subset [B(\mathcal{H})^n]_{\delta_0}$ is an open set in $[B(\mathcal{H})^n]_{\delta_0}$, we deduce that D is an open set in $B(\mathcal{H})^n$. The proof is complete. $\square$

In Theorem 1.4, we conjecture that the condition that g has a non-zero radius of convergence is a consequence of the fact that $f = (f_1, \ldots, f_n)$ has nonzero radius of convergence such that $f(0) = 0$ and $\det J_f(0) \neq 0$. We also remark that there is a converse for Theorem 1.4. Let D, G be open neighborhoods of 0 in $B(\mathcal{H})^n$ and let $\varphi : D \to G$ and $\psi : G \to D$ be free holomorphic functions such that $\psi = (\psi_1, \ldots, \psi_n)$ is the inverse of $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then the associated formal power series are inverses to each other with respect to composition. Indeed, assume that φ_i has the representation $\sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} X_\alpha$ on D, and ψ_i has the representation $\sum_{k=1}^\infty \sum_{|\alpha|=k} c_\alpha^{(i)} X_\alpha$ on G. Then, we ca find $0 < r < 1$ such that $[B(\mathcal{H})^n]^-_r \subset G$ and $\varphi(\psi(X)) = X$ for any $X \in [B(\mathcal{H})^n]^-_r$, where the convergence of the series defining $\psi(X)$ and $\varphi(\psi(X))$ are in the operator norm topology. Hence, we deduce that $\varphi(\psi(rS_1, \ldots, rS_n)) = (rS_1, \ldots, rS_n)$. Since $\varphi_i(\psi(rS_1, \ldots, rS_n))$ is in the noncommutative disc algebra $\mathcal{A}_n$, it has a unique Fourier representation $\sum_{k=1}^\infty \sum_{|\alpha|=k} c_\alpha^{(i)} r^{|\alpha|} S_\alpha$, where the coefficients $c_\alpha^{(i)}$ are exactly those of the formal power series $\varphi_i \circ \psi$. The equality above shows that $c_\alpha^{(i)} = 0$ if $|\alpha| \geqslant 2$ and $c_{g_j}^{(i)} = \delta_{ij}$. Therefore, $\varphi \circ \psi = id$. Due to Theorem 1.3, we also deduce that $\psi \circ \varphi = id$, which proves our assertion.

2. Hilbert spaces of noncommutative formal power series

In this section, we introduce three classes: $(\mathcal{A})$, $(\mathcal{S})$, and $(\mathcal{F})$, of n-tuples $f = (f_1, \ldots, f_n)$ of formal power series, and the corresponding Hilbert space $\mathbb{H}^2(f)$. The associated domain $\mathbb{B}_f(\mathcal{H})$ has a universal model $(M_{Z_1}, \ldots, M_{Z_n})$ of multiplication operators acting on $\mathbb{H}^2(f)$, which plays a crucial role in the dilation theory on the noncommutative domain $\mathbb{B}_f(\mathcal{H})$.

Hilbert spaces associated with polynomial automorphisms of $B(\mathcal{H})^n$. Let $\mathbb{C}[Z_1, \ldots, Z_n]$ be the algebra of noncommutative polynomials over $\mathbb{C}$ (complex numbers) and noncommuting indeterminates $Z_1, \ldots, Z_n$. We say that an n-tuple $p = (p_1, \ldots, p_n)$ of polynomials is invertible in $\mathbb{C}[Z_1, \ldots, Z_n]^n$ with respect to composition if there exists an n-tuple $q = (q_1, \ldots, q_n)$ of polynomials such that $p \circ q = q \circ p = id$. We remark that such an n-tuple of polynomials induces a free holomorphic automorphism of $B(\mathcal{H})^n$, i.e., the map $\Phi_p : B(\mathcal{H})^n \to B(\mathcal{H})^n$ defined by

$$\Phi_p(X) := \big(p_1(X), \ldots, p_n(X)\big), \qquad X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n.$$

We say that Φ_p is a polynomial automorphism of $B(\mathcal{H})^n$ and write $\Phi_p \in \mathrm{Aut}(B(\mathcal{H})^n)$. Note that if p, p' are n-tuples of polynomials and $\Phi_p, \Phi_{p'}$ are in $\mathrm{Aut}(B(\mathcal{H})^n)$, then so is $\Phi_{p\circ p'}$ and $\Phi_{p\circ p'} = \Phi_p \Phi_{p'}$.

Theorem 2.1. *If $p = (p_1, \ldots, p_n)$ is an n-tuple of noncommutative polynomials in $Z_1, \ldots, Z_n$, then the following statements are equivalent.*

(i) *p is invertible in $\mathbb{C}[Z_1, \ldots, Z_n]^n$ with respect to composition.*
(ii) *There exists an n-tuple $q = (q_1, \ldots, q_n)$ of noncommutative polynomials in $Z_1, \ldots, Z_n$ such that $q \circ p = id$.*
(iii) *$Z_1, \ldots, Z_n$ are contained in the linear span of $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ (where $p_0 := I$).*
(iv) *The set $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is a linear basis in $\mathbb{C}[Z_1, \ldots, Z_n]$.*

Proof. First we consider the case when $p_i(0) = 0$, $i = 1, \ldots, n$. The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are obvious. To prove that (iii) $\Longrightarrow$ (iv), assume that condition (iii) holds. Since $Z_1, \ldots, Z_n$ are contained in the linear span of $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$, there are some complex numbers $\{a_\alpha^{(i)}\}_{\alpha\in\mathbb{F}_n^+, |\alpha|\leqslant m}$ such that $Z_i = \sum_{|\alpha|\leqslant m} a_\alpha^{(i)} p_\alpha(Z)$, $i = 1, \ldots, n$. Setting $q = (q_1, \ldots, q_n)$ with $q_i(Z) := \sum_{|\alpha|\leqslant m} a_\alpha^{(i)} Z_\alpha$, we have $q(0) = 0$ and $q \circ p = id$. Due to Lemma 1.1, we obtain $\det J_p(0) \det J_q(0) = 1$, which implies $\det J_p(0) \neq 0$. Using now Theorem 1.2, we deduce that the set $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is a linearly independent in $\mathbb{C}[Z_1, \ldots, Z_n]$. On the other hand, condition (iii) also implies that $\mathbb{C}[Z_1, \ldots, Z_n]$ is spanned by $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$. Therefore, condition (iv) holds.

Since (iv) $\Longrightarrow$ iii) is obvious, it remains to prove that (iii) $\Longrightarrow$ (i). As above, if (iii) holds, then there is an n-tuple $q = (q_1, \ldots, q_n)$ of polynomials with $q_i(0) = 0$ such that $q \circ p = id$ and the set $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is a linearly independent in $\mathbb{C}[Z_1, \ldots, Z_n]$. The latter property shows that p is not a right zero divisor with respect to the composition of polynomials, that is, if $\psi \in \mathbb{C}[Z_1, \ldots, Z_n]^n$ and $\psi \circ p = 0$, then $\psi = 0$. Due to relation $q \circ p = id$, we obtain $(p \circ q - id) \circ p = 0$. Since p is not a right zero divisor, we deduce that $p \circ q = id$, which completes the proof.

Now, we consider the case when $p(0) \neq 0$. Assume that (iii) holds. Then $p_i' := p_i - p_i(0)I$, $i = 1, \ldots, n$, has the property that $p_i'(0) = 0$ and $Z_1, \ldots, Z_n$ are contained in the linear span of $\{p_\alpha'\}_{\alpha\in\mathbb{F}_n^+}$. Applying the first part of the proof to $p' := (p_1', \ldots, p_n')$, we deduce that the set $\{p_\alpha'\}_{\alpha\in\mathbb{F}_n^+}$ is a linear basis for $\mathbb{C}[Z_1, \ldots, Z_n]$. Consequently, setting $\mathcal{M}_k := \mathrm{span}\{p_\alpha'\}_{|\alpha|\leqslant k}$, $k \geqslant 0$, we have $\dim \mathcal{M}_k = 1 + n + n^2 + \cdots + n^k$. Now, assume that $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is not linearly independent in $\mathbb{C}[Z_1, \ldots, Z_n]$. Then there exists $m \geqslant 1$ such that $\{p_\alpha\}_{|\alpha|\leqslant m}$ is not linearly independent. This shows that the space $\mathcal{N}_m := \mathrm{span}\{p_\alpha\}_{|\alpha|\leqslant m}$ has $\dim \mathcal{N}_m$ strictly less than $\dim \mathcal{M}_m = 1 + n + n^2 + \cdots + n^m$. On the other hand, note that for each $\alpha \in \mathbb{F}_n^+$, p_α is a linear combination of p_β' with $\beta \in \mathbb{F}_n^+$, $|\beta| \leqslant |\alpha|$, and each p_α' is a linear combination of p_β with $\beta \in \mathbb{F}_n^+$, $|\beta| \leqslant |\alpha|$. Consequently, $\mathcal{N}_m = \mathcal{M}_m$ and, therefore, $\dim \mathcal{N}_m = \dim \mathcal{M}_m$, which is in contradiction with the strict inequality above. Therefore, the set $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is a linearly independent in $\mathbb{C}[Z_1, \ldots, Z_n]$. Since $\mathbb{C}[Z_1, \ldots, Z_n]$ is spanned by $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$, we deduce that $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is a linear basis in $\mathbb{C}[Z_1, \ldots, Z_n]$, which shows that condition (iv) holds. Moreover, it shows that p is not a right zero divisor with respect to the composition of polynomials. Since $Z_1, \ldots, Z_n$ are contained in the linear span of $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$, we find $q \in \mathbb{C}[Z_1,\dots,Z_n]$ such that $q \circ p = id$. Hence, we obtain $(p \circ q - id) \circ p = 0$. Since p is not a right zero divisor, we deduce that $p \circ q = id$, which implies (i). The proof is complete. $\square$

We say that $p = (p_1,\dots,p_n)$ has property $(\mathcal{A})$ if any of the equivalences of Theorem 2.1 holds.

Example 2.2. If

$$\begin{aligned} p_1 &= a_0 I + a_1 Z_1 + a_2 Z_2 + a_3 Z_3 Z_2, \\ p_2 &= b_0 I + b_1 Z_2 + b_2 Z_3 + b_3 Z_3^2, \\ p_3 &= c_0 I + c_1 Z_3 \end{aligned}$$

are polynomials with complex coefficients such that $a_1 b_1 c_1 \neq 0$ then $p = (p_1, p_2, p_3)$ has property $(\mathcal{A})$.

In what follows we present a large class of polynomial automorphisms of $B(\mathcal{H})^n$.

Proposition 2.3. *Let $p = (p_1,\dots,p_n)$ be an n-tuple of noncommutative polynomials in $Z_1,\dots,Z_n$ of the form*

$$\begin{aligned} [p_1,\dots,p_n] = {} & [a_1 I,\dots,a_n I] + [Z_1,\dots,Z_n]A \\ & + \big[q_1(Z_2,\dots,Z_n), q_2(Z_3,\dots,Z_n),\dots,q_{n-1}(Z_n), 0\big]A, \end{aligned}$$

where $a_i \in \mathbb{C}$, $A \in M_{n\times n}$ is an invertible scalar matrix, and $q_1,\dots,q_{n-1}$ are arbitrary noncommutative polynomials in the specified indeterminates. Then p has property $(\mathcal{A})$.

Proof. According to Theorem 2.1, it is enough to show that $Z_1,\dots,Z_n$ are contained in the linear span of $\{p_\alpha\}_{\alpha\in\mathbb{F}_n^+}$. To solve the formal system, multiply (to the right) both sides of the equality by A^{-1} and solve for the indeterminates $Z_n, Z_{n-1},\dots,Z_1$ in this order. $\square$

As we saw in the proof of Theorem 2.1, if $p = (p_1,\dots,p_n)$ is an n-tuple of noncommutative polynomials with property $(\mathcal{A})$, then the Jacobian matrix

$$J_p(0) := \left[\frac{\partial p_i}{\partial Z_j}\bigg|_{Z=0} \right]_{1\leqslant i,j\leqslant n}$$

is invertible. Moreover, for the class of noncommutative polynomials considered in Proposition 2.3, we have that $J_p(X)$ is an invertible operator for any $X \in B(\mathcal{H})^n$. This leads to the following question. Is the Jacobian conjecture true in our noncommutative setting? In other words, assuming that $p = (p_1,\dots,p_n)$ is an n-tuple of noncommutative polynomials such that the Jacobian matrix $J_p(X)$ is invertible for any $X \in B(\mathcal{H})^n$ (or only for $X = 0$), does this imply that Φ_p is a polynomial automorphism of $B(\mathcal{H})^n$? Of course, this is true if each polynomial p_i has degree 1.

Let $p = (p_1, \ldots, p_n)$ be an n-tuple of noncommutative polynomials in $Z_1, \ldots, Z_n$ with property $(\mathcal{A})$. We introduce an inner product on $\mathbb{C}[Z_1, \ldots, Z_n]$ by setting $\langle p_\alpha, p_\beta \rangle := \delta_{\alpha\beta}$, $\alpha, \beta \in \mathbb{F}_n^+$. Let $\mathbb{H}^2(p)$ be the completion of the linear space $\bigvee \{p_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ with respect to this inner product. It is easy to see that, due to Theorem 2.1, the noncommutative polynomials $\mathbb{C}[Z_1, \ldots, Z_n]$ are dense in $\mathbb{H}^2(p)$. We define the noncommutative domain

$$\mathbb{B}_p(\mathcal{H}) := \left\{ (X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon \sum_{j=1}^n p_j(X) p_j(X)^* \leqslant I \right\},$$

which will be studied in the next sections.

Hilbert spaces of noncommutative formal power series. We recall (see [30]) that the algebra $H_{\mathbf{ball}}$ of free holomorphic functions on the open operatorial n-ball of radius one is defined as the set of all power series $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ with radius of convergence $\geqslant 1$, i.e., $\{a_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ are complex numbers with $\limsup_{k\to\infty} (\sum_{|\alpha|=k} |a_\alpha|^2)^{1/2k} \leqslant 1$. In this case, the mapping

$$\left[B(\mathcal{H})^n\right]_1 \ni (X_1, \ldots, X_n) \mapsto f(X_1, \ldots, X_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha \in B(\mathcal{H})$$

is well defined, where the convergence is in the operator norm topology. Moreover, the series converges absolutely, i.e., $\sum_{k=0}^\infty \| \sum_{|\alpha|=k} a_\alpha X_\alpha \| < \infty$, and uniformly on any ball $[B(\mathcal{H})^n]_\gamma$ with $0 \leqslant \gamma < 1$.

Another case when the evaluation of f can be defined is the following. Assume that there exists an n-tuple $\rho = (\rho_1, \ldots, \rho_n)$ of strictly positive numbers such that

$$\limsup_{k\to\infty} \left(\sum_{|\alpha|=k} |a_\alpha| \rho_\alpha \right)^{1/k} \leqslant 1.$$

Then the series $f(X_1, \ldots, X_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha$ converges absolutely and uniformly on any noncommutative polydisc

$$P(\mathbf{r}) := \left\{ (X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon \|X_j\| \leqslant r_j,\ j = 1, \ldots, n \right\}$$

of multiradius $\mathbf{r} = (r_1, \ldots, r_n)$ with $r_j < \rho_j$, $j = 1, \ldots, n$.

We should also remark that, when $(X_1, \ldots, X_n) \in B(\mathcal{H})^n$ is a nilpotent n-tuple of operators, i.e., there is $m \geqslant 1$ such that $X_\alpha = 0$ for all $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = m$, then $f(X_1, \ldots, X_n)$ makes sense since the series defining it has only finitely many nonzero terms.

We need a few more definitions. Let $g = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Z_\alpha$ be a formal power series in indeterminates $Z_1, \ldots, Z_n$. We denote by $\mathcal{C}_g(\mathcal{H})$ (resp. $\mathcal{C}_g^a(\mathcal{H})$, $\mathcal{C}_g^{SOT}(\mathcal{H})$) the set of all $Y := (Y_1, \ldots, Y_n) \in B(\mathcal{H})^n$ such that the series $g(Y_1, \ldots, Y_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha Y_\alpha$ is norm (resp. absolutely, SOT) convergent. These sets are called sets of norm (resp. absolutely, SOT) convergence for the power series g. We also introduce the set $\mathcal{C}_g^{rad}(\mathcal{H})$ of all $Y := (Y_1, \ldots, Y_n) \in B(\mathcal{H})^n$ such that there exists $\delta \in (0, 1)$ with the property that $rY \in \mathcal{C}_g(\mathcal{H})$ for any $r \in (\delta, 1)$ and

$$\widehat{g}(Y_1, \ldots, Y_n) := \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha r^{|\alpha|} Y_\alpha$$

exists. Note that

$$\mathcal{C}_g^a(\mathcal{H}) \subseteq \mathcal{C}_g(\mathcal{H}) \subseteq \mathcal{C}_g^{SOT}(\mathcal{H}) \quad \text{and} \quad \mathcal{C}_g^{rad}(\mathcal{H}) \subseteq \overline{\mathcal{C}_g(\mathcal{H})}^{SOT}.$$

Now, consider an n-tuple of formal power series $f = (f_1, \ldots, f_n)$ in indeterminates $Z_1, \ldots, Z_n$ with the property that the Jacobian

$$\det J_f(0) := \det\left[\frac{\partial f_i}{\partial Z_j}\bigg|_{Z=0}\right]_{i,j=1}^n \neq 0.$$

Due to Theorem 1.2, the set $\{f_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ (where $f_0 := I$) is linearly independent in $\mathbf{S}[Z_1, \ldots, Z_n]$. We introduce an inner product on the linear span of $\{f_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ by setting $\langle f_\alpha, f_\beta\rangle := \delta_{\alpha\beta}$, $\alpha, \beta \in \mathbb{F}_n^+$. Let $\mathbb{H}^2(f)$ be the completion of the linear space $\bigvee\{f_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ with respect to this inner product. Assume now that $f(0) = 0$. Theorem 1.3 shows that f is not a *right zero divisor* with respect to the composition of power series, i.e., there is no non-zero power series G in $\mathbf{S}[Z_1, \ldots, Z_n]$ such that $G \circ f = 0$. Consequently, the elements of $\mathbb{H}^2(f)$ can be seen as formal power series in $\mathbf{S}[Z_1, \ldots, Z_n]$ of the form $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha f_\alpha$, where $\sum_{\alpha\in\mathbb{F}_n^+} |a_\alpha|^2 < \infty$.

Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series in $Z_1, \ldots, Z_n$ such that $f(0) = 0$. We say that f has property $(\mathcal{S})$ if the following conditions hold.

$(\mathcal{S}_1)$ The n-tuple f has nonzero radius of convergence and $\det J_f(0) \neq 0$.

$(\mathcal{S}_2)$ The indeterminates $Z_1, \ldots, Z_n$ are in the Hilbert space $\mathbb{H}^2(f)$ and each left multiplication operator $M_{Z_i} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$ defined by

$$M_{Z_i}\psi := Z_i\psi, \quad \psi \in \mathbb{H}^2(f),$$

is a bounded multiplier of $\mathbb{H}^2(f)$.

$(\mathcal{S}_3)$ The left multiplication operators $M_{f_j} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$, $M_{f_j}\psi = f_j\psi$, satisfy the equations

$$M_{f_j} = f_j(M_{Z_1}, \ldots, M_{Z_n}), \quad j = 1, \ldots, n,$$

where $(M_{Z_1}, \ldots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$.

We remark that if f is an n-tuple of noncommutative polynomials, then the condition $(\mathcal{S}_3)$ is automatically satisfied. We should also mention that, in case $(M_{Z_1}, \ldots, M_{Z_n})$ is in the set $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$, then the condition $(\mathcal{S}_3)$ should be understood as

$$M_{f_j} = \widehat{f_j}(M_{Z_1}, \ldots, M_{Z_n}) := \text{SOT-}\lim_{r\to 1} f_j(rM_{Z_1}, \ldots, rM_{Z_n}), \quad j = 1, \ldots, n.$$

Remark 2.4. If $p = (p_1, \ldots, p_n)$ is an n-tuple of noncommutative polynomials with property $(\mathcal{A})$, then it has property $(\mathcal{S})$.

Proposition 2.5. *If $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series with $f(0) = 0$ and property $(\mathcal{S})$, then $\mathbb{C}[Z_1, \ldots, Z_n]$ is dense in the Hilbert space $\mathbb{H}^2(f)$.*

Proof. Since $Z_i \in \mathbb{H}^2(f)$ and M_{Z_i} are bounded multipliers of $\mathbb{H}^2(f)$, we deduce that $Z_\alpha \in \mathbb{H}^2(f)$ for any $\alpha \in \mathbb{F}_n^+$ and, therefore, $\mathbb{C}[Z_1, \dots, Z_n] \subset \mathbb{H}^2(f)$. Let f_j, $j = 1, \dots, n$, have the representation $f_j(Z_1, \dots, Z_n) = \sum_{k=0}^\infty \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} Z_\alpha$. First, we assume that $(M_{Z_1}, \dots, M_{Z_n})$ is in the set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and

$$f_j(M_{Z_1}, \dots, M_{Z_n}) = \sum_{k=0}^\infty \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} M_{Z_\alpha},$$

where the convergence of the series is in the strong operator topology. Then, for any $\epsilon > 0$ and any polynomial $\psi \in \mathbb{C}[Z_1, \dots, Z_n]$, there exists $N_j \geqslant 1$ such that

$$\left\| f_j(M_{Z_1}, \dots, M_{Z_n})\psi - \sum_{k=0}^{N_j} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} M_{Z_\alpha} \psi \right\|_{\mathbb{H}^2(f)} < \epsilon, \quad j = 1, \dots, n. \tag{2.1}$$

Fix $i, j \in \{1, \dots, n\}$. By (2.1), we can find polynomials p and q such that

$$\left\| f_j(M_{Z_1}, \dots, M_{Z_n})1 - p \right\|_{\mathbb{H}^2(f)} < \frac{\epsilon}{2\|M_{f_j}\|} \quad \text{and} \quad \left\| f_j(M_{Z_1}, \dots, M_{Z_n})p - qp \right\|_{\mathbb{H}^2(f)} < \frac{\epsilon}{2}.$$

Hence, and using condition $(\mathcal{S}_2)$, we deduce that

$$\begin{aligned} \|f_j f_i - qp\|_{\mathbb{H}^2(f)} &\leqslant \left\| f_j(M_{Z_1}, \dots, M_{Z_n}) f_i(M_{Z_1}, \dots, M_{Z_n})1 - f_j(M_{Z_1}, \dots, M_{Z_n})p \right\| \\ &\quad + \left\| f_j(M_{Z_1}, \dots, M_{Z_n})p - qp \right\| \\ &\leqslant \left\| f_j(M_{Z_1}, \dots, M_{Z_n}) \right\| \frac{\epsilon}{2\|M_{f_j}\|} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

An inductive argument shows that each power series f_α, $\alpha \in \mathbb{F}_n^+$, can be approximated in $\mathbb{H}^2(f)$ by polynomials in $\mathbb{C}[Z_1, \dots, Z_n]$. Taking into account that $\operatorname{span}\{f_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is dense in $\mathbb{H}^2(f)$, we deduce that $\mathbb{C}[Z_1, \dots, Z_n]$ is dense in $\mathbb{H}^2(f)$.

Now, we consider the case when $(M_{Z_1}, \dots, M_{Z_n})$ is in the set $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$ and

$$\widehat{f_j}(M_{Z_1}, \dots, M_{Z_n}) = \text{SOT-}\lim_{r \to 1} \sum_{k=0}^\infty \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} r^{|\alpha|} M_{Z_\alpha},$$

where the convergence of the series is in the operator norm topology for each $0 \leqslant r < 1$. Hence, we deduce that, for any $\epsilon > 0$ and any polynomial $\psi \in \mathbb{C}[Z_1, \dots, Z_n]$, there exists $r_0 \in (0, 1)$ such that

$$\left\| \widehat{f_j}(M_{Z_1}, \dots, M_{Z_n})\psi - \sum_{k=0}^\infty \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} r_0^{|\alpha|} M_{Z_\alpha} \psi \right\|_{\mathbb{H}^2(f)} < \epsilon, \quad j = 1, \dots, n.$$

Using the convergence of the series in the operator norm topology, we find $N_j \geqslant 1$ such that

$$\left\| \widehat{f}_j(M_{Z_1}, \dots, M_{Z_n})\psi - \sum_{k=0}^{N_j} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} c_\alpha^{(j)} r_0^{|\alpha|} M_{Z_\alpha}\psi \right\|_{\mathbb{H}^2(f)} < \epsilon, \quad j = 1, \dots, n.$$

Now, one can proceed as in the first part of the proof to show that $\mathbb{C}[Z_1, \dots, Z_n]$ is dense in the Hilbert space $\mathbb{H}^2(f)$. The proof is complete. □

According to [30] and [32], the noncommutative Hardy space $H^\infty_{\mathbf{ball}}(B(\mathcal{E}, \mathcal{G}))$ can be identified to the operator space $F_n^\infty \bar{\otimes} B(\mathcal{E}, \mathcal{G})$ (the weakly closed operator space generated by the spatial tensor product), where F_n^∞ is the noncommutative analytic Toeplitz algebra. More precisely, a bounded free holomorphic function F is uniquely determined by its (*model*) *boundary function* $\widetilde{F} \in F_n^\infty \bar{\otimes} B(\mathcal{E}, \mathcal{G})$ defined by $\widetilde{F} :=$ SOT-$\lim_{r\to 1} F(rS_1, \dots, rS_n)$. Moreover, F is the noncommutative Poisson transform [26] of $\widetilde{F}$ at $X \in [B(\mathcal{H})^n]_1$, i.e., $F(X) = (P_X \otimes I)[\widetilde{F}]$. Similar results hold for bounded free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$.

The next result provides a characterization for the n-tuples of formal power series with property $(\mathcal{S})$.

Lemma 2.6. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series with $f(0) = 0$. Then f has property $(\mathcal{S})$ if and only if the following conditions hold*:

(i) *the n-tuple f has nonzero radius of convergence and* $\det J_f(0) \neq 0$;
(ii) *the inverse of f, say $g = (g_1, \dots, g_n)$, is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$*;
(iii) *the model boundary function $\widetilde{g} = (\widetilde{g}_1, \dots, \widetilde{g}_n)$ satisfies either one of the following conditions*:
  (a) *$\widetilde{g}$ is in $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and $S_i = f_i(\widetilde{g}_1, \dots, \widetilde{g}_n)$, $i = 1, \dots, n$*;
  (b) *$\widetilde{g}$ is in $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$ and $S_i = SOT$-$\lim_{r\to 1} f_j(r\widetilde{g}_1, \dots, r\widetilde{g}_n)$, $i = 1, \dots, n$, where $(S_1, \dots, S_n)$ are the left creation operators on the full Fock space $F^2(H_n)$.*

If f is an n-tuple of noncommutative polynomials, then condition (iii) *is automatically satisfied.*

Proof. Since condition $(\mathcal{S}_1)$ coincides with (i), we show that condition $(\mathcal{S}_2)$ holds if and only if f satisfies condition (ii). To prove the direct implication note that, by Theorem 1.3, the composition map $C_f : \mathbf{S}[Z_1, \dots, Z_n] \to \mathbf{S}[Z_1, \dots, Z_n]$ defined by $C_f\psi := \psi \circ f$ is an isomorphism. Therefore, there is an n-tuple $g = (g_1, \dots, g_n)$ of power series such that $f \circ g = g \circ f = id$. On the other hand, condition $(\mathcal{S}_2)$ implies the existence of an n-tuple $\chi = (\chi_1, \dots, \chi_n)$ of formal power series with $\chi(0) = 0$ and $\chi_i \in H^2_{\mathbf{ball}}$, i.e., $\chi_i = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ for some $a_\alpha^{(i)} \in \mathbb{C}$ with $\sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha^{(i)}|^2 < \infty$, and such that $\chi \circ f = id$. Consequently, $(f \circ \chi - id) \circ f = 0$ and, using the injectivity of C_f, we deduce that $f \circ \chi = id$. Since the inverse of f is unique, we must have $g = \chi$.

Due to condition $(\mathcal{S}_2)$, the left multiplication operator $M_{Z_i} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$ defined by

$$M_{Z_i}\psi := Z_i\psi, \quad \psi \in \mathbb{H}^2(f),$$

is a bounded multiplier of $\mathbb{H}^2(f)$. Let $U : \mathbb{H}^2(f) \to F^2(H_n)$ be the unitary operator defined by $U(f_\alpha) := e_\alpha$, $\alpha \in \mathbb{F}_n^+$. Note that $Z_i = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} f_\alpha = U^{-1}(\varphi_i)$, where $\varphi := \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} e_\alpha \in F^2(H_n)$. One can easily see that M_{Z_i} is a bounded multiplier of $\mathbb{H}^2(f)$ if and only if φ_i is a bounded multiplier of $F^2(H_n)$. Moreover, $M_{Z_i} = U^{-1}\varphi_i(S_1,\ldots,S_n)U$, where $\varphi_i(S_1,\ldots,S_n)$ is in the noncommutative Hardy algebra F_n^∞ and has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} S_\alpha$. According to Theorem 3.1 from [30], we deduce that $g_i = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ is a bounded free holomorphic function on the unit ball $[B(\mathcal{H})^n]_1$ and has its model boundary function $\widetilde{g_i} = \varphi_i(S_1,\ldots,S_n)$. Therefore, condition $(\mathcal{S}_2)$ is equivalent to item (ii). Since each $g \in \mathbb{H}^2(f)$ has a unique representation $g = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha f_\alpha$ with $\sum_{\alpha\in\mathbb{F}_n^+} |a_\alpha|^2 < \infty$, the multiplication operator $M_{f_j} : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$ defined by

$$M_{f_j}\left(\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha f_\alpha\right) = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha f_j f_\alpha$$

satisfies the equation

$$M_{f_j} = U^{-1} S_j U, \quad j = 1,\ldots,n, \tag{2.2}$$

where $S_1,\ldots,S_n$ are the left creation operators on $F^2(H_n)$. Consequently, $M_{f_\alpha} = U^{-1}S_\alpha U$, $\alpha \in \mathbb{F}_n^+$. Since $M_{Z_i} = U^{-1}\widetilde{g}_i U$, where $\widetilde{g}_i$ is the model boundary function of $g_i \in H^\infty_{\mathbf{ball}}$, it is easy to see that the equality $M_{f_j} = f_j(M_{Z_1},\ldots,M_{Z_n})$, $j = 1,\ldots,n$, of $(\mathcal{S}_3)$ is equivalent to condition (iii). This completes the proof. $\square$

Let $g = (g_1,\ldots,g_n)$ be the n-tuple of power series, as in Lemma 2.6, having the representations

$$g_i := \sum_{k=0}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)} Z_\alpha, \quad i = 1,\ldots,n,$$

where the sequence $\{a_\alpha^{(i)}\}_{\alpha\in\mathbb{F}_n^+}$ is uniquely determined by the condition $g \circ f = id$. We say that an n-tuple of operators $X = (X_1,\ldots,X_n) \in B(\mathcal{H})^n$ satisfies the equation $g(f(X)) = X$ if either one of the following conditions hold:

(a) $X \in \mathcal{C}_f^{SOT}(\mathcal{H})$ and either $X_i = \sum_{k=1}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)} [f(X)]_\alpha$, $i = 1,\ldots,n$, where the convergence of the series is in the strong operator topology, or

$$X_i = \text{SOT-}\lim_{r\to1} \sum_{k=1}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} \big[f(X)\big]_\alpha, \quad i = 1,\ldots,n;$$

(b) $X \in \mathcal{C}_f^{rad}(\mathcal{H})$ and either $X_i = \sum_{k=1}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)} [\widehat{f}(X)]_\alpha$, $i = 1,\ldots,n$, where the convergence of the series is in the strong operator topology, or

$$X_i = \text{SOT-}\lim_{r\to1} \sum_{k=1}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} \big[\widehat{f}(X)\big]_\alpha, \quad i = 1,\ldots,n.$$

We consider the noncommutative domains

$$\mathbb{B}_f(\mathcal{H}) := \{X = (X_1, \dots, X_n) \in B(\mathcal{H})^n\colon\ g\big(f(X)\big) = X \text{ and } \|f(X)\| \leqslant 1\}$$

and

$$\mathbb{B}_f^<(\mathcal{H}) := \{X = (X_1, \dots, X_n) \in B(\mathcal{H})^n\colon\ g\big(f(X)\big) = X \text{ and } \|f(X)\| < 1\}.$$

We say that $(T_1, \dots, T_n) \in B(\mathcal{H})^n$ is a pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$ if

$$\text{SOT-}\lim_{k\to\infty} \sum_{\alpha\in\mathbb{F}_n,\, |\alpha|=k} \big[f(T)\big]_\alpha \big[f(T)\big]_\alpha^* = 0.$$

The set of all pure elements of $\mathbb{B}_f(\mathcal{H})$ is denoted by $\mathbb{B}_f^{pure}(\mathcal{H})$. Note that

$$\mathbb{B}_f^<(\mathcal{H}) \subseteq \mathbb{B}_f^{pure}(\mathcal{H}) \subseteq \mathbb{B}_f(\mathcal{H}).$$

An n-tuple of operators $X := (X_1, \dots, X_n) \in B(\mathcal{H})^n$ is called nilpotent if there is $m \geqslant 1$ such that $X_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = m$. We denote by $\mathbb{B}_f^{nil}(\mathcal{H})$ the set of all nilpotent n-tuples in $\mathbb{B}_f(\mathcal{H})$.

Proposition 2.7. *Let $g \in H^\infty(\mathbb{D})$ be such that $g(0) = 0$ and $g'(0) \neq 0$, and let f be its inverse power series with respect to composition. If S is the unilateral shift on the Hardy space $H^2(\mathbb{D})$ and*

$$f\big(g(S)\big) = S$$

for an appropriate evaluation of f at $g(S)$ (where $g(S)$ is defined using the Nagy–Foias functional calculus), then f has the property $(\mathcal{S})$.

Proof. According to [6], the power series associated with g has an inverse f, with respect to composition, with nonzero radius of convergence. Using the fact that $S = f(g(S))$ and applying Lemma 2.6 when $n = 1$, we deduce that f has the property $(\mathcal{S})$. □

In what follows, we present several examples of n-tuples of formal power series with property $(\mathcal{S})$. First, we consider the single variable case.

Example 2.8. The power series defined by

$$f = Z\left(I + \frac{1}{a}Z\right)^{-1}, \quad a > 2,$$

has property $(\mathcal{S})$ and

$$\big[B(\mathcal{H})\big]_1^- \subsetneq \big[B(\mathcal{H})\big]_{\frac{a}{a-1}} \subset \mathbb{B}_f(\mathcal{H}).$$

Proof. A straightforward computation shows that the inverse power series of f is $g = Z(I - \frac{1}{a}Z)^{-1}$. The corresponding function $z \mapsto g(z)$ is analytic and bounded on $\mathbb{D}$. Moreover,

$$g(S) = S - \frac{1}{a}S^2 + \frac{1}{a^2}S^3 + \cdots$$

is a bounded operator, where the convergence is in the operator norm topology, and $\|g(S)\| < 2$. Taking into account that $\|\frac{1}{a}g(S)\| < 1$, we deduce that

$$f\big(g(S)\big) = g(S)\left(I + \frac{1}{a}g(S)\right)^{-1} = S\left(I - \frac{1}{a}S\right)^{-1}\left(I + \frac{1}{a}S\left(I - \frac{1}{a}S\right)^{-1}\right)^{-1} = S.$$

Therefore, f has property $(\mathcal{S})$. Consider the noncommutative domain

$$\mathbb{B}_f(\mathcal{H}) := \big\{X \in B(\mathcal{H}):\ X = g\big(f(X)\big) \text{ and } \big\|f(X)\big\| \leqslant 1\big\}.$$

Note that if $\|X\| < a$, then $f(X) := X(I + \frac{1}{a}X)^{-1}$ is well defined. If, in addition, $\|f(X)\| \leqslant 1$, then one can easily see that

$$g\big(f(X)\big) = f(X)\left(I - \frac{1}{a}f(X)\right)^{-1} = X.$$

Hence

$$\big\{X \in B(\mathcal{H}):\ \|X\| < a \text{ and } \big\|f(X)\big\| \leqslant 1\big\} \subset \mathbb{B}_f(\mathcal{H}).$$

Note also that if $\|X\| \leqslant \frac{a}{a-1}$, then

$$\big\|f(X)\big\| \leqslant \|X\|\frac{1}{1 + \frac{\|X\|}{a}} \leqslant 1.$$

Since $a > 2$, we have $1 < \frac{a}{a-1} < a$ and

$$\big[B(\mathcal{H})\big]_1^- \subsetneq \big[B(\mathcal{H})\big]_{\frac{a}{a-1}} \subset \mathbb{B}_f(\mathcal{H}).$$

This completes the proof. □

Now we consider some tuples of noncommutative polynomials with the property $(\mathcal{S})$.

Example 2.9. If

$$\begin{cases} p_1 = Z_1 - Z_2 - \frac{1}{2}Z_1Z_2, \\ p_2 = Z_2 \end{cases} \quad \text{and} \quad \begin{cases} q_1 = Z_1 - \frac{1}{3}Z_1Z_2, \\ q_2 = Z_2 - \frac{1}{2}Z_3Z_2, \\ q_3 = Z_3, \end{cases}$$

then $p = (p_1, p_2)$ and $q = (q_1, q_2, q_3)$ have property $(\mathcal{S})$.

Proof. Note that

$$\begin{cases} Z_1 = (p_1 + p_2)\left(I + \frac{p_2}{2} + \left(\frac{p_2}{2}\right)^2 + \cdots\right), \\ Z_2 = p_2. \end{cases}$$

Setting $g_1 := (Z_1 + Z_2)(I + \frac{Z_2}{2} + (\frac{Z_2}{2})^2 + \cdots)$ and $g_2 = Z_2$, it is easy to see that $p \circ g = g \circ p = id$. On the other hand, $g = (g_1, g_2)$ is a bounded free holomorphic function on $[B(\mathcal{H})^2]_1$ and the model boundary function $\widetilde{g} = (\widetilde{g}_1, \widetilde{g}_2)$ is given by $\widetilde{g}_1 := (S_1 + S_2)(I + \frac{1}{2}S_2 + (\frac{1}{2}S_2)^2 + \cdots)$ and $\widetilde{g}_2 = S_2$. According to Lemma 2.6, $p = (p_1, p_2)$ has property $(\mathcal{S})$. The second example can be treated similarly. Setting $r = (r_1, r_2, r_3)$, where

$$\begin{cases} r_1 = Z_1\left[I - \frac{1}{3}Z_2\left(I - \frac{1}{2}Z_3\right)^{-1}\right]^{-1}, \\ r_2 = Z_2\left(I - \frac{1}{2}Z_3\right)^{-1}, \\ r_3 = Z_3, \end{cases}$$

one can check that $r \circ q = q \circ r$ and the model boundary functions $\widetilde{r}_1 = S_1[I - \frac{1}{3}S_2(I - \frac{1}{2}S_3)^{-1}]^{-1}$, $\widetilde{r}_2 = S_2(I - \frac{1}{2}S_3)^{-1}$ and $\widetilde{r}_3 = S_3$ are in noncommutative disc algebra $\mathcal{A}_3$. Applying again Lemma 2.6, we deduce that $q = (q_1, q_2)$ has property $(\mathcal{S})$. □

Example 2.10. Let $\gamma > 0$ and $a \in \mathbb{C}$ with $|a| > 1$ and let

$$f_1 = \frac{1}{\gamma}Z_1 - \frac{1}{\gamma}Z_2 - \left(\frac{1}{\gamma}Z_2\right)^2 - \cdots$$
$$f_2 = \frac{a}{\gamma}Z_2.$$

Then $f = (f_1, f_2)$ has property $(\mathcal{S})$.

Proof. First note that $f = (f_1, f_2)$ satisfies condition $(\mathcal{S}_1)$. Since $Z_1 = \gamma f_1 + \gamma\sum_{j=1}^{\infty}(\frac{1}{a}f_2)^j$, $Z_2 = \frac{\gamma}{a}f_2$, and $\sum_{j=1}^{\infty}\frac{1}{|a|^{2j}} < \infty$, we deduce that $Z_1, \ldots, Z_n$ are in $\mathbb{H}^2(f)$. Let $U : \mathbb{H}^2(f) \to F^2(H_2)$ be the unitary operator defined by $U(f_\alpha) := e_\alpha$, $\alpha \in \mathbb{F}_2^+$. Note that the multiplication operator $M_{Z_1} \in B(\mathbb{H}^2(f))$ is unitarily equivalent to the operator $\varphi_1(S_1, S_2) \in B(F^2(H_2))$ defined by

$$\varphi_1(S_1, S_2) := \gamma S_1 + \gamma\sum_{j=1}^{\infty}\left(\frac{1}{a}S_2\right)^j,$$

which is in the noncommutative disc algebra $\mathcal{A}_2$. Similarly, the operator $M_{Z_2} \in B(\mathbb{H}^2(\psi))$ is unitarily equivalent to $\varphi_2(S_1, S_2) := \frac{\gamma}{a} S_2 \in \mathcal{A}_2$. Therefore, condition $(\mathcal{S}_2)$ is satisfied. It remains to check condition $(\mathcal{S}_3)$. Since $|a| > 1$, we have $\|M_{Z_2}\| < \gamma$ and, therefore,

$$f(M_{Z_1}, M_{Z_2}) = \frac{1}{\gamma} M_{Z_1} - \sum_{j=1}^{\infty} \left(\frac{1}{\gamma} M_{Z_2} \right)^j,$$

where the convergence is in the operator norm topology. On the other hand, since the operator $M_{f_1} \in B(\mathbb{H}^2(f))$ is unitarily equivalent to the left creation operator S_1 on $F^2(H_2)$, the condition $M_{f_1} = \lim_{m \to \infty} [\frac{1}{\gamma} M_{Z_1} - \sum_{j=1}^{m} (\frac{1}{\gamma} M_{Z_2})^j]$ is equivalent to

$$S_1 = \lim_{m \to \infty} \left[S_1 + \sum_{j=1}^{\infty} \left(\frac{1}{a} S_2 \right)^j - \sum_{j=1}^{m} \left(\frac{1}{a} S_2 \right)^j \right],$$

which is obviously true. This completes the proof. $\square$

Similarly, one can treat the following

Example 2.11. If

$$\begin{aligned} f_1 &= Z_1 - Z_2 - Z_2 Z_1 - Z_2^2 - Z_2^3 \cdots \\ f_2 &= 2 Z_2, \end{aligned}$$

then $f = (f_1, f_2)$ has property $(\mathcal{S})$.

Hilbert spaces of free holomorphic functions. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be an n-tuple of free holomorphic functions on $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, with range in $[B(\mathcal{H})^n]_1$. We say that φ is not a *right zero divisor* with respect to the composition with free holomorphic functions on $[B(\mathcal{H})^n]_1$ if for any non-zero free holomorphic function G on $[B(\mathcal{H})^n]_1$, the composition $G \circ \varphi$ is not identically zero. We recall (see [33]) that $G \circ \varphi$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$. Consider the vector space of free holomorphic functions

$$\mathbb{H}^2(\varphi) := \left\{ G \circ \varphi \colon\ G \in H^2_{\mathbf{ball}} \right\},$$

where the noncommutative Hardy space $H^2_{\mathbf{ball}}$ is the Hilbert space of all free holomorphic functions on $[B(\mathcal{H})^n]_1$ of the form

$$f(X_1, \ldots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha, \qquad \sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha|^2 < \infty,$$

with the inner product $\langle f, g \rangle := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \bar{b}_\alpha$, where $g = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} b_\alpha X_\alpha$ is another free holomorphic function in $H^2_{\mathbf{ball}}$. Note that each element $\psi \in \mathbb{H}^2(\varphi)$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has a unique representation of the form $\psi = G \circ \varphi$ for some $G \in H^2_{\mathbf{ball}}$. We introduce an inner product on $\mathbb{H}^2(\varphi)$ by setting

$$\langle F \circ \varphi, G \circ \varphi \rangle_{\mathbb{H}^2(\varphi)} := \langle F, G \rangle_{H^2_{\mathbf{ball}}}.$$

It is easy to see that $\mathbb{H}^2(\varphi)$ is a Hilbert space with respect to this inner product. We make the following assumptions:

($\mathcal{F}_1$) the n-tuple $\varphi = (\varphi_1, \ldots, \varphi_n)$ of free holomorphic functions on $[B(\mathcal{H})^n]_\gamma$ has range in $[B(\mathcal{H})^n]_1$ and it is not a right zero divisor with respect to the composition with free holomorphic functions on $[B(\mathcal{H})^n]_1$.

($\mathcal{F}_2$) The coordinate functions $X_1, \ldots, X_n$ on $[B(\mathcal{H})^n]_\gamma$ are contained in $\mathbb{H}^2(\varphi)$ and the left multiplication by X_i is a bounded multiplier of $\mathbb{H}^2(\varphi)$, for each $i = 1, \ldots, n$.

($\mathcal{F}_3$) For each $i = 1, \ldots, n$, the left multiplication operator $M_{\varphi_i} : \mathbb{H}^2(\varphi) \to \mathbb{H}^2(\varphi)$ satisfies the equation

$$M_{\varphi_i} = \varphi_i(M_{Z_1}, \ldots, M_{Z_n}),$$

where $(M_{Z_1}, \ldots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_\varphi^{SOT}(\mathbb{H}^2(\varphi))$ or $\mathcal{C}_\varphi^{rad}(\mathbb{H}^2(\varphi))$.

If φ is an n-tuple of noncommutative polynomials, then the condition ($\mathcal{F}_3$) is automatically satisfied. Under the above-mentioned conditions, the free holomorphic function φ is said to have property ($\mathcal{F}$). We remark that, unlike the power series with property ($\mathcal{S}$), $\varphi(0)$ could be different from 0.

Using Theorem 2.1 from [33], we can show that φ has property ($\mathcal{F}$) if and only if there exists $g = (g_1, \ldots, g_n)$ a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$ such that

$$g\bigl(\varphi(X)\bigr) = X, \quad X \in \bigl[B(\mathcal{H})^n\bigr]_\gamma, \tag{2.3}$$

where $\varphi(X)$ is in the set of norm-convergence of g, and the model boundary function $\widetilde{g} = (\widetilde{g}_1, \ldots, \widetilde{g}_n)$ satisfies either one of the following conditions:

(a) $\widetilde{g}$ is in $\mathcal{C}_\varphi^{SOT}(\mathbb{H}^2(\varphi))$ and $S_i = \varphi_i(\widetilde{g}_1, \ldots, \widetilde{g}_n)$, $i = 1, \ldots, n$;

(b) $\widetilde{g}$ is in $\mathcal{C}_\varphi^{rad}(\mathbb{H}^2(\varphi))$ and $S_i = \text{SOT-}\lim_{r\to 1} \varphi_j(r\widetilde{g}_1, \ldots, r\widetilde{g}_n)$, $i = 1, \ldots, n$, where $(S_1, \ldots, S_n)$ are the left creation operators on the full Fock space $F^2(H_n)$.

Example 2.12. If

$$\varphi_1 = \frac{1}{6}Z_1 - \frac{1}{8}Z_2 - \left(\frac{1}{8}Z_2\right)^2 - \cdots$$
$$\varphi_2 = \frac{1}{3}Z_2.$$

Then $\varphi = (\varphi_1, \varphi_2)$ is a free holomorphic function on $[B(\mathcal{H})^2]_2$ and has property ($\mathcal{F}$). In this case, $\mathbb{H}^2(\varphi)$ is a Hilbert space of free holomorphic functions on $[B(\mathcal{H})^2]_2$.

The theory of noncommutative characteristic functions for row contractions [19] was used in [33] to determine the group $\mathrm{Aut}(B(\mathcal{H})_1^n)$ of all free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$. We showed that any $\Psi \in \mathrm{Aut}(B(\mathcal{H})_1^n)$ has the form

$$\Psi = \Phi_U \circ \Psi_\lambda,$$

where Φ_U is an automorphism implemented by a unitary operator U on $\mathbb{C}^n$, i.e.,

$$\Phi_U(X_1, \ldots, X_n) := [X_1, \ldots, X_n]U, \quad (X_1, \ldots, X_n) \in \big[B(\mathcal{H})^n\big]_1,$$

and Ψ_λ is an involutive free holomorphic automorphism associated with $\lambda := \Psi^{-1}(0) \in \mathbb{B}_n$. The automorphism $\Psi_\lambda : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ is given by

$$\Psi_\lambda(X_1, \ldots, X_n) := \lambda - \Delta_\lambda \left(I_{\mathcal{H}} - \sum_{i=1}^n \bar{\lambda}_i X_i \right)^{-1} [X_1, \ldots, X_n]\Delta_{\lambda^*}, \quad (X_1, \ldots, X_n) \in \big[B(\mathcal{H})^n\big]_1,$$

where Δ_λ and Δ_{λ^*} are the defect operators associated with the row contraction $\lambda := (\lambda_1, \ldots, \lambda_n)$. Note that, when $\lambda = 0$, we have $\Psi_0(X) = -X$. We recall that if $\lambda \in \mathbb{B}_n \setminus \{0\}$ and $\gamma := \frac{1}{\|\lambda\|_2}$, then Ψ_λ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has the following properties:

(i) $\Psi_\lambda(0) = \lambda$ and $\Psi_\lambda(\lambda) = 0$;
(ii) Ψ_λ is an involution, i.e., $\Psi_\lambda(\Psi_\lambda(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$;
(iii) Ψ_λ is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$;
(iv) Ψ_λ is a homeomorphism of $[B(\mathcal{H})^n]_1^-$ onto $[B(\mathcal{H})^n]_1^-$;
(v) the model boundary function $\tilde{\Psi}_\lambda$ is unitarily equivalent to the row contraction $[S_1, \ldots, S_n]$.

Proposition 2.13. *Any free holomorphic automorphism of* $[B(\mathcal{H})^n]_1$ *has property* $(\mathcal{F})$.

Proof. Let $\varphi \in \mathrm{Aut}(B(\mathcal{H})_1^n)$. Since the composition of free holomorphic functions is a free holomorphic function, one can easily show, by contradiction, that condition $(\mathcal{F}_1)$ is satisfied by φ. Now, taking into account the properties of the free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$ and the remarks above, we have $\varphi \in H^\infty_{\mathbf{ball}}$ and $\varphi(\varphi(X)) = X$ for all $X \in [B(\mathcal{H})^n]_1$, which shows that condition $(\mathcal{F}_2)$ holds. Moreover, since the multiplication $M_{X_i} : \mathbb{H}^2(\varphi) \to \mathbb{H}^2(\varphi)$ is unitarily equivalent to the model boundary function $\widetilde{\varphi}$ acting on $F^2(H_n)$, and $M_{\varphi_i} : \mathbb{H}^2(\varphi) \to \mathbb{H}^2(\varphi)$ is unitarily equivalent to $S_i \in B(F^2(H_n))$, the equation $M_{\varphi_i} = \varphi_i(M_{Z_1}, \ldots, M_{Z_n})$ is equivalent to the equation $S_i = \varphi_i(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$, where $(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ is in the convergence set $\mathcal{C}_\varphi^{rad}(\mathbb{H}^2(\varphi))$. Due to the functional calculus for row contractions [22], the latter equality holds for any $\varphi \in \mathrm{Aut}(B(\mathcal{H})_1^n)$. Therefore, φ satisfies condition $(\mathcal{F}_3)$, which proves our assertion. $\square$

We saw above that, due to condition $(\mathcal{F}_2)$, there is a bounded free holomorphic function $g : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})^n$ such that $X = g(\varphi(X))$ for any $X \in [B(\mathcal{H})^n]_\gamma$. We consider the noncommutative domain

$$\mathbb{B}_\varphi(\mathcal{H}) := \big\{ Y = (Y_1, \ldots, Y_n) \in B(\mathcal{H})^n \colon\ g\big(\varphi(Y)\big) = Y \text{ and } \big\|\varphi(Y)\big\| \leqslant 1 \big\}$$

which will be studied in the next sections. Note that the ball $[B(\mathcal{H})^n]_\gamma$ is included in $\mathbb{B}_\varphi(\mathcal{H})$.

3. Noncommutative domains and the universal model $(M_{Z_1},\dots,M_{Z_n})$

Throughout this section, unless otherwise specified, we assume that $f=(f_1,\dots,f_n)$ is either one of the following:

(i) an n-tuple of polynomials with property $(\mathcal{A})$;
(ii) an n-tuple of formal power series with $f(0)=0$ and property $(\mathcal{S})$;
(iii) an n-tuple of free holomorphic functions with property $(\mathcal{F})$.

In this case, we say that f has the *model property*. We denote by $\mathcal{M}$ the set of all n-tuples f with the model property. The noncommutative domain associated with f is

$$\mathbb{B}_f(\mathcal{H}):=\bigl\{X=(X_1,\dots,X_n)\in B(\mathcal{H})^n\colon\ g\bigl(f(X)\bigr)=X \text{ and } \bigl\|f(X)\bigr\|\leqslant 1\bigr\},$$

where $g:=(g_1,\dots,g_n)$ is the inverse power series of f with respect to composition, and the evaluations are well-defined (see previous section). We recall that the condition $g(f(X))=X$ is automatically satisfied when f is an n-tuple of polynomials with property $(\mathcal{A})$.

In this section, we present some of the basic properties of the universal model $(M_{Z_1},\dots,M_{Z_n})$ associated with the noncommutative domain $\mathbb{B}_f$.

Two n-tuples $(A_1,\dots,A_n)\in B(\mathcal{H})$ and $(B_1,\dots,B_n)\in B(\mathcal{K})$ are said to be unitarily equivalent if there is a unitary operator $U:\mathcal{H}\to\mathcal{K}$ such that $A_i=U^*B_iU$ for all $i=1,\dots,n$.

Theorem 3.1. *Let $T:=(T_1,\dots,T_n)$ be an n-tuple of operators in $B(\mathcal{H})^n$ and let f have the model property. Then the following statements are equivalent*:

(i) *$T=(T_1,\dots,T_n)$ is a pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$*;
(ii) *there exists a Hilbert space $\mathcal{D}$ and a co-invariant subspace $\mathcal{M}\subseteq\mathbb{H}^2(f)\otimes\mathcal{D}$ under each operator $M_{Z_1}\otimes I_{\mathcal{D}},\dots,M_{Z_n}\otimes I_{\mathcal{D}}$ such that the n-tuple $(T_1,\dots,T_n)$ is unitarily equivalent to*

$$\bigl(P_{\mathcal{M}}(M_{Z_1}\otimes I_{\mathcal{D}})|_{\mathcal{M}},\dots,P_{\mathcal{M}}(M_{Z_n}\otimes I_{\mathcal{D}})|_{\mathcal{M}}\bigr).$$

Proof. We shall prove the theorem when f is an n-tuple of formal power series with $f(0)=0$ and has property $(\mathcal{S})$. The other two cases can be treated similarly. Let $g=(g_1,\dots,g_n)$ be the inverse of f with respect to composition. Note that condition $(\mathcal{S}_3)$ implies

$$\sum_{j=1}^n f_j(M_{Z_1},\dots,M_{Z_n})f_j(M_{Z_1},\dots,M_{Z_n})^*=\sum_{j=1}^n M_{f_j}M_{f_j}^*=U^{-1}\left(\sum_{j=1}^n S_jS_j^*\right)U\leqslant I,$$

where $U:\mathbb{H}^2(f)\to F^2(H_n)$ is the unitary operator defined by $U(f_\alpha):=e_\alpha$, $\alpha\in\mathbb{F}_n^+$. Since $M_{f_\alpha}=[f(M_{Z_1},\dots,M_{Z_n})]_\alpha$, $M_{f_\alpha}=U^{-1}S_\alpha U$, $\alpha\in\mathbb{F}_n^+$, and SOT-$\lim_{p\to\infty}\sum_{|\alpha|=p}S_\alpha S_\alpha^*=0$, we deduce that the n-tuple $M_Z:=(M_{Z_1},\dots,M_{Z_n})$ is a pure element with $\|f(M_Z)\|\leqslant 1$. Now let us show that M_Z is in the noncommutative domain $\mathbb{B}_f(\mathbb{H}^2(f))$. It remains to prove that $g(f(M_Z))=M_Z$ which, due to condition $(\mathcal{S}_3)$, is equivalent to

$$g_i(M_{f_1},\dots,M_{f_n})=M_{Z_i},\quad i=1,\dots,n. \tag{3.1}$$

According to Lemma 2.6, if $g_i = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$, then $M_{Z_i} = U^{-1}\varphi_i(S_1,\dots,S_n)U$, where $\varphi_i(S_1,\dots,S_n)\in F_n^\infty$ has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} S_\alpha$. Proving the equality above is equivalent to showing that

$$\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} S_\alpha = \varphi_i(S_1,\dots,S_n),\quad i=1,\dots,n.$$

The latter relation is well known (see [22]). Therefore, $M_Z\in\mathbb{B}_f(\mathbb{H}^2(f))$. If $\mathcal{D}$ is a Hilbert space and $\mathcal{M}\subseteq\mathbb{H}^2(f)\otimes\mathcal{D}$ is a co-invariant subspace under $M_{Z_1}\otimes I_{\mathcal{D}},\dots,M_{Z_n}\otimes I_{\mathcal{D}}$, then

$$\big[f\big(P_{\mathcal{M}}(M_{Z_1}\otimes I_{\mathcal{D}})|_{\mathcal{M}},\dots,P_{\mathcal{M}}(M_{Z_n}\otimes I_{\mathcal{D}})|_{\mathcal{M}}\big)\big]_\alpha = P_{\mathcal{M}}\big\{\big[f(M_{Z_1},\dots,M_{Z_n})\big]_\alpha\otimes I_{\mathcal{D}}\big\}\big|_{\mathcal{M}}$$

for any $\alpha\in\mathbb{F}_n^+$. Due to relation

$$g_i\big(f_1(M_{Z_1},\dots,M_{Z_n}),\dots,f_n(M_{Z_1},\dots,M_{Z_n})\big) = M_{Z_i},\quad i=1,\dots,n,$$

we deduce that

$$\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|}\big[f(M_{Z_1},\dots,M_{Z_n})\big]_\alpha = M_{Z_i},\quad i=1,\dots,n.$$

Taking the compression to the subspace $\mathcal{M}\subseteq\mathbb{H}^2(f)\otimes\mathcal{D}$, we deduce that

$$g_i\big(f\big(P_{\mathcal{M}}(M_{Z_1}\otimes I_{\mathcal{D}})|_{\mathcal{M}},\dots,P_{\mathcal{M}}(M_{Z_n}\otimes I_{\mathcal{D}})|_{\mathcal{M}}\big)\big) = P_{\mathcal{M}}(M_{Z_i}\otimes I_{\mathcal{D}})|_{\mathcal{M}}$$

for each $i=1,\dots,n$. Since $(M_{Z_1},\dots,M_{Z_n})$ is a pure element in $\mathbb{B}_f(\mathbb{H}^2(f))$, we deduce that the n-tuple $(P_{\mathcal{M}}(M_{Z_1}\otimes I_{\mathcal{D}})|_{\mathcal{M}},\dots,P_{\mathcal{M}}(M_{Z_n}\otimes I_{\mathcal{D}})|_{\mathcal{M}})$ is a pure element in $\mathbb{B}_f(\mathcal{M})$. Therefore, the implication (ii) $\Longrightarrow$ (i) holds.

Now, we prove the implication (i) $\Longrightarrow$ (ii). Assume that condition (i) holds. Let $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ be a pure n-tuple of operators. Consider the defect operator

$$\Delta_{f,T} := \left(I-\sum_{j=1}^n f_j(T)f_j(T)^*\right)^{1/2}$$

and the defect space $\mathcal{D}_{f,T} := \overline{\Delta_f(T)\mathcal{H}}$. Define the noncommutative Poisson kernel $K_{f,T}:\mathcal{H}\to\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}$ by setting

$$K_{f,T}h := \sum_{\alpha\in\mathbb{F}_n^+} f_\alpha\otimes\Delta_{f,T}\big[f(T)\big]_\alpha^* h,\quad h\in\mathcal{H}. \tag{3.2}$$

We need to prove that $K_{f,T}$ is an isometry and $K_{f,T}T_i^* = (M_{Z_i}\otimes I_{\mathcal{D}_{f,T}})K_{f,T}$ for any $i=1,\dots,n$. Indeed, a straightforward calculation reveals that

$$\begin{aligned}\Big\|\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|\leqslant q} f_\alpha\otimes\Delta_{f,T}\big[f(T)\big]_\alpha^* h\Big\|^2_{\mathbb{H}^2(f)\otimes\mathcal{H}} &= \sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|\leqslant q}\big\|\Delta_{f,T}\big[f(T)\big]_\alpha^* h\big\|^2_{\mathcal{H}}\\ &= \sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|\leqslant q}\big\langle\big[f(T)\big]_\alpha\Delta_{f,T}^2\big[f(T)\big]_\alpha^* h, h\big\rangle\\ &= \|h\| - \Big\langle\Big(\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=q}\big[f(T)\big]_\alpha\big[f(T)\big]_\alpha^*\Big)h, h\Big\rangle\end{aligned}$$

for any $q\in\mathbb{N}$. Since $T=(T_1,\ldots,T_n)$ is a pure n-tuple in $\mathbb{B}_f(\mathcal{H})$ we have

$$\text{SOT-}\lim_{q\to\infty}\sum_{\alpha\in\mathbb{F}_n,\,|\alpha|=q}\big[f(T)\big]_\alpha\big[f(T)\big]_\alpha^* = 0.$$

Consequently, we obtain $\|K_{f,T}h\|=\|h\|$ for any $h\in\mathcal{H}$. On the other hand, for any $h,h'\in\mathcal{H}$ and $\alpha\in\mathbb{F}_n^+$, we have

$$\begin{aligned}\big\langle K_{f,T}^*(f_\alpha\otimes h), h'\big\rangle &= \big\langle f_\alpha\otimes h, K_{f,T}h'\big\rangle\\ &= \big\langle h, \Delta_{f,T}\big[f(T)\big]_\alpha^* h'\big\rangle\\ &= \big\langle\big[f(T)\big]_\alpha\Delta_{f,T}h, h'\big\rangle.\end{aligned}$$

Therefore,

$$K_{f,T}^*(f_\alpha\otimes h) = \big[f(T)\big]_\alpha\Delta_{f,T}h,\quad h\in\mathcal{H}. \tag{3.3}$$

Since the n-tuple f has property $(\mathcal{S})$, for each $i=1,\ldots,n$, $Z_i\in\mathbb{H}^2(f)$, i.e., there is a sequence $\{a_\alpha^{(i)}\}_{\alpha\in\mathbb{F}_n^+}$ with $\sum_{\alpha\in\mathbb{F}_n^+}|a_\alpha^{(i)}|^2<\infty$ such that

$$Z_i = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)}\big[f(Z)\big]_\alpha.$$

Taking into account that $T:=(T_1,\ldots,T_n)\in\mathbb{B}_f(\mathcal{H})$, we have either $T\in\mathcal{C}_f^{SOT}(\mathcal{H})$ or $T\in\mathcal{C}_f^{rad}(\mathcal{H})$. Let us consider first the case when $T\in\mathcal{C}_f^{SOT}(\mathcal{H})$. The equation $T=g(f(T))$ shows that either

$$T_i = \sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,|\alpha|=k} a_\alpha^{(i)}\big[f(T)\big]_\alpha,\quad i=1,\ldots,n, \tag{3.4}$$

where the convergence of the series is in the strong operator topology, or

$$T_i = \text{SOT-}\lim_{r\to1}\sum_{k=0}^\infty\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|}\big[f(T)\big]_\alpha,\quad i=1,\ldots,n. \tag{3.5}$$

When relation (3.4) holds, we have

$$T_i\big[f(T)\big]_\beta = \sum_{k=0}^\infty \sum_{\alpha\in\mathbb{F}_n^+, |\alpha|=k} a_\alpha^{(i)}\big[f(T)\big]_\alpha\big[f(T)\big]_\beta, \quad i=1,\dots,n,\ \beta\in\mathbb{F}_n^+, \tag{3.6}$$

where the convergence of the series is in the strong operator topology. Using relation (3.3), we deduce that

$$K_{f,T}^*\left(\sum_{k=0}^p \sum_{|\alpha|=k} a_\alpha^{(i)} f_\alpha f_\beta\otimes h\right) = \left(\sum_{k=0}^p \sum_{|\alpha|=k} a_\alpha^{(i)}\big[f(T)\big]_\alpha\big[f(T)\big]_\beta\right)\Delta_{f,T}h, \quad h\in\mathcal{H},$$

for any $p\in\mathbb{N}$. Hence, due to relation (3.6) and the fact that $M_{Z_i} f_\beta = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^{(i)} f_\alpha f_\beta$ in $\mathbb{H}^2(f)$, we obtain

$$K_{f,T}^*(M_{Z_i} f_\beta\otimes h) = T_i\big[f(T)\big]_\beta \Delta_{f,T}h, \quad h\in\mathcal{H},$$

which, combined with relation (3.3), implies

$$K_{f,T}^*(M_{Z_i}\otimes I)(f_\beta\otimes h) = T_i K_{f,T}^*(f_\beta\otimes h)$$

for any $\beta\in\mathbb{F}_n^+$ and $i=1,\dots,n$. Consequently

$$K_{f,T}T_i^* = \big(M_{Z_i}^*\otimes I\big)K_{f,T}$$

for any $i=1,\dots,n$. Now, we assume that relation (3.5) holds. Then, using relation (3.3), we deduce that

$$K_{f,T}^*\left(\sum_{k=0}^p \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} f_\alpha f_\beta\otimes h\right) = \left(\sum_{k=0}^p \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|}\big[f(T)\big]_\alpha\big[f(T)\big]_\beta\right)\Delta_{f,T}h, \quad h\in\mathcal{H},$$

for any $p\in\mathbb{N}$ and $r\in[0,1)$. Taking first $p\to\infty$ and then $r\to 1$, we obtain $K_{f,T}^*(M_{Z_i} f_\beta\otimes h) = T_i[f(T)]_\beta\Delta_{f,T}h$, $h\in\mathcal{H}$. This implies $K_{f,T}T_i^* = (M_{Z_i}^*\otimes I)K_{f,T}$ for any $i=1,\dots,n$. The case when $T\in\mathcal{C}_f^{rad}(\mathcal{H})$ can be treated similarly. The proof is complete. □

Any n-tuple $(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ gives rise to a Hilbert module over $\mathbb{C}[Z_1,\dots,Z_n]$ by setting

$$p\cdot h := p(T_1,\dots,T_n)h, \quad p\in\mathbb{C}[Z_1,\dots,Z_n] \text{ and } h\in\mathcal{H},$$

which we call $\mathbb{B}_f$-Hilbert module. The homomorphisms in this category are the contractive operators intertwining the module action. If $\mathcal{K}\subseteq\mathcal{H}$ is a closed subspace of $\mathcal{H}$ which is invariant under the action of the associated operators with $\mathcal{H}$, i.e., $T_1,\dots,T_n$, then $\mathcal{K}$ and the quotient $\mathcal{H}/\mathcal{K}$ have natural $\mathbb{B}_f$-Hilbert module structure coming from that of $\mathcal{H}$. More precisely, the canonical n-tuples associated with $\mathcal{K}$ and $\mathcal{H}/\mathcal{K}$ are $(T_1|_\mathcal{K},\dots,T_n|_\mathcal{K})\in\mathbb{B}_f(\mathcal{K})$ and

$(P_{\mathcal{K}^\perp}T_1|_{\mathcal{K}^\perp},\dots,P_{\mathcal{K}^\perp}T_n|_{\mathcal{K}^\perp})\in\mathbb{B}_f(\mathcal{K}^\perp)$, respectively, where $P_{\mathcal{K}^\perp}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}^\perp:=\mathcal{H}\ominus\mathcal{K}$.

Each noncommutative domain $\mathbb{B}_f$ has a universal model $(M_{Z_1},\dots,M_{Z_n})\in\mathbb{B}_f(\mathbb{H}^2(f))$. The module structure defined by $M_{Z_1},\dots,M_{Z_n}$ on the Hilbert space $\mathbb{H}^2(f)$ occupies the position of the rank-one free module in the algebraic theory [11]. More precisely, the free $\mathbb{B}_f$-Hilbert module of rank one $\mathbb{H}^2(f)$ has a universal property in the category of pure $\mathbb{B}_f$-Hilbert modules of finite rank. Indeed, it is a consequence of Theorem 3.1 that if $\mathcal{H}$ is a pure finite rank $\mathbb{B}_f$-Hilbert module over $\mathbb{C}[Z_1,\dots,Z_n]$, then there exist $m\in\mathbb{N}$ and a closed submodule $\mathcal{M}$ of $\mathbb{H}^2(f)\otimes I_{\mathbb{C}^m}$ such that $(\mathbb{H}^2(f)\otimes I_{\mathbb{C}^m})/\mathcal{M}$ is isomorphic to $\mathcal{H}$. To clarify our terminology, we mention that the rank of a $\mathbb{B}_f$-Hilbert module $\mathcal{H}$ is the rank of the defect operator $\Delta_{f,T}$, while $\mathcal{H}$ is called pure if T is a pure n-tuple in $\mathbb{B}_f(\mathcal{H})$.

We introduce the *dilation index* of $T=(T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$, denoted by dil-ind$(T)$, to be the minimum dimension of the Hilbert space $\mathcal{D}$ in Theorem 3.1. According to the proof of the latter theorem, we deduce that dil-ind$(T)\leqslant\dim\mathcal{D}_{f,T}=\operatorname{rank}\Delta_{f,T}$. On the other hand, let $\mathcal{G}$ be a Hilbert space such that $\mathcal{H}$ can be identified with a co-invariant subspace of $\mathbb{H}^2(f)\otimes\mathcal{G}$ under $M_{Z_i}\otimes I_{\mathcal{G}}$, $i=1,\dots,n$, and such that $T_i=P_{\mathcal{H}}(M_{Z_i}\otimes I_{\mathcal{G}})|\mathcal{H}$ for $i=1,\dots,n$. Then

$$\begin{aligned}I_{\mathcal{H}}-\sum_{i=1}^n f_i(T)f_i(T)^* &= P_{\mathcal{H}}\left[\left(I_{\mathbb{H}^2(f)}-\sum_{i=1}^n f_i(M_{Z_1},\dots,M_{Z_n})f_i(M_{Z_1},\dots,M_{Z_n})^*\right)\otimes I_{\mathcal{G}}\right]\Bigg|_{\mathcal{H}}\\ &= P_{\mathcal{H}}\big(\Delta_{f,M_Z}^2\otimes I_{\mathcal{G}}\big)\big|_{\mathcal{H}}=P_{\mathcal{H}}(P_{\mathbb{C}}\otimes I_{\mathcal{G}})|_{\mathcal{H}}.\end{aligned}$$

Hence, we obtain $\operatorname{rank}\Delta_{f,T}\leqslant\dim\mathcal{G}$. Therefore, we have proved that dil-ind$(T)=\operatorname{rank}\Delta_{f,T}$.

Corollary 3.2. *If $(T_1,\dots,T_n)$ is a pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$, then*

$$T_\alpha T_\beta^*=K_{f,T}^*\big[\big(M_{Z_\alpha}M_{Z_\beta}^*\big)\otimes I\big]K_{f,T},\quad \alpha,\beta\in\mathbb{F}_n^+,$$

and

$$\left\|\sum_{i=1}^m q_i(T_1,\dots,T_n)q_i(T_1,\dots,T_n)^*\right\|\leqslant\left\|\sum_{i=1}^m q_i(M_{Z_1},\dots,M_{Z_n})q_i(M_{Z_1},\dots,M_{Z_n})^*\right\|$$

for any $q_i\in\mathbb{C}[Z_1,\dots,Z_n]$ and $m\in\mathbb{N}$.

Theorem 3.3. *Let $f=(f_1,\dots,f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1},\dots,M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Then the C^*-algebra $C^*(M_{Z_1},\dots,M_{Z_n})$ is irreducible and coincides with*

$$\overline{\operatorname{span}}\big\{M_{Z_\alpha}M_{Z_\beta}^*:\ \alpha,\beta\in\mathbb{F}_n^+\big\}.$$

Proof. Let $\mathcal{M}\subset\mathbb{H}^2(f)$ be a nonzero subspace which is jointly reducing for $M_{Z_1},\dots,M_{Z_n}$, and let $y=\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha f_\alpha$ be a nonzero power series in $\mathcal{M}$. Then there is $\beta\in\mathbb{F}_n^+$ such that $a_\beta\neq 0$. Since $f=(f_1,\dots,f_n)$ is an n-tuple of formal power series with the model property, we have

$M_{f_i} = f_i(M_Z)$, where $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$. Consequently, we obtain

$$a_\beta = P_{\mathbb{C}} M_{f_\beta}^* y = \left(I - \sum_{i=1}^n f_i(M_Z) f_i(M_Z)^*\right) \left[f(M_Z)\right]_\beta^* y.$$

Taking into account that $\mathcal{M}$ is reducing for $M_{Z_1}, \ldots, M_{Z_n}$ and $a_\beta \neq 0$, we deduce that $1 \in \mathcal{M}$. Using again that $\mathcal{M}$ is invariant under $M_{Z_1}, \ldots, M_{Z_n}$, we obtain $\mathbb{C}[Z_1, \ldots, Z_n] \subset \mathcal{M}$. Since, according to Proposition 2.5, $\mathbb{C}[Z_1, \ldots, Z_n]$ is dense in $\mathbb{H}^2(f)$, we conclude that $\mathcal{M} = \mathbb{H}^2(f)$, which shows that $C^*(M_{Z_1}, \ldots, M_{Z_n})$ is irreducible.

Since f has the model property, we have $Z_i = \sum_{\alpha \in \mathbb{F}^+} a_\alpha^{(i)} f_\alpha \in \mathbb{H}^2(f)$ and the multiplication M_{Z_i} is a bounded multiplier of $\mathbb{H}^2(f)$ which satisfies the equation

$$M_{Z_i} = \text{SOT-}\lim_{r \to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} M_{f_\alpha}, \quad i = 1, \ldots, n.$$

Hence, and taking into account that

$$f_i(M_Z)^* f_j(M_Z) = M_{f_i}^* M_{f_j} = \delta_{ij} I, \quad i, j \in \{1, \ldots, n\},$$

we deduce that, for any $x, y \in \mathbb{H}^2(f)$,

$$\begin{aligned}
\langle M_{Z_i}^* M_{Z_j} x, y \rangle &= \lim_{r \to 1} \left\langle \sum_{k=0}^\infty \sum_{|\beta|=k} a_\beta^{(j)} r^{|\beta|} \left[f(M_Z)\right]_\beta x, \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} r^{|\alpha|} \left[f(M_Z)\right]_\alpha y \right\rangle \\
&= \lim_{r \to 1} \lim_{m \to \infty} \left\langle \sum_{|\alpha| \leqslant m} \sum_{k=0}^\infty \sum_{|\beta|=k} \overline{a_\alpha^{(i)}} a_\beta^{(j)} r^{|\alpha|+|\beta|} \left[f(M_Z)\right]_\alpha^* \left[f(M_Z)\right]_\beta x, y \right\rangle \\
&= \lim_{r \to 1} \lim_{m \to \infty} \left\langle \sum_{|\alpha| \leqslant m} \sum_{k=0}^\infty \sum_{|\beta|=k} \overline{a_\alpha^{(i)}} a_\beta^{(j)} r^{|\alpha|+|\beta|} \delta_{\alpha\beta} x, y \right\rangle \\
&= \lim_{r \to 1} \lim_{m \to \infty} \sum_{|\alpha| \leqslant m} \overline{a_\alpha^{(i)}} a_\alpha^{(j)} r^{2|\alpha|} \langle x, y \rangle \\
&= \lim_{r \to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} \overline{a_\alpha^{(i)}} a_\alpha^{(j)} r^{2|\alpha|} \langle x, y \rangle = \langle Z_j, Z_i \rangle_{\mathbb{H}^2(f)} \langle x, y \rangle_{\mathbb{H}^2(f)}.
\end{aligned}$$

Hence, we deduce that

$$M_{Z_i}^* M_{Z_j} = \langle Z_j, Z_i \rangle_{\mathbb{H}^2(f)} I_{\mathbb{H}^2(f)}, \quad i, j \in \{1, \ldots, n\},$$

and, therefore, $C^*(M_{Z_1}, \ldots, M_{Z_n})$ coincides with

$$\overline{\operatorname{span}} \left\{ M_{Z_\alpha} M_{Z_\beta}^* : \alpha, \beta \in \mathbb{F}_n^+ \right\}.$$

The proof is complete. $\square$

Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property. We say that f has the radial approximation property, and write $f \in \mathcal{M}_{rad}$, if there is $\delta \in (0, 1)$ such that $(rf_1, \ldots, rf_n)$ has the model property for any $r \in (\delta, 1]$. Denote by $\mathcal{M}^{\|}$ the set of all formal power series $f = (f_1, \ldots, f_n)$ having the model property and such that the universal model $(M_{Z_1}, \ldots, M_{Z_n})$ associated with the noncommutative domain $\mathbb{B}_f$ is in the set of norm-convergence (or radial norm-convergence) of f. We also introduce the class $\mathcal{M}^{\|}_{rad}$ of all formal power series $f = (f_1, \ldots, f_n)$ with the property that there is $\delta \in (0, 1)$ such that $rf \in \mathcal{M}^{\|}$ for any $r \in (\delta, 1]$.

Lemma 3.4. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $g = (g_1, \ldots, g_n)$ be its inverse with respect to the composition. Setting $g_i = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$, the following statements are equivalent.*

(i) *The n-tuple f has the radial approximation property.*
(ii) *There is $\delta \in (0, 1)$ with the property that $g_i(\frac{1}{r}S) := \sum_{\alpha \in \mathbb{F}_n^+} \frac{a_\alpha^{(i)}}{r^{|\alpha|}} S_\alpha$ is the Fourier representation of an element in F_n^∞ and*

$$\frac{1}{r}S_j = f_j\left(g_1\left(\frac{1}{r}S\right), \ldots, g_n\left(\frac{1}{r}S\right)\right), \quad i, j \in \{1, \ldots, n\},\ r \in (\delta, 1],$$

where $g(\frac{1}{r}S)$ is either in the convergence set $\mathcal{C}_f^{SOT}(F^2(H_n))$ or $\mathcal{C}_f^{rad}(F^2(H_n))$, and $S = (S_1, \ldots, S_n)$ is the n-tuple of left creation operators on $F^2(H_n)$. If f is an n-tuple of noncommutative polynomials, then the later condition is automatically satisfied.

Moreover, $f \in \mathcal{M}^{\|}_{rad}$ if and only if item (ii) *holds and $g(\frac{1}{r}S)$ is in the set of norm-convergence* (*or radial norm-convergence*) *of f.*

Proof. The proof is straightforward if one uses Lemma 2.6 (and the proof) and its analogues when f is an n-tuple of polynomials with property $(\mathcal{A})$ or a free holomorphic function with property $(\mathcal{F})$. □

Remark 3.5. In all the examples presented in this paper, the corresponding n-tuple $f = (f_1, \ldots, f_n)$ is in the class $\mathcal{M}^{\|}_{rad}$. Moreover, any n-tuple of polynomials with property $(\mathcal{A})$ is also in the class $\mathcal{M}^{\|}_{rad}$.

Proposition 3.6. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with $f(0) = 0$ and $\det J_f(0) \neq 0$, and let $g = (g_1, \ldots, g_n)$ be its inverse. Assume that f and g have nonzero radius of convergence. Then*

(i) *$f(g(X)) = X$ for any $X \in [B(\mathcal{H})^n]_{\gamma_1}$, where $0 < \gamma_1 < r(g)$ and $g([B(\mathcal{H})^n]_{\gamma_1}) \subset [B(\mathcal{H})^n]_{r(f)}$.*
(ii) *$g(f(X)) = X$ for any $X \in [B(\mathcal{H})^n]_{\gamma_2}$, where $0 < \gamma_2 < r(f)$ and $f([B(\mathcal{H})^n]_{\gamma_2}) \subset [B(\mathcal{H})^n]_{r(g)}$.*

If $\gamma_1 > 1$, then $f \in \mathcal{M}^{\|}_{rad}$, and, if $0 < \gamma < \gamma_1 \leqslant 1$, then $\frac{1}{\gamma} f$ has the same property.

Proof. Since g has nonzero radius of convergence and $g(0) = 0$, the Schwartz lemma for free holomorphic functions implies that there is $\gamma_1 \in (0, r(g))$ such that $\|g(X)\| < r(f)$ for any $X \in [B(\mathcal{H})^n]_{\gamma_1}$. On the other hand, using Theorem 1.2 from [33], the composition $f \circ g$ is a free holomorphic function on $[B(\mathcal{H})^n]_{\gamma_1}$. Due to the uniqueness theorem for free holomorphic functions and the fact that $f \circ g = id$, as formal power series, we deduce that $f(g(X)) = X$ for any $X \in [B(\mathcal{H})^n]_{\gamma_1}$. Item (ii) can be proved similarly. Now, using Lemma 3.4, we can deduce the last part of the proposition. □

We remark that Proposition 3.6 does not imply the existence of a free biholomorphic function from $[B(\mathcal{H})^n]_{\gamma_1}$ to $[B(\mathcal{H})^n]_{\gamma_2}$ (see the examples presented in this paper).

Let $f = (f_1, \ldots, f_n)$ be an n-tuple with the model property and let $T := (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$. We say that an n-tuple $V := (V_1, \ldots, V_n)$ of operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal dilation of T if the following properties are satisfied:

(i) $(V_1, \ldots, V_n) \in \mathbb{B}_f(\mathcal{K})$;
(ii) there is a $*$-representation $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$ such that $\pi(M_{Z_i}) = V_i$, $i = 1, \ldots, n$;
(iii) $V_i^*|_{\mathcal{H}} = T_i^*$ for $i = 1, \ldots, n$;
(iv) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

Without the condition (iv), the n-tuple V is called dilation of T. For information on completely bounded (resp. positive) maps, we refer to Paulsen's book [17].

Theorem 3.7. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the radial approximation property and let $T := (T_1, \ldots, T_n)$ be an n-tuple of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$. Then the following statements hold.*

(i) *There is a unique unital completely contractive linear map*

$$\Psi_{f,T} : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{H})$$

such that

$$\Psi_{f,T}(M_{Z_\alpha} M_{Z_\beta}^*) = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

(ii) *If $f \in \mathcal{M}_{rad} \cap \mathcal{M}^{\|}$, then there is a minimal dilation of T which is unique up to an isomorphism.*

Proof. According to Lemma 3.4, there is $\delta \in (0, 1)$ such that, for each $r \in (\delta, 1]$ and $i = 1, \ldots, n$, the multiplication operator $M_{Z_i}^{(r)} : \mathbb{H}^2(rf) \to \mathbb{H}^2(rf)$, defined by $M_{Z_i}^{(r)}\psi := Z_i\psi$, is unitarily equivalent to an operator $\varphi_i(\frac{1}{r}S) \in F_n^\infty$ having the Fourier representation $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} \frac{1}{r^{|\alpha|}} S_\alpha$. Therefore, for any $a_{\alpha,\beta} \in \mathbb{C}$,

$$\left\| \sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha,\beta} M_{Z_\alpha}^{(r)} M_{Z_\beta}^{(r)*} \right\| = \left\| \sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha,\beta} \varphi_\alpha\left(\frac{1}{r}S\right) \varphi_\beta\left(\frac{1}{r}S\right)^* \right\|. \tag{3.7}$$

Note that $(T_1, \ldots, T_n)$ is a pure n-tuple in $\mathbb{B}_{rf}(\mathcal{H})$ for any $r \in (\delta, 1)$. Applying Theorem 3.1, we deduce that

$$\left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} T_\alpha T_\beta^* \right\| \leqslant \left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} M_{Z_\alpha}^{(r)} M_{Z_\beta}^{(r)*} \right\|. \tag{3.8}$$

On the other hand, according to [30], $\varphi_i(\frac{t}{r}S)$ is in the noncommutative disc algebra $\mathcal{A}_n$ for any $t \in (0,1)$, and the map $(0,1) \ni t \to \varphi_i(\frac{t}{r}S)$ is continuous in the operator norm topology. Consequently,

$$\begin{aligned}\lim_{r\to 1}\left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta}\varphi_\alpha\left(\frac{1}{r}S\right)\varphi_\beta\left(\frac{1}{r}S\right)^* \right\| &= \left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta}\varphi_\alpha(S)\varphi_\beta(S)^* \right\| \\ &= \left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} M_{Z_\alpha} M_{Z_\beta}^* \right\|.\end{aligned}$$

Combining this with relations (3.7) and (3.8), we have

$$\left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} T_\alpha T_\beta^* \right\| \leqslant \left\| \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} M_{Z_\alpha} M_{Z_\beta}^* \right\|.$$

A similar inequality can be obtained if we pass to matrices with entries in $C^*(M_{Z_1}, \ldots, M_{Z_n})$. Now, an approximation argument shows that the map

$$\sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} M_{Z_\alpha} M_{Z_\beta}^* \mapsto \sum_{|\alpha|,|\beta|\leqslant m} a_{\alpha,\beta} T_\alpha T_\beta^*$$

can be extended to a unique unital completely contractive map on $\overline{\operatorname{span}}\{M_{Z_\alpha} M_{Z_\beta}^*\colon \alpha, \beta \in \mathbb{F}_n^+\}$. Since, due to Theorem 3.3, the latter span coincides with $C^*(M_{Z_1}, \ldots, M_{Z_n})$, item (i) follows. Now, we assume that $f \in \mathcal{M}_{rad} \cap \mathcal{M}^{\|}$. Applying Stinespring's dilation [36] to the unital completely positive linear map $\Psi_{f,T}$ and taking into account that $C^*(M_{Z_1}, \ldots, M_{Z_n}) = \overline{\operatorname{span}}\{M_{Z_\alpha} M_{Z_\beta}^*\colon \alpha, \beta \in \mathbb{F}_n^+\}$, we find a unique representation $\pi : C^*(M_{Z_1}, \ldots, M_{Z_n}) \to B(\mathcal{K})$, where $\mathcal{K} \supseteq \mathcal{H}$, such that $\pi(M_{Z_i})^*|_{\mathcal{H}} = T_i^*$, $i = 1, \ldots, n$, and $\mathcal{K} = \bigvee_{\alpha\in\mathbb{F}_n^+} \pi(M_{Z_\alpha})\mathcal{H}$. Setting $V_i := \pi(M_{Z_i})$, $i = 1, \ldots, n$, it remains to prove that $(V_1, \ldots, V_n) \in \mathbb{B}_f(\mathcal{K})$. To this end, note that since $(M_{Z_1}, \ldots, M_{Z_n}) \in \mathcal{M}^{\|}$, we have $f_i(M_{Z_1}, \ldots, M_{Z_n}) \in C^*(M_{Z_1}, \ldots, M_{Z_n})$ for $i = 1, \ldots, n$. Consequently, the inequality $\sum_{i=1}^n f_i(M_{Z_1}, \ldots, M_{Z_n}) f_i(M_{Z_1}, \ldots, M_{Z_n})^* \leqslant I_{\mathbb{H}^2(f)}$ implies

$$\sum_{i=1}^n f_i\big(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})\big) f_i\big(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n})\big)^* \leqslant I_{\mathcal{K}}.$$

On the other hand, since $g(f(M_{Z_1}, \ldots, M_{Z_n})) = M_{Z_i}$, where the convergence is in the operator norm topology, we deduce that $g_i(f(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))) = \pi(M_{Z_i})$, $i = 1, \ldots, n$. Therefore, the n-tuple $(\pi(M_{Z_1}), \ldots, \pi(M_{Z_n}))$ is in the noncommutative domain $\mathbb{B}_f(\mathcal{K})$. The proof is complete. $\square$

Let $\mathcal{S} \subset C^*(M_{Z_1}, \ldots, M_{Z_n})$ be the operator system defined by

$$\mathcal{S} := \left\{ p(M_{Z_1}, \ldots, M_{Z_n}) + q(M_{Z_1}, \ldots, M_{Z_n})^* :\ p, q \in \mathbb{C}[Z_1, \ldots, Z_n] \right\}.$$

Theorem 3.8. *Let $f = (f_1, \ldots, f_n) \in \mathcal{M}_{rad} \cap \mathcal{M}^{\|}$ and $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$. Then the following statements are equivalent*:

(i) $(T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$;
(ii) *the map $q(M_{Z_1}, \ldots, M_{Z_n}) \mapsto q(T_1, \ldots, T_n)$ is completely contractive*;
(iii) *The map $\Psi : \mathcal{S} \to B(\mathcal{H})$ defined by*

$$\Psi\left(p(M_{Z_1}, \ldots, M_{Z_n}) + q(M_{Z_1}, \ldots, M_{Z_n})^*\right) := p(T_1, \ldots, T_n) + q(T_1, \ldots, T_n)^*$$

is completely positive.

Proof. The implication (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are due to Theorem 3.7. Since the implication (iii) $\Longrightarrow$ (ii) follows from the theory of completely positive (resp. contractive) maps, it remains to prove the implication (ii) $\Longrightarrow$ (i). To this end, assume that the map $q(M_{Z_1}, \ldots, M_{Z_n}) \mapsto q(T_1, \ldots, T_n)$ is completely contractive. For each $j = 1, \ldots, n$, assume that f_j has the representation $\sum_{\alpha \in \mathbb{F}_n^+} c_\alpha^{(j)} Z_\alpha$ and let $q_m^{(j)} := \sum_{k=0}^m \sum_{|\alpha|=k} c_\alpha^{(j)} Z_\alpha$, $m \in \mathbb{N}$. Since the universal model $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ is in the set of norm convergence for the n-tuple f, we have $f_j(M_Z) = \lim_{m\to\infty} q_m^{(j)}(M_Z)$ with the convergence in the operator norm topology. On the other hand, due to Theorem 3.7, we have

$$\left\| q_m^{(j)}(T_1, \ldots, T_n) - q_k^{(j)}(T_1, \ldots, T_n) \right\| \leqslant \left\| q_m^{(j)}(M_{Z_1}, \ldots, M_{Z_n}) - q_k^{(j)}(M_{Z_1}, \ldots, M_{Z_n}) \right\|$$

for any $m, k \in \mathbb{N}$. Consequently, $\{q_m^{(j)}(T_1, \ldots, T_n)\}_{m=1}^\infty$ is a Cauchy sequence in $B(\mathcal{H})$ and, therefore, $f_j(T_1, \ldots, T_n) := \lim_{m\to\infty} q_m^{(j)}(T_1, \ldots, T_n)$ exists in the operator norm. Now, since

$$\begin{aligned} &\left\| \left[q_m^{(1)}(T_1, \ldots, T_n), \ldots, q_m^{(n)}(T_1, \ldots, T_n) \right] \right\| \\ &\quad \leqslant \left\| \left[q_m^{(1)}(M_{Z_1}, \ldots, M_{Z_n}), \ldots, q_m^{(n)}(M_{Z_1}, \ldots, M_{Z_n}) \right] \right\|, \end{aligned}$$

taking the limit as $m \to \infty$, we obtain $\|f(T)\| \leqslant \|f(M_Z)\| \leqslant 1$. Since $f = (f_1, \ldots, f_n)$ has the radial approximation property, relation (3.1) and Lemma 3.4 show that the sequence $p_m^{(i)} := \sum_{k=0}^m \sum_{|\alpha|=k} a_\alpha^{(j)} Z_\alpha$ of noncommutative polynomials satisfies the relation

$$M_{Z_i} = g_i\left(f(M_Z)\right) = \lim_{m\to\infty} p_m^{(i)}\left(f(M_Z)\right),$$

where the limit is in the operator norm. Therefore, we have $\|p_m^{(i)}(f(M_Z)) - M_{Z_i}\| \to 0$ as $m \to \infty$. Using the von Neumann type inequality

$$\left\| p_m^{(i)}\left(f(T)\right) - T_i \right\| \leqslant \left\| p_m^{(i)}\left(f(M_Z)\right) - M_{Z_i} \right\|, \quad m \in \mathbb{N},$$

we deduce that $T_i = \lim_{m\to\infty} p_m^{(i)}(f(T))$ in the operator norm and, therefore, $g_i(f(T)) = T_i$ for all $i = 1, \ldots, n$. This shows that $(T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ and completes the proof. $\square$

We introduce the noncommutative domain algebra $\mathcal{A}(\mathbb{B}_f)$ as the norm closure of all polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity.

Theorem 3.9. *Let $f = (f_1, \ldots, f_n) \in \mathcal{M}_{rad} \cap \mathcal{M}^{\|}$ and $(A_1, \ldots, A_n) \in B(\mathcal{H})^n$. Then there is an n-tuple of operators $(T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ and an invertible operator X such that*

$$A_i = X^{-1} T_i X, \quad \text{for any } i = 1, \ldots, n,$$

if and only if the n-tuple $(A_1, \ldots, A_n)$ is completely polynomially bounded with respect to the noncommutative domain algebra $\mathcal{A}(\mathbb{B}_f)$.

Proof. Using Theorem 3.8 and Paulsen's similarity result [16], the result follows. □

Lemma 3.10. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series in the class $\mathcal{M}^{\|}$, and let $g = (g_1, \ldots, g_n)$ be the inverse of f. Then the following statements hold.*

(i) *The set $\mathbb{B}_f^<(\mathcal{H})$ coincides with $g([B(\mathcal{H})^n]_1)$. When $\mathcal{H} = \mathbb{C}$, the result holds true when f has only the model property.*
(ii) *The set $\mathbb{B}_f^{pure}(\mathcal{H})$ coincides with the image of all pure row contractions under g.*
(iii) *If $f(0) = 0$, then $\mathbb{B}_f^<(\mathcal{H})$ contains an open ball in $B(\mathcal{H})^n$ centered at 0, and*

$$\{X \in B(\mathcal{H})^n\colon X \text{ is nilpotent and } \|f(X)\| \leqslant 1\} = \mathbb{B}_f^{nil}(\mathcal{H}) \subset \mathbb{B}_f^{pure}(\mathcal{H}).$$

Proof. We shall prove items (i) and (ii) when f is an n-tuple of formal power series with property $(\mathcal{S})$. The other two cases (when f has property $(\mathcal{A})$ or property $(\mathcal{F})$) can be treated similarly. First, note that $\mathbb{B}_f^<(\mathcal{H}) \subseteq g([B(\mathcal{H})^n]_1)$. To prove the reversed inclusion let $Y = g(X)$, where $X \in [B(\mathcal{H})^n]_1$. According to Lemma 2.6 part (iii), we have either $\widetilde{g} \in \mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and

$$S_i = f_i(\widetilde{g}_1, \ldots, \widetilde{g}_n), \quad i = 1, \ldots, n, \tag{3.9}$$

or $\widetilde{g} \in \mathcal{C}_f^{rad}(\mathbb{H}^2(f))$ and

$$S_i = \text{SOT-}\lim_{r \to 1} f_j(r\widetilde{g}_1, \ldots, r\widetilde{g}_n), \quad i = 1, \ldots, n. \tag{3.10}$$

Since $f \in \mathcal{M}^{\|}$, the n-tuple $(M_{Z_1}, \ldots, M_{Z_n})$ is in the set of norm-convergence (or radial norm-convergence) for the n-tuple of formal power series $f = (f_1, \ldots, f_n)$. This implies that the convergence above is in the operator topology. Applying the noncommutative Poisson transform P_X, we deduce that $X_i = f_i(g_1(X), \ldots, g_n(X))$, $i = 1, \ldots, n$. This implies that $f(Y) = f(g(X)) = X$ and $g(f(Y)) = g(X) = Y$, which shows that $Y \in \mathbb{B}_f^<(\mathcal{H})$. Therefore, $\mathbb{B}_f^<(\mathcal{H}) = g([B(\mathcal{H})^n]_1)$, the function g is one-to-one on $[B(\mathcal{H})^n]_1$ and f is its inverse on $\mathbb{B}_f^<(\mathcal{H})$. Now consider the case when $\mathcal{H} = \mathbb{C}$ and assume that f has the model property. Since $\mathbb{B}_f^<(\mathbb{C}) \subseteq g(\mathbb{B}_n)$, we prove the reverse inclusion. Let $\mu = g(\lambda)$ for some $\lambda \in \mathbb{B}_n$ and assume that one of the relations (3.9) or (3.10) holds, say the latter. Setting $z_\lambda := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_\alpha e_\alpha \in F^2(H_n)$, we deduce that

$$\begin{aligned}
\lambda_j = \langle S_j(1), z_\lambda \rangle &= \lim_{r \to 1} \langle f_j(r\widetilde{g}_1, \ldots, r\widetilde{g}_n)(1), z_\lambda \rangle \\
&= \lim_{r \to 1} f_j\big(rg_1(\lambda), \ldots, rg_n(\lambda)\big) = f_j\big(g(\lambda)\big).
\end{aligned}$$

This implies that $f(\mu) = f(g(\lambda)) = \lambda$ and $g(f(\mu)) = g(\lambda) = \mu$, which shows that $\mu \in \mathbb{B}_f^<(\mathbb{C})$. Therefore, $\mathbb{B}_f^<(\mathbb{C}) = g(\mathbb{B}_n)$, the function g is one-to-one on $\mathbb{B}_n$ and f is its inverse on $\mathbb{B}_f^<(\mathbb{B}_n)$. Similarly, one can assume that relation (3.9) holds and reach the same conclusion.

To prove item (ii), set $[B(\mathcal{H})^n]_1^{pure} := \{X \in [B(\mathcal{H})^n]_1^-\colon\ X \text{ is a pure row contraction}\}$ and note that $\mathbb{B}_f^{pure}(\mathcal{H}) \subseteq \{g(X)\colon\ X \in [B(\mathcal{H})^n]_1^{pure}\}$. The reversed inclusion follows similarly to the proof of item (i) using the noncommutative Poison transform P_X, where X is a pure row contraction. In this case, we also show that $f(g(X)) = X$ and deduce that $g : [B(\mathcal{H})^n]_1^{pure} \to \mathbb{B}_f^{pure}(\mathcal{H})$ is a bijection with inverse $f : \mathbb{B}_f^{pure}(\mathcal{H}) \to [B(\mathcal{H})^n]_1^{pure}$. Now we prove part (iii). Since f has nonzero radius of convergence and $f(0) = 0$, the Schwartz lemma for free holomorphic functions implies that there is $\gamma > 0$ such that $\|f(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_\gamma$. On the other hand, using Theorem 1.2 from [33], the composition $g \circ f$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$. Due to the uniqueness theorem for free holomorphic functions and the fact that $g \circ f = id$, as formal power series, we deduce that $g(f(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$.

If $X \in B(\mathcal{H})^n$ is a nilpotent n-tuple with $\|f(X)\| \leqslant 1$, then taking into account that $f(0) = 0$, we deduce that $[f_1(X), \ldots, f_n(X)]$ is a nilpotent n-tuple. Hence and using that $g \circ f = id$, we deduce that $g(f(X)) = X$, which completes the proof. □

Lemma 3.11. *If $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series in the class $\mathcal{M}_{rad}^{\|}$, then*

$$\mathbb{B}_f(\mathcal{H}) = g\big(\big[B(\mathcal{H})^n\big]_1^-\big),$$

where $g = (g_1, \ldots, g_n)$ is the inverse of f with respect to the composition of power series. Moreover, the function $g : [B(\mathcal{H})^n]_1^- \to \mathbb{B}_f(\mathcal{H})$ is a bijection with inverse $f : \mathbb{B}_f(\mathcal{H}) \to [B(\mathcal{H})^n]_1^-$. When $\mathcal{H} = \mathbb{C}$, the result holds true when f has only the radial approximation property.

Proof. First, note that $\mathbb{B}_f(\mathcal{H}) \subseteq g([B(\mathcal{H})^n]_1^-)$. To prove the reverse inclusion, let $Y := g(X)$ and $X = (X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1^-$. Since f has the radial approximation property, $g = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ for some $\gamma > 1$. Moreover, according to Lemma 3.4, there is $\delta \in (0, 1)$ with the property that for any $r \in (\delta, 1]$, the series $g_i(\frac{1}{r}S) := \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{a_\alpha^{(i)}}{r^{|\alpha|}} S_\alpha$ is convergent in the operator norm topology and represents an element in the noncommutative disc algebra $\mathcal{A}_n$, and

$$\frac{1}{r} S_j = f_j\left(g_1\left(\frac{1}{r}S\right), \ldots, g_n\left(\frac{1}{r}S\right)\right), \quad j \in \{1, \ldots, n\},\ r \in (\delta, 1], \tag{3.11}$$

where $g(\frac{1}{r}S)$ is in the norm-convergence (or radial norm-convergence) of f. Applying now the noncommutative Poisson transform P_{rX}, we deduce that $X_j = f_j(g(X))$ for $j = 1, \ldots, n$. This also shows that g is one-to-one on $[B(\mathcal{H})^n]_1^-$. On the other hand, the relation above implies $Y = g(X) = g(f(g(X))) = g(f(Y))$ and $\|f(Y)\| \leqslant 1$, which shows that $Y \in \mathbb{B}_f(\mathcal{H})$. Therefore, $\mathbb{B}_f(\mathcal{H}) = g([B(\mathcal{H})^n]_1^-)$ and f is one-to-one on $\mathbb{B}_f(\mathcal{H})$.

Now consider the case when $\mathcal{H} = \mathbb{C}$ and assume that f has only the radial approximation property. Since $\mathbb{B}_f(\mathbb{C}) \subseteq g(\overline{\mathbb{B}}_n)$, we prove the reverse inclusion. Let $\mu = g(\lambda)$ for some $\lambda \in \overline{\mathbb{B}}_n$ and assume that relation (3.11) holds, where $g(\frac{1}{r}S)$ is either in the convergence set $\mathcal{C}_f^{SOT}(F^2(H_n))$ or $\mathcal{C}_f^{rad}(F^2(H_n))$. For example, assume that $g(\frac{1}{r}S) \in \mathcal{C}_f^{SOT}(F^2(H_n))$. For each $r \in (\delta, 1)$, consider $z_{r\lambda} := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_\alpha r^{|\alpha|} e_\alpha \in F^2(H_n)$, and note that

$$\lambda_j = \left\langle \frac{1}{r} S_j(1), z_{r\lambda} \right\rangle = \left\langle f_j\left(g_1\left(\frac{1}{r}S\right), \ldots, g_n\left(\frac{1}{r}S\right)\right)(1), z_{r\lambda} \right\rangle$$
$$= f_j\big(g_1(\lambda), \ldots, g_n(\lambda)\big) = f_j\big(g(\lambda)\big).$$

This implies that $f(\mu) = f(g(\lambda)) = \lambda$ and $g(f(\mu)) = g(\lambda) = \mu$, which shows that $\mu \in \mathbb{B}_f(\mathbb{C})$. Therefore, $\mathbb{B}_f(\mathbb{C}) = g(\overline{\mathbb{B}}_n)$, the function g is one-to-one on $\overline{\mathbb{B}}_n$ and f is its inverse on $\mathbb{B}_f(\mathbb{C})$. Similarly, one can treat the case when $g(\frac{1}{r}S) \in \mathcal{C}_f^{rad}(F^2(H_n))$. The proof is complete. □

In what follows, we identify the characters of the noncommutative domain algebra $\mathcal{A}(\mathbb{B}_f)$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in $\mathbb{B}_f(\mathbb{C})$ and define the evaluation functional

$$\Phi_\lambda : \mathcal{P}(M_{Z_1}, \ldots, M_{Z_n}) \to \mathbb{C}, \qquad \Phi_\lambda\big(p(M_Z)\big) = p(\lambda),$$

where $\mathcal{P}(M_{Z_1}, \ldots, M_{Z_n})$ denotes the algebra of all polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity. According to Theorem 3.7, we have $|p(\lambda)| = \|p(\lambda I_{\mathbb{C}})\| \leqslant \|p(M_Z)\|$. Hence, Φ_λ has a unique extension to the domain algebra $\mathcal{A}(\mathbb{B}_f)$. Therefore Φ_λ is a character of $\mathcal{A}(\mathbb{B}_f)$.

Theorem 3.12. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the radial approximation property and let $M_{\mathcal{A}(\mathbb{B}_f)}$ be the set of all characters of $\mathcal{A}(\mathbb{B}_f)$. Then the map*

$$\Psi : \mathbb{B}_f(\mathbb{C}) \to M_{\mathcal{A}(\mathbb{B}_f)}, \qquad \Psi(\lambda) := \Phi_\lambda,$$

is a homeomorphism and $\mathbb{B}_f(\mathbb{C})$ is homeomorphic to the closed unit ball $\overline{\mathbb{B}}_n$.

Proof. First, notice that Ψ is injective. To prove that Ψ is surjective, assume that $\Phi : \mathcal{A}(\mathbb{B}_f) \to \mathbb{C}$ is a character. Setting $\lambda_i := \Phi(M_{Z_i})$, $i = 1, \ldots, n$, we deduce that $\Phi(p(M_Z)) = p(\lambda)$ for any polynomial $p(M_{Z_1}, \ldots, M_{Z_n})$ in $\mathcal{A}(\mathbb{B}_f)$. Since Φ is a character it follows that it is completely contractive. Applying Theorem 3.8 in the particular case when $A_i := \lambda_i I_{\mathbb{C}}$, $i = 1, \ldots, n$, it follows that $(\lambda_1 I_{\mathbb{C}}, \ldots, \lambda_n I_{\mathbb{C}}) \in \mathbb{B}_f(\mathbb{C})$. Moreover, since

$$\Phi\big(p(M_Z)\big) = p(\lambda) = \Phi_\lambda\big(p(M_Z)\big)$$

for any polynomial $p(M_Z)$ in $\mathcal{A}(\mathbb{B}_f)$, we must have $\Phi = \Phi_\lambda$. Suppose now that $\lambda^\alpha := (\lambda_1^\alpha, \ldots, \lambda_n^\alpha)$; $\alpha \in J$, is a net in $\mathbb{B}_f(\mathbb{C})$ such that $\lim_{\alpha\in J} \lambda^\alpha = \lambda := (\lambda_1, \ldots, \lambda_n)$. It is clear that

$$\lim_{\alpha\in J} \Phi_{\lambda^\alpha}\big(p(M_Z)\big) = \lim_{\alpha\in J} p\big(\lambda^\alpha\big) = p(\lambda) = \Phi_\lambda\big(p(M_Z)\big)$$

for every polynomial $p(M_Z)$. Since the set of all polynomials $\mathcal{P}(M_{Z_1}, \ldots, M_{Z_n})$ is dense in $\mathcal{A}(\mathbb{B}_f)$ and $\sup_{\alpha\in J} \|\Phi_{\lambda^\alpha}\| \leqslant 1$, it follows that Ψ is continuous. According to Lemma 3.11, $\mathbb{B}_f(\mathbb{C}) = g(\overline{\mathbb{B}}_n)$ is a compact subset of $\mathbb{C}^n$ and $g : \overline{\mathbb{B}}_n \to \mathbb{B}_f(\mathbb{C})$ is a bijection. Since both $\mathbb{B}_f(\mathbb{C})$ and $M_{\mathcal{A}(\mathbb{B}_f)}$ are compact Hausdorff spaces and Ψ is also one-to-one and onto, we deduce that Ψ is a homeomorphism. On the other hand, since the map $\lambda \mapsto g(\lambda)$ is holomorphic on a ball $(\mathbb{C}^n)_\gamma$ for some $\gamma > 1$, one can see that $\mathbb{B}_f(\mathbb{C})$ is homeomorphic to the closed unit ball $\overline{\mathbb{B}}_n$. The proof is complete. □

4. The invariant subspaces under $M_{Z_1}, \ldots, M_{Z_n}$

In this section we obtain a Beurling type characterization of the joint invariant subspaces under the multiplication operators $M_{Z_1}, \ldots, M_{Z_n}$ associated with the noncommutative domain $\mathbb{B}_f$ and a minimal dilation theorem for pure n-tuples of operators in $\mathbb{B}_f(\mathcal{H})$.

An operator $A : \mathbb{H}^2(f) \otimes \mathcal{H} \to \mathbb{H}^2(f) \otimes \mathcal{K}$ is called multi-analytic with respect to $M_{Z_1}, \ldots, M_{Z_n}$ if $A(M_{Z_i} \otimes I_{\mathcal{H}}) = (M_{Z_i} \otimes I_{\mathcal{K}})A$ for any $i = 1, \ldots, n$. If, in addition, A is a partial isometry, we call it inner.

Theorem 4.1. *Let $f = (f_1, \ldots, f_n)$ be n-tuple of formal power series with the model property and let $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with $\mathbb{B}_f$. If $Y \in B(\mathbb{H}^2(f) \otimes \mathcal{H})$, then the following statements are equivalent.*

- (i) *There is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ with respect to the multiplication operators $M_{Z_1}, \ldots, M_{Z_n}$ such that $Y = \Psi\Psi^*$.*
- (ii) *$\Phi_{f,M_Z\otimes I}(Y) \leqslant Y$, where the positive linear mapping $\Phi_{f,M_Z\otimes I} : B(\mathbb{H}^2(f) \otimes \mathcal{H}) \to B(\mathbb{H}^2(f) \otimes \mathcal{H})$ is defined by*

$$\Phi_{f,M_Z\otimes I}(Y) := \sum_{i=1}^n \bigl(f_i(M_Z) \otimes I_{\mathcal{H}}\bigr) Y \bigl(f_i(M_Z) \otimes I_{\mathcal{H}}\bigr)^*.$$

Proof. First, assume that condition (ii) holds and note that $Y - \Phi^m_{f,M_Z\otimes I}(Y) \geqslant 0$ for any $m = 1, 2, \ldots.$ Since $(M_{Z_1}, \ldots, M_{Z_n})$ is a pure n-tuple with respect to the noncommutative domain $\mathbb{B}_f(\mathbb{H}^2(f))$, we deduce that SOT-$\lim_{m\to\infty} \Phi^m_{f,M_Z\otimes I}(Y) = 0$, which implies $Y \geqslant 0$. Denote $\mathcal{M} := \overline{\operatorname{range} Y^{1/2}}$ and define

$$Q_i\bigl(Y^{1/2}x\bigr) := Y^{1/2}\bigl(f_i(M_Z)^* \otimes I_{\mathcal{H}}\bigr)x, \quad x \in \mathbb{H}^2(f) \otimes \mathcal{H}, \tag{4.1}$$

for any $i = 1, \ldots, n$. We have

$$\sum_{i=1}^n \bigl\| Q_i\bigl(Y^{1/2}x\bigr)\bigr\|^2 \leqslant \sum_{i=1}^n \bigl\| Y^{1/2}\bigl(f_i(M_Z)^* \otimes I_{\mathcal{H}}\bigr)x\bigr\|^2 = \bigl\langle \Phi_{f,M_Z\otimes I}(Y)x, x\bigr\rangle \leqslant \bigl\|Y^{1/2}x\bigr\|^2$$

for any $x \in \mathbb{H}^2(f) \otimes \mathcal{H}$, which implies $\|Q_i Y^{1/2}x\|^2 \leqslant \|Y^{1/2}x\|^2$, for any $x \in \mathbb{H}^2(f) \otimes \mathcal{H}$. Consequently, Q_i can be uniquely be extended to a bounded operator (also denoted by Q_i) on the subspace $\mathcal{M}$. Setting $A_i := Q_i^*$, $i = 1, \ldots, n$, we deduce that $\sum_{i=1}^n A_i A_i^* \leqslant I_{\mathcal{M}}$. Denoting $\varphi_A(Y) := \sum_{i=1}^n A_i Y A_i^*$ and using relation (4.1), we have

$$\bigl\langle \varphi_A^m(I) Y^{1/2}x, Y^{1/2}x\bigr\rangle = \bigl\langle \Phi^m_{f,M_Z\otimes I}(Y)x, x\bigr\rangle \leqslant \|Y\| \bigl\langle \Phi^m_{f,M_Z\otimes I}(I)x, x\bigr\rangle$$

for any $x \in \mathbb{H}^2(f) \otimes \mathcal{H}$. Since SOT-$\lim_{m\to\infty} \Phi^m_{f,M_Z\otimes I}(I) = 0$, we have SOT-$\lim_{m\to\infty} \varphi_A^m(I) = 0$. Therefore $A := (A_1, \ldots, A_n)$ is a pure row contraction. According to [26], the Poisson kernel $K_A : \mathcal{M} \to \mathbb{H}^2(f) \otimes \mathcal{E}$ ($\mathcal{E}$ is an appropriate Hilbert space) defined by

$$K_A h := \sum_{\alpha\in\mathbb{F}_n^+} f_\alpha \otimes \Delta_A A_\alpha^* h, \quad h \in \mathcal{M},$$

where $\Delta_A := (I - A_1 A_1^* - \ldots, A_n A_n^*)^{1/2}$ is an isometry with the property that

$$A_i K_A^* = K_A^*(M_{f_i} \otimes I_{\mathcal{E}}), \quad i = 1, \ldots, n. \tag{4.2}$$

Let $\Gamma := Y^{1/2} K_A^* : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ and note that, due to the fact that f has the model property, $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$ for $i = 1, \ldots, n$. Consequently, we have

$$\begin{aligned} \Gamma(M_{f_i} \otimes I_{\mathcal{E}}) &= Y^{1/2} K_A^*(M_{f_i} \otimes I_{\mathcal{E}}) = Y^{1/2} A_i K_A^* \\ &= \big(f_i(M_Z) \otimes I_{\mathcal{H}}\big) Y^{1/2} K_A^* = (M_{f_i} \otimes I_{\mathcal{H}})\Psi \end{aligned}$$

for any $i = 1, \ldots, n$. Now, let $g = (g_1, \ldots, g_n)$ be the inverse of $f = (f_1, \ldots, f_n)$ with respect to the composition of power series. In the proof of Theorem 3.1, we showed that $g_i(M_{f_1}, \ldots, M_{f_n}) = M_{Z_i}$ for all $i = 1, \ldots, n$. Hence, we deduce that the operator M_{Z_i} is in the SOT-closure of all polynomials in $M_{f_1}, \ldots, M_{f_n}$ and the identity. Consequently, the relation $\Gamma(M_{f_i} \otimes I_{\mathcal{E}}) = (M_{f_i} \otimes I_{\mathcal{H}})\Gamma$ implies $\Gamma(M_{Z_i} \otimes I_{\mathcal{H}}) = (M_{Z_i} \otimes I_{\mathcal{H}})\Gamma$ for $i = 1, \ldots, n$, which shows that Γ is a multi-analytic with respect to $M_{Z_1}, \ldots, M_{Z_n}$. Note that we also have $\Gamma\Gamma^* = Y^{1/2} K_A^* K_A Y^{1/2} = Y$. The proof is complete. □

The next result is a Beurling [5] type characterization of the invariant subspaces under the multiplication operators $M_{Z_1}, \ldots, M_{Z_n}$ associated with the noncommutative domain $\mathbb{B}_f$.

Theorem 4.2. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the multiplication operators associated with the noncommutative domain $\mathbb{B}_f$. A subspace $\mathcal{N} \subseteq \mathbb{H}^2(f) \otimes \mathcal{H}$ is invariant under each operator $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$ if and only if there exists an inner multi-analytic operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ with respect to $M_{Z_1}, \ldots, M_{Z_n}$ such that*

$$\mathcal{N} = \Psi\big[\mathbb{H}^2(f) \otimes \mathcal{E}\big].$$

Proof. Assume that $\mathcal{N} \subseteq \mathbb{H}^2(f) \otimes \mathcal{H}$ is invariant under each operator $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$. Since $P_{\mathcal{N}}(M_{Z_i} \otimes I_{\mathcal{H}})P_{\mathcal{N}} = (M_{Z_i} \otimes I_{\mathcal{H}})P_{\mathcal{N}}$ for any $i = 1, \ldots, n$, and $M_Z := (M_{Z_1}, \ldots, M_{Z_n}) \in \mathbb{B}_f(\mathbb{H}^2(f))$, we have

$$\begin{aligned} \Phi_{f, M_Z \otimes I_{\mathcal{H}}}(P_{\mathcal{N}}) &= P_{\mathcal{N}}\left[\sum_{i=1}^n \big(f_i(M_Z) \otimes I_{\mathcal{H}}\big) P_{\mathcal{N}} \big(f_i(M_Z)^* \otimes I_{\mathcal{H}}\big)\right] P_{\mathcal{N}} \\ &\leqslant P_{\mathcal{N}}\left[\sum_{i=1}^n \big(f_i(M_Z) \otimes I_{\mathcal{H}}\big)\big(f_i(M_Z)^* \otimes I_{\mathcal{H}}\big)\right] P_{\mathcal{N}} \\ &= P_{\mathcal{N}}\left(\sum_{i=1}^n M_{f_i} M_{f_i}^* \otimes I_{\mathcal{H}}\right) P_{\mathcal{N}} \leqslant P_{\mathcal{N}}. \end{aligned}$$

Here, we also used the fact that $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$. Applying now Theorem 4.1, we find a multi-analytic operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ with respect to the operators $M_{Z_1}, \ldots, M_{Z_n}$ such that $P_{\mathcal{N}} = \Psi\Psi^*$. Since $P_{\mathcal{N}}$ is an orthogonal projection, we deduce that

Ψ is a partial isometry and $\mathcal{N} = \Psi[\mathbb{H}^2(f) \otimes \mathcal{E}]$. Since the converse is obvious, the proof is complete. □

Theorem 4.3. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. If $\mathcal{N} \subseteq \mathbb{H}^2(f) \otimes \mathcal{H}$ is a coinvariant subspace under $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$, then there is a subspace $\mathcal{E} \subseteq \mathcal{H}$ such that*

$$\overline{\operatorname{span}}\left\{(M_{Z_\alpha} \otimes I_{\mathcal{H}})\mathcal{N}\colon \alpha \in \mathbb{F}_n^+\right\} = \mathbb{H}^2(f) \otimes \mathcal{E}.$$

In particular, $\mathcal{N}$ is cyclic for the operators $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$ if and only if $(P_{\mathbb{C}} \otimes I_{\mathcal{H}})\mathcal{N} = \mathcal{H}$, where $P_{\mathbb{C}}$ is the orthogonal projection on $\mathbb{C}$.

Proof. Let $\mathcal{E} := (P_{\mathbb{C}} \otimes I_{\mathcal{H}})\mathcal{N} \subset \mathcal{H}$, where $1 \otimes \mathcal{H}$ is identified with $\mathcal{H}$, and let $h \in \mathcal{N}$ be a nonzero vector with representation $h = \sum_{\alpha \in \mathbb{F}_n^+} f_\alpha \otimes h_\alpha$, $h_\alpha \in \mathcal{H}$. Choose $\beta \in \mathbb{F}_n^+$ with $h_\beta \neq 0$. Since $\mathcal{N}$ is a co-invariant subspace under $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$, and f has the model property, we have $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$ for $i = 1, \ldots, n$, and deduce that

$$(P_{\mathbb{C}} \otimes I_{\mathcal{H}})\left(\left[f(M_Z)\right]_\alpha \otimes I_{\mathcal{H}}\right)h = (P_{\mathbb{C}} M_{f_\alpha} \otimes I_{\mathcal{H}})h = h_\beta \in \mathcal{E}.$$

This implies $(M_{f_\beta} \otimes I_{\mathcal{H}})(1 \otimes h_\beta) = f_\beta \otimes h_\beta \in \mathbb{H}^2(f) \otimes \mathcal{E}$ for any $\beta \in \mathbb{F}_n^+$. Hence, we deduce that $h = \sum_{\alpha \in \mathbb{F}_n^+} f_\alpha \otimes h_\alpha \in \mathbb{H}^2(f) \otimes \mathcal{E}$. Therefore, $\mathcal{N} \subset \mathbb{H}^2(f) \otimes \mathcal{E}$, which implies

$$\mathcal{G} := \overline{\operatorname{span}}\left\{(M_\alpha \otimes I_{\mathcal{H}})\mathcal{N}\colon \alpha \in \mathbb{F}_n^+\right\} \subseteq \mathbb{H}^2(f) \otimes \mathcal{E}.$$

Now, we prove the reverse inclusion. Let $h_0 \in \mathcal{E}$, $h_0 \neq 0$. Due to the definition of the subspace $\mathcal{E}$, there exists $x \in \mathcal{M}$ such that $x = 1 \otimes h_0 + \sum_{|\alpha| \geqslant 1} f_\alpha \otimes h_\alpha$. Hence, we obtain

$$h_0 = (P_{\mathbb{C}} \otimes I_{\mathcal{H}})x = \left(I - \sum_{i=1}^n M_{f_i} M_{f_i}^* \otimes I_{\mathcal{H}}\right)x.$$

Since M_{f_i} is a SOT-limit of polynomials in $M_{Z_1}, \ldots, M_{Z_n}$, and $\mathcal{N}$ is a co-invariant subspace under $M_{Z_1} \otimes I_{\mathcal{H}}, \ldots, M_{Z_n} \otimes I_{\mathcal{H}}$, we deduce that $h_0 \in \mathcal{G}$. Therefore, $\mathcal{E} \subset \mathcal{G}$ and $(M_{Z_\alpha} \otimes I_{\mathcal{H}}) \times (1 \otimes \mathcal{E}) \subset \mathcal{G}$ for $\alpha \in \mathbb{F}_n^+$. Since, due to Proposition 2.5, $\mathbb{C}[Z_1, \ldots, Z_n]$ is dense in $\mathbb{H}^2(f)$, we deduce that $\mathbb{H}^2(f) \otimes \mathcal{E} \subseteq \mathcal{G}$. The last part of the theorem is now obvious. The proof is complete. □

A simple consequence of Theorem 4.3 is the following result.

Corollary 4.4. *A subspace $\mathcal{N} \subseteq \mathbb{H}^2(f) \otimes \mathcal{H}$ is reducing under each operator $M_{Z_i} \otimes I_{\mathcal{H}}$, $i = 1, \ldots, n$, if and only if there is a subspace $\mathcal{E} \subseteq \mathcal{H}$ such that $\mathcal{N} = \mathbb{H}^2(f) \otimes \mathcal{E}$.*

We remark that, in Theorem 4.2, the inner multi-analytic operator $\Psi : \mathbb{H}^2(f) \otimes \mathcal{E} \to \mathbb{H}^2(f) \otimes \mathcal{H}$ with respect to $M_{Z_1}, \ldots, M_{Z_n}$ and with the property that $\mathcal{N} = \Psi[\mathbb{H}^2(f) \otimes \mathcal{E}]$ can be chosen to be an isometry. Indeed, let $\mathcal{M} := \{x \in \mathbb{H}^2(f) \otimes \mathcal{E}\colon \|\Psi(x)\| = \|x\|\}$. Since f has the model property, we deduce that $f_i(M_{Z_1},\dots,M_{Z_n}) = M_{f_i}$ is an isometry for each $i = 1,\dots,n$. Consequently, we have

$$\begin{aligned}\left\|\Psi(M_{f_i}\otimes I_{\mathcal{E}})x\right\| &= \left\|\Psi f_i(M_{Z_1}\otimes I_{\mathcal{E}},\dots,M_{Z_n}\otimes I_{\mathcal{E}})x\right\|\\ &= \left\|f_i(M_{Z_1}\otimes I_{\mathcal{H}},\dots,M_{Z_n}\otimes I_{\mathcal{H}})\Psi(x)\right\|\\ &= \left\|\Psi(x)\right\| = \|x\| = \left\|f_i(M_{Z_1}\otimes I_{\mathcal{E}},\dots,M_{Z_n}\otimes I_{\mathcal{E}})x\right\| = \left\|(M_{f_i}\otimes I_{\mathcal{E}})x\right\|\end{aligned}$$

for any $x\in\mathcal{M}$ and $i = 1,\dots,n$. This implies that $\mathcal{M}$ is an invariant subspace under $M_{f_i}\otimes I_{\mathcal{E}}$, $i = 1,\dots,n$. Using the fact that $M_{Z_i} = g_i(M_{f_1},\dots,M_{f_n})$, $i = 1,\dots,n$, where $g = (g_1,\dots,g_n)$ is the inverse of $f = (f_1,\dots,f_n)$, we deduce that $\mathcal{M}$ is invariant under $M_{Z_1}\otimes I_{\mathcal{E}},\dots,M_{Z_n}\otimes I_{\mathcal{E}}$. On the other hand, since $\mathcal{M}^\perp = \ker\Psi$ and $\Psi(M_{Z_i}\otimes I_{\mathcal{E}}) = (M_{Z_i}\otimes I_{\mathcal{H}})\Psi$, it is clear that $\mathcal{M}^\perp$ is also invariant under $M_{Z_1}\otimes I_{\mathcal{E}},\dots,M_{Z_n}\otimes I_{\mathcal{E}}$, which shows that $\mathcal{M}$ is a reducing subspace for $M_{Z_1}\otimes I_{\mathcal{E}},\dots,M_{Z_n}\otimes I_{\mathcal{E}}$. Now, due to Corollary 4.4, $\mathcal{M} = \mathbb{H}^2(f)\otimes\mathcal{G}$ for some subspace $\mathcal{G}\subseteq\mathcal{E}$. Therefore, we have

$$\mathcal{N} = \Psi\left[\mathbb{H}^2(f)\otimes\mathcal{E}\right] = \Psi(\mathcal{M}) = \Psi\left[\mathbb{H}^2(f)\otimes\mathcal{G}\right]$$

and the restriction of Ψ to $\mathbb{H}^2(f)\otimes\mathcal{G}$ is an isometric multi-analytic operator, which proves our assertion.

The next result can be viewed as a continuation of Theorem 3.1.

Theorem 4.5. *Let $T = (T_1,\dots,T_n)\in\mathbb{B}_f(\mathcal{H})$ be a pure n-tuple of operators and let $f = (f_1,\dots,f_n)$ have the model theory. Then the noncommutative Poisson kernel $K_{f,T}:\mathcal{H}\to\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}$ defined by relation* (3.2) *is an isometry, the subspace $K_{f,T}(\mathcal{H})$ is co-invariant under $M_{Z_1}\otimes I_{\mathcal{H}},\dots,M_{Z_n}\otimes I_{\mathcal{H}}$, and*

$$T_i = K_{f,T}^*(M_{Z_i}\otimes I_{\mathcal{D}_{f,T}})K_{f,T},\quad i = 1,\dots,n.$$

Moreover, the dilation above is minimal, i.e.,

$$\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T} = \bigvee_{\alpha\in\mathbb{F}_n^+}(M_{Z_\alpha}\otimes I_{\mathcal{D}_{f,T}})K_{f,T}(\mathcal{H}),$$

and unique up to an isomorphism.

Proof. The first part of the theorem was proved in Theorem 3.1. Due to the definition of the noncommutative Poisson kernel $K_{f,T}$, we have $(P_{\mathbb{C}}\otimes I_{\mathcal{D}_{f,T}})K_{f,T}(\mathcal{H}) = \mathcal{D}_{f,T}$. Applying Theorem 4.3, we deduce the minimality of the dilation. To prove the uniqueness, consider another minimal dilation of $(T_1,\dots,T_n)$, that is,

$$T_i = V^*(M_{Z_i}\otimes I_{\mathcal{E}})V,\quad i = 1,\dots,n, \tag{4.3}$$

where $V:\mathcal{H}\to\mathbb{H}^2(f)\otimes\mathcal{E}$ is an isometry, $V(\mathcal{H})$ is co-invariant under $M_{Z_i}\otimes I_{\mathcal{E}}$, $i = 1,\dots,n$, and

$$\mathbb{H}^2(f)\otimes\mathcal{E} = \bigvee_{\alpha\in\mathbb{F}_n^+}(M_{Z_\alpha}\otimes I_{\mathcal{E}})V(\mathcal{H}).$$

According to Theorem 3.3 and Theorem 3.1, we have

$$\overline{\operatorname{span}}\{M_{Z_\alpha} M_{Z_\beta}^*\colon \alpha, \beta \in \mathbb{F}_n^+\} = C^*(M_{Z_1}, \dots, M_{Z_n})$$

and there is a completely positive linear map $\Phi : C^*(M_{Z_1}, \dots, M_{Z_n}) \to B(\mathcal{H})$ such that $\Phi(M_{Z_\alpha} M_{Z_\beta}^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. Note that relation (4.3) and the fact that $V(\mathcal{H})$ is co-invariant under $M_{Z_i} \otimes I_{\mathcal{E}}$, $i = 1, \dots, n$, imply that

$$\Phi(X) = K_{f,T}^* \pi_1(X) K_{f,T} = V^* \pi_2(X) V, \quad X \in C^*(M_{Z_1}, \dots, M_{Z_n}),$$

where π_1,π_2 are the $*$-representations of $C^*(M_{Z_1}, \dots, M_{Z_n})$ on $\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ and $\mathbb{H}^2(f) \otimes \mathcal{E}$ given by $\pi_1(X) := X \otimes I_{\mathcal{D}_{f,T}}$ and $\pi_2(X) := X \otimes I_{\mathcal{E}}$, respectively. Since π_1, π_2 are minimal Stinespring dilations of Φ, due to the uniqueness [36], there exists a unitary operator $W : \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T} \to \mathbb{H}^2(f) \otimes \mathcal{E}$ such that

$$W(M_{Z_i} \otimes I_{\mathcal{D}_{f,T}}) = (M_{Z_i} \otimes I_{\mathcal{E}})W, \quad i = 1, \dots, n,$$

and $WK_{f,T} = V$. Hence, we also deduce that $W(M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}}) = (M_{Z_i}^* \otimes I_{\mathcal{E}})W$ for $i = 1, \dots, n$. Since, due to Theorem 3.3, the C^*-algebra $C^*(M_{Z_1}, \dots, M_{Z_n})$ is irreducible, we must have $W = I_{\mathbb{H}^2(f)} \otimes \Gamma$, where $\Gamma \in B(\mathcal{D}_{f,T}, \mathcal{E})$ is a unitary operator. Consequently, we have $\dim \mathcal{D}_{f,T} = \dim \mathcal{E}$ and $WK_{f,T}V(\mathcal{H}) = V(\mathcal{H})$, which proves that the two minimal dilations are unitarily equivalent. The proof is complete. □

Corollary 4.6. *Let $(M_{Z_1}, \dots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. The n-tuples $(M_{Z_1} \otimes I_{\mathcal{H}}, \dots, M_{Z_n} \otimes I_{\mathcal{H}})$ and $(M_{Z_1} \otimes I_{\mathcal{K}}, \dots, M_{Z_n} \otimes I_{\mathcal{K}})$ are unitarily equivalent if and only if $\dim \mathcal{H} = \dim \mathcal{K}$.*

Proof. Let $W : \mathbb{H}^2(f) \otimes \mathcal{H} \to \mathbb{H}^2(f) \otimes \mathcal{K}$ be a unitary operator such that $W(M_{Z_i} \otimes I_{\mathcal{H}}) = (M_{Z_i} \otimes I_{\mathcal{K}})W$ for $i = 1, \dots, n$. Since W is unitary, we have $W(M_{Z_i}^* \otimes I_{\mathcal{H}}) = (M_{Z_i}^* \otimes I_{\mathcal{K}})W$, $i = 1, \dots, n$. Using the fact that $C^*(M_{Z_1}, \dots, M_{Z_n})$ is irreducible, we deduce that $W = I_{\mathbb{H}^2(f)} \otimes \Gamma$ for a unitary operator $\Gamma \in B(\mathcal{H}, \mathcal{K})$, which shows that $\dim \mathcal{H} = \dim \mathcal{K}$. The converse is obvious, so the proof is complete. □

5. The Hardy algebra $H^\infty(\mathbb{B}_f)$ and the eigenvectors of $M_{Z_1}^*, \dots, M_{Z_n}^*$

Let $f = (f_1, \dots, f_n)$ be an n-tuple with the model property. We define the noncommutative Hardy algebra $H^\infty(\mathbb{B}_f)$ to be the WOT-closure of all noncommutative polynomials in $M_{Z_1}, \dots, M_{Z_n}$ and the identity. Assume that $f \in \mathcal{M}^{\|}$. We say that $F : \mathbb{B}_f^<(\mathcal{H}) \to B(\mathcal{H})$ is a free holomorphic function on $\mathbb{B}_f^<(\mathcal{H})$ if there are some coefficients $c_\alpha \in \mathbb{C}$ such that

$$F(Y) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha \big[f(Y)\big]_\alpha, \quad Y \in \mathbb{B}_f^<(\mathcal{H}),$$

where the convergence of the series is in the operator norm topology. Since, according to Lemma 3.10, we have $\mathbb{B}_f^<(\mathcal{H}) = g([B(\mathcal{H})^n]_1)$ and $f(g(X)) = X$, $X \in [B(\mathcal{H})^n]_1$, the uniqueness of the representation of F follows from the uniqueness of the representation of free holomorphic functions on $[B(\mathcal{H})^n]_1$.

Theorem 5.1. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $\mathbb{B}_f$ be the corresponding noncommutative domain. Then the following statements hold.*

(i) *$H^\infty(\mathbb{B}_f)$ coincides with the algebra of bounded left multipliers of $\mathbb{H}^2(f)$.*
(ii) *If $f \in \mathcal{M}^{\|}$, then $H^\infty(\mathbb{B}_f)$ can be identified with the algebra $\mathbb{H}^\infty(\mathbb{B}_f^<)$ of all bounded free holomorphic functions on the noncommutative domain $\mathbb{B}_f^<(\mathcal{H})$, which coincides with*

$$\left\{\varphi \circ f : \mathbb{B}_f^<(\mathcal{H}) \to B(\mathcal{H}) \colon \varphi \in H^\infty_{\mathbf{ball}}\right\}.$$

(iii) *If $\psi \in H^\infty(\mathbb{B}_f)$, then there is a unique $\varphi = \sum_\alpha c_\alpha S_\alpha$ in the noncommutative analytic Toeplitz algebra F_n^∞ such that*

$$\psi = \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha r^{|\alpha|}\left[f(M_Z)\right]_\alpha, \quad c_\alpha \in \mathbb{C},$$

where $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ and the convergence of the series is in the operator norm topology.

Proof. According to the proof of Lemma 2.6, $M_{f_j} = U^{-1} S_j U$, $j = 1, \ldots, n$, where $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$, and $M_{Z_j} = U^{-1}\varphi_j(S_1, \ldots, S_n)U$, where $\varphi_j(S_1, \ldots, S_n)$ is in the noncommutative Hardy algebra F_n^∞. We recall that F_n^∞ is the WOT closure of the noncommutative polynomials in $S_1, \ldots, S_n$ and the identity. Since $H^\infty(\mathbb{B}_f)$ is the WOT-closure of all noncommutative polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity, we deduce that $H^\infty(\mathbb{B}_f) \subseteq U^{-1}F_n^\infty U$. On the other hand, using again Lemma 2.6, the creation operator S_j is in the WOT-closure of polynomials in $\varphi_1(S_1, \ldots, S_n), \ldots, \varphi_n(S_1, \ldots, S_n)$ and the identity. Consequently, we have $U^{-1}S_j U \in H^\infty(\mathbb{B}_f)$, $j = 1, \ldots, n$, which implies $U^{-1}F_n^\infty U \subseteq H^\infty(\mathbb{B}_f)$. Thus, we have proved that

$$H^\infty(\mathbb{B}_f) = U^{-1}F_n^\infty U. \tag{5.1}$$

Taking into account that $U(\mathbb{H}^2(f)) = F^2(H_n)$ and that the algebra of bounded left multipliers on $F^2(H_n)$ coincides with F_n^∞, we deduce item (i).

To prove (ii), we recall (see [30]) that if $\varphi \in H^\infty_{\mathbf{ball}}$, then $\varphi(X) = \sum_{k=0}^\infty \sum_{|\alpha=k} a_\alpha X_\alpha$, $X \in [B(\mathcal{H})^n]_1$, where the convergence is in the operator norm topology. Moreover, $\sup_{X\in[B(\mathcal{H})^n]_1} \|\varphi(X)\| < \infty$, and the model boundary function $\widetilde{\varphi} := \text{SOT-}\lim_{r\to 1}\sum_{k=0}^\infty \times \sum_{|\alpha=k} a_\alpha r^{|\alpha|} S_\alpha$ exists in F_n^∞. Since, according to Lemma 3.10, $\mathbb{B}_f^<(\mathcal{H}) = g([B(\mathcal{H})^n]_1)$ and $f(g(X)) = X$ for $X \in [B(\mathcal{H})^n]_1$, the map $F : \mathbb{B}_f^<(\mathcal{H}) \to B(\mathcal{H})$ defined by

$$F(Y) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha\left[f(Y)\right]_\alpha, \quad Y \in \mathbb{B}_f^<(\mathcal{H}),$$

is well defined with the convergence in the operator norm topology. Consequently, $F = \varphi \circ f$ is a bounded free holomorphic function on $\mathbb{B}_f^<(\mathcal{H})$. Now, let $G \in \mathbb{H}^\infty(\mathbb{B}_f^<)$. Then there are coefficients $c_\alpha \in \mathbb{C}$ such that

$$G(Y) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha \big[f(Y)\big]_\alpha, \quad Y \in \mathbb{B}_f^<(\mathcal{H}),$$

where the convergence is in the norm topology and $\sup_{Y \in \mathbb{B}_f^<(\mathcal{H})} \|G(Y)\| < \infty$. Taking $Y = g(rS_1, \ldots, rS_n)$, we deduce that $\sup_{r\in[0,1)} \|\sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} S_\alpha\| < \infty$, which shows that the map $\varphi : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$, defined by $\varphi(X) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha X_\alpha$ is in $H^\infty_{\mathbf{ball}}$, and $G = \varphi \circ f$. This shows that

$$\mathbb{H}^\infty(\mathbb{B}_f^<) = \big\{\varphi \circ f : \mathbb{B}_f^<(\mathcal{H}) \to B(\mathcal{H}) \colon \varphi \in H^\infty_{\mathbf{ball}}\big\}.$$

Hence, using relation (5.1) and the fact that F_n^∞ can be identified with $H^\infty_{\mathbf{ball}}$, we deduce item (ii).

To prove part (iii), let $\psi \in H^\infty(\mathbb{B}_f)$. Due to relation (5.1), the operator $U\psi U^{-1}$ is in the Hardy algebra F_n^∞ and, therefore, there are coefficients $c_\alpha \in \mathbb{C}$ such that

$$U\psi U^{-1} = \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha=k} a_\alpha r^{|\alpha|} S_\alpha,$$

where the convergence of the series is in norm. Since f has the model property, we have $M_{f_j} = f_j(M_{Z_1}, \ldots, M_{Z_n})$, $j = 1, \ldots, n$. Using now relation $M_{f_j} = U^{-1} S_j U$, we deduce that item (iii) holds. The proof is complete. □

Theorem 5.2. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. The eigenvectors for $M_{Z_1}^*, \ldots, M_{Z_n}^*$ are precisely the noncommutative Poisson kernels*

$$\Gamma_\lambda := \left(1 - \sum_{i=1}^n \big|f_i(\lambda)\big|^2\right)^{1/2} \sum_{\alpha \in \mathbb{F}_n} \big[\overline{f(\lambda)}\big]_\alpha f_\alpha, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_f^<(\mathbb{C}).$$

They satisfy the equations

$$M_{Z_i}^* \Gamma_\lambda = \bar{\lambda}_i \Gamma_\lambda, \quad i = 1, \ldots, n.$$

If $\lambda \in \mathbb{B}_f^<(\mathbb{C})$ and $\varphi(M_Z)$ is in $H^\infty(\mathbb{B}_f)$, then the map

$$\Phi_\lambda : H^\infty(\mathbb{B}_f) \to \mathbb{C}, \qquad \Phi_\lambda\big(\varphi(M_Z)\big) := \varphi(\lambda),$$

is WOT-continuous and multiplicative and $\varphi(\lambda) = \langle \varphi(M_Z)\Gamma_\lambda, \Gamma_\lambda\rangle$. Moreover, $\varphi(M_Z)^*\Gamma_\lambda = \overline{\varphi(\lambda)}\Gamma_\lambda$ and $\lambda \mapsto \varphi(\lambda)$ is a bounded holomorphic function on $\mathbb{B}_f^<(\mathbb{C}) \subset \mathbb{C}^n$.

Proof. Assume that $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_f^<(\mathbb{C})$. According to Theorem 3.1, the noncommutative Poisson kernel associated with the noncommutative domain $\mathbb{B}_f$ at λ, which is a pure element, is the operator $K_{f,\lambda} : \mathbb{C} \to \mathbb{H}^2(f) \otimes \mathbb{C}$ defined by

$$K_{f,\lambda}(z) = \sum_{\alpha \in \mathbb{F}_n^+} f_\alpha \otimes \left(1 - \sum_{i=1}^n \left| f_i(\lambda)\right|^2\right)^{1/2} \left[\overline{f(\lambda)}\right]_\alpha z, \quad z \in \mathbb{C},$$

which satisfies the equation $(M_{Z_i}^* \otimes I_{\mathbb{C}}) K_{f,\lambda} = K_{f,\lambda}(\bar{\lambda}_i I_{\mathbb{C}})$ for $i = 1, \dots, n$. Under the natural identification of $\mathbb{H}^2(f) \otimes \mathbb{C}$ with $\mathbb{H}^2(f)$, we deduce that $\Gamma_\lambda = K_{f,\lambda}$ and

$$M_{Z_i}^* \Gamma_\lambda = \bar{\lambda}_i \Gamma_\lambda, \quad i = 1, \dots, n.$$

Conversely, let $\xi := \sum_{\beta \in \mathbb{F}_n^+} c_\beta f_\beta(Z)$ be a formal power series in $\mathbb{H}^2(f)$ such that $\xi \neq 0$ and assume that $M_{Z_i}^* \xi = \bar{\lambda}_i \xi$, $i = 1, \dots, n$, for some $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Let f_i have the representation $f_i = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^{(i)} Z_\alpha$. Since $f = (f_1, \dots, f_n)$ has the model property, we have $M_{f_i} = f_i(M_{Z_1}, \dots, M_{Z_n}) = \sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha^{(i)} M_{Z_\alpha}$, where $(M_{Z_1}, \dots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$. We shall consider just one case since the other can be treated similarly. For example, assume that $(M_{Z_1}, \dots, M_{Z_n}) \in \mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ and let $\eta \in \mathbb{H}^2(f)$. Then we have

$$\begin{aligned} \left\langle f_i(M_{Z_1}, \dots, M_{Z_n})^* \xi, \eta\right\rangle &= \lim_{m\to\infty} \left\langle \xi, \sum_{k=1}^m \sum_{|\alpha|=k} a_\alpha^{(i)} M_{Z_\alpha} \eta \right\rangle = \lim_{m\to\infty} \left\langle \sum_{k=1}^m \sum_{|\alpha|=k} \overline{a_\alpha^{(i)}} M_{Z_\alpha}^* \xi, \eta \right\rangle \\ &= \lim_{m\to\infty} \left\langle \sum_{k=1}^m \sum_{|\alpha|=k} \overline{a_\alpha^{(i)}} \bar{\lambda}_\alpha \xi, \eta \right\rangle = \lim_{m\to\infty} \sum_{k=1}^m \sum_{|\alpha|=k} \overline{a_\alpha^{(i)}} \bar{\lambda}_\alpha \langle \xi, \eta\rangle \\ &= \left\langle \overline{f_i(\lambda)} \xi, \eta\right\rangle, \end{aligned}$$

which shows that

$$f_i(M_{Z_1}, \dots, M_{Z_n})^* \xi = \overline{f_i(\lambda)} \xi, \quad i = 1, \dots, n. \tag{5.2}$$

Hence, and using the fact that $M_{f_i} = f_i(M_{Z_1}, \dots, M_{Z_n})$, $i = 1, \dots, n$, we deduce that

$$\begin{aligned} c_\beta &= \langle \xi, M_{f_\beta} 1\rangle = \left\langle \xi, \left[f(M_{Z_1}, \dots, M_{Z_n})\right]_\beta 1\right\rangle \\ &= \left\langle \left[f(M_{Z_1}, \dots, M_{Z_n})\right]_\beta^* \xi, 1\right\rangle = \left[\overline{f(\lambda)}\right]_\beta \langle \xi, 1\rangle \\ &= c_0 \left[\overline{f(\lambda)}\right]_\beta \end{aligned}$$

for any $\beta \in \mathbb{F}_n^+$. Therefore, we have

$$\xi = c_0 \sum_{\beta \in \mathbb{F}_n^+} \left[\overline{f(\lambda)}\right]_\beta f_\beta.$$

Since $\xi \in \mathbb{H}^2(f)$, we must have

$$\sum_{k=0}^{\infty}\left(\left|f_1(\lambda)\right|^2+\cdots+\left|f_n(\lambda)\right|^2\right)^k=\sum_{\beta\in\mathbb{F}_n^+}\left|\left[f(\lambda)\right]_\beta\right|^2<\infty.$$

Hence, we deduce that $|f_1(\lambda)|^2+\cdots+|f_n(\lambda)|^2<1$.

Now, due to relation (5.2) and using again that $M_{f_i}=f_i(M_{Z_1},\ldots,M_{Z_n})$, $i=1,\ldots,n$, we deduce that $M_{f_i}^*\xi=\overline{f_i(\lambda)}\xi$. On the other, according to the proof of Theorem 3.1 (see relation (3.1)), we have

$$M_{Z_i}=g_i(M_{f_1},\ldots,M_{f_n})=\text{SOT-}\lim_{r\to 1}g_i(rM_{f_1},\ldots,rM_{f_n}).$$

As above, one can show that $g_i(M_{f_1},\ldots,M_{f_n})^*\xi=\overline{g_i(f(\lambda))}\xi$ for $i=1,\ldots,n$. Combining this relation with the fact that $M_{Z_i}^*\xi=\bar{\lambda}_i\xi$, $i=1,\ldots,n$, we conclude that $\lambda=g(f(\lambda))$. Therefore, $\lambda\in\mathbb{B}_f^<(\mathbb{C})$.

According to Theorem 5.1, part (iii), we have $\varphi(M_Z)=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}c_\alpha r^{|\alpha|}\times[f(M_Z)]_\alpha$ for some coefficients $c_\alpha\in\mathbb{C}$. Using relation (5.2), we deduce that

$$\begin{aligned}\left\langle\varphi(M_Z)\Gamma_\lambda,\Gamma_\lambda\right\rangle&=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}c_\alpha r^{|\alpha|}\left\langle\Gamma_\lambda,\left[f(M_Z)\right]_\alpha^*\Gamma_\lambda\right\rangle\\&=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}c_\alpha r^{|\alpha|}\left\langle\Gamma_\lambda,\overline{\left[f(\lambda)\right]}_\alpha\Gamma_\lambda\right\rangle\\&=\text{SOT-}\lim_{r\to 1}\|\Gamma_\lambda\|^2\sum_{k=0}^{\infty}\sum_{|\alpha|=k}c_\alpha r^{|\alpha|}\left[f(\lambda)\right]_\alpha=\varphi(\lambda).\end{aligned}$$

Similarly, one can show that $\varphi(M_Z)^*\Gamma_\lambda=\overline{\varphi(\lambda)}\Gamma_\lambda$. According to Lemma 3.10 part (i), the mapping $f|_{\mathbb{B}_f^<(\mathbb{C})}:\mathbb{B}_f^<(\mathbb{C})\to\mathbb{B}_n$ is the inverse of $g|_{\mathbb{B}_n}:\mathbb{B}_n\to\mathbb{B}_f^<(\mathbb{C})$. Since g is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$, the map $\mathbb{B}_n\ni\lambda\mapsto g(\lambda)\in\mathbb{B}_f^<(\mathbb{C})$ is holomorphic on $\mathbb{B}_n$ and its inverse $\mathbb{B}_f^<(\mathbb{C})\ni\lambda\mapsto f(\lambda)\in\mathbb{B}_n$ is also holomorphic. On the other hand, according to Theorem 5.1, part (iii), there is $\psi\in H_{\mathbf{ball}}^\infty$ such that $\varphi(\lambda)=\psi(f(\lambda))$ for $\lambda\in\mathbb{B}_f^<(\mathbb{C})$. Hence, we deduce that $\lambda\mapsto\varphi(\lambda)$ is a bounded holomorphic function on $\mathbb{B}_f^<(\mathbb{C})$. This completes the proof. □

Theorem 5.2 can be used to prove the following result. Since the proof is similar to the corresponding result from [9], we shall omit it.

Corollary 5.3. *A map $\Phi:H^\infty(\mathbb{B}_f)\to\mathbb{C}$ is a WOT-continuous multiplicative linear functional if and only if there exists $\lambda\in\mathbb{B}_f^<(\mathbb{C})$ such that*

$$\Phi(A)=\Phi_\lambda(A):=\langle A\Gamma_\lambda,\Gamma_\lambda\rangle,\quad A\in H^\infty(\mathbb{B}_f),$$

where Γ_λ is the noncommutative Poisson kernel associated with the domain $\mathbb{B}_f$ at λ.

Assume that $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series with the model property. Using Theorem 5.1, one can prove that J is a WOT-closed two-sided ideal of $H^\infty(\mathbb{B}_f)$ if and only if there is a WOT-closed two-sided ideal $\mathcal{I}$ of F_n^∞ such that

$$J = \{\varphi(f(M_Z)): \varphi \in \mathcal{I}\}.$$

We mention that if $\varphi(S_1, \ldots, S_n) \in F_n^\infty$ has the Fourier representation $\varphi(S_1, \ldots, S_n) = \sum_{\alpha \in \mathbb{F}_n^+} c_\alpha S_\alpha$, then

$$\varphi(f(M_Z)) = \text{SOT-}\lim_{r \to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} [f(M_Z)]_\alpha$$

exists. Denote by $H^\infty(\mathcal{V}_{f,J})$ the WOT-closed algebra generated by the operators $B_i := P_{\mathcal{N}_J} M_{Z_i} |\mathcal{N}_J$, for $i = 1, \ldots, n$, and the identity, where

$$\mathcal{N}_J := \mathbb{H}^2(f) \ominus \mathcal{M}_J \quad \text{and} \quad \mathcal{M}_J := \overline{J\mathbb{H}^2(f)}.$$

The following result is a consequence of Theorem 4.1 from [2] and the above-mentioned remarks.

Theorem 5.4. *Let J be a WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$. Then the map*

$$\Gamma : H^\infty(\mathbb{B}_f)/J \to B(\mathcal{N}_J) \quad \textit{defined by } \Gamma(\varphi + J) = P_{\mathcal{N}_J} \varphi|_{\mathcal{N}_J}$$

is a completely isometric representation.

Since the set of all polynomials in $M_{Z_1}, \ldots, M_{Z_n}$ and the identity is WOT-dense in $H^\infty(\mathbb{B}_f)$, Theorem 5.4 implies that $P_{\mathcal{N}_J} H^\infty(\mathbb{B}_f)|_{\mathcal{N}_J}$ is a WOT-closed subalgebra of $B(\mathcal{N}_J)$ and, moreover, $H^\infty(\mathcal{V}_{f,J}) = P_{\mathcal{N}_J} H^\infty(\mathbb{B}_f)|_{\mathcal{N}_J}$.

We need a few more definitions. For each $\lambda = (\lambda_1, \ldots, \lambda_n)$ and each n-tuple $\mathbf{k} := (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 := \{0, 1, \ldots\}$, let $\lambda^{\mathbf{k}} := \lambda_1^{k_1} \cdots \lambda_n^{k_n}$. If $\mathbf{k} \in \mathbb{N}_0$, we denote

$$\Lambda_{\mathbf{k}} := \{\alpha \in \mathbb{F}_n^+: \lambda_\alpha = \lambda^{\mathbf{k}} \text{ for all } \lambda \in \mathbb{C}^n\}.$$

For each $\mathbf{k} \in \mathbb{N}_0^n$, define the formal power series

$$\omega^{\mathbf{k})} := \frac{1}{\gamma_{\mathbf{k}}} \sum_{\alpha \in \Lambda_{\mathbf{k}}} f_\alpha \in \mathbb{H}^2(f), \quad \text{where } \gamma_{\mathbf{k}} := \operatorname{card} \Lambda_{\mathbf{k}} = \binom{|\mathbf{k}|!}{k_1! \cdots k_n!}.$$

Note that the set $\{\omega^{(\mathbf{k})}: \mathbf{k} \in \mathbb{N}_0^n\}$ consists of orthogonal power series in $\mathbb{H}^2(f)$ and $\|\omega^{(\mathbf{k})}\| = \frac{1}{\sqrt{\gamma_{\mathbf{k}}}}$. We denote by $\mathbb{H}_s^2(f)$ the closed span of these formal power series, and call it the symmetric Hardy space associated with the noncommutative domain $\mathbb{B}_f$.

Theorem 5.5. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$.*

Let J_c be the WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ generated by the commutators

$$M_{Z_i}M_{Z_j} - M_{Z_j}M_{Z_i}, \quad i, j = 1, \dots, n.$$

Then the following statements hold.

(i) $\mathbb{H}_s^2(f) = \overline{\operatorname{span}}\{\Gamma_\lambda\colon \lambda \in \mathbb{B}_f^<(\mathbb{C})\} = \mathcal{N}_{J_c} := \mathbb{H}^2(f) \ominus \overline{J_c(1)}$.

(ii) *The symmetric Hardy space $\mathbb{H}_s^2(f)$ can be identified with the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$ of all holomorphic functions $\psi : \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ which admit a series representation $\psi(\lambda) = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}} f(\lambda)^{\mathbf{k}}$ with*

$$\|\psi\|_2 = \sum_{\mathbf{k}\in\mathbb{N}_0} |c_{\mathbf{k}}|^2 \frac{1}{\gamma_{\mathbf{k}}} < \infty.$$

More precisely, every element $\psi = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}}\omega^{(\mathbf{k})}$ in $\mathbb{H}_s^2(f)$ has a functional representation on $\mathbb{B}_f^<(\mathbb{C})$ given by

$$\psi(\lambda) := \langle \psi, \Omega_\lambda\rangle = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}} f(\lambda)^{\mathbf{k}}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_f^<(\mathbb{C}),$$

where $\Omega_\lambda := \frac{1}{\sqrt{1-\sum_{i=1}^n |f_i(\lambda)|^2}}\Gamma_\lambda$ and

$$\bigl|\psi(\lambda)\bigr| \leqslant \frac{\|\psi\|_2}{\sqrt{1-\sum_{i=1}^n |f_i(\lambda)|^2}}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_f^<(\mathbb{C}).$$

(iii) *The mapping $\Lambda_f : \mathbb{B}_f^<(\mathbb{C}) \times \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ defined by*

$$\Lambda_f(\mu, \lambda) := \langle \Omega_\lambda, \Omega_\mu\rangle = \frac{1}{1-\sum_{i=1}^n f_i(\mu)\overline{f_i(\lambda)}}, \quad \lambda, \mu \in \mathbb{B}_f^<(\mathbb{C}),$$

is positive definite.

Proof. First, note that $\Omega_\lambda = \sum_{\mathbf{k}\in\mathbb{N}_0^n} \overline{f(\lambda)^{\mathbf{k}}}\gamma_{\mathbf{k}}\omega^{(\mathbf{k})}$, $\lambda \in \mathbb{B}_f^<(\mathbb{C})$, and, therefore,

$$\overline{\operatorname{span}}\bigl\{\Gamma_\lambda\colon \lambda \in \mathbb{B}_f^<(\mathbb{C})\bigr\} \subseteq \mathbb{H}_s^2(f).$$

Now, we prove that $\omega^{(\mathbf{k})} \in \mathcal{N}_{J_c} := F^2(H_n) \ominus \overline{J_c(1)}$. First, we show that J_c coincides with the WOT-closed commutator ideal of $H^\infty(\mathbb{B}_f)$. Indeed, since $M_{Z_i}M_{Z_j} - M_{Z_j}M_{Z_i} \in J_c$ and every permutation of k objects is a product of transpositions, it is clear that $M_{Z_\alpha}M_{Z_\beta} - M_{Z_\beta}M_{Z_\alpha} \in J_c$ for any $\alpha, \beta \in \mathbb{F}_n^+$. Consequently, $M_{Z_\gamma}(M_{Z_\alpha}M_{Z_\beta} - M_{Z_\beta}M_{Z_\alpha})M_{Z_\omega} \in J_c$ for any $\alpha, \beta, \gamma, \omega \in \mathbb{F}_n^+$. Since the polynomials in $M_{Z_1}, \dots, M_{Z_n}$ are WOT dense in $H^\infty(\mathbb{B}_f)$, the result follows. Note also that $\overline{J_c(1)} \subset \mathbb{H}^2(f)$ coincides with

$$\overline{\operatorname{span}}\bigl\{Z_{\gamma g_j g_i \beta} - Z_{\gamma g_i g_j \beta}\colon \gamma, \beta \in \mathbb{F}_n^+,\ i, j = 1, \dots, n\bigr\}.$$

Similarly, one can prove that the WOT-closed two-sided ideal generated by the commutators $M_{f_j}M_{f_i} - M_{f_i}M_{f_j}$, $i, j \in \{1, \dots, n\}$ coincides with the WOT-closed commutator ideal of $H^\infty(\mathbb{B}_f)$. Combining these results, we deduce that J_c coincides with the WOT-closed two-sided ideal generated by the commutators $M_{f_j}M_{f_i} - M_{f_i}M_{f_j}$, $i, j \in \{1, \dots, n\}$ and

$$\overline{J_c(1)} = \overline{\operatorname{span}}\{f_{\gamma g_j g_i \beta} - f_{\gamma g_i g_j \beta} \colon \gamma, \beta \in \mathbb{F}_n^+,\ i, j = 1, \dots, n\}.$$

Consequently, since

$$\left\langle \sum_{\alpha \in \Lambda_{\mathbf{k}}} f_\alpha, M_{f_\gamma}(M_{f_j}M_{f_i} - M_{f_i}M_{f_j})M_{f_\beta}(1)\right\rangle = 0$$

for any $\mathbf{k} \in \mathbb{N}_0^n$, we deduce that $\omega^{(\mathbf{k})} \in \mathcal{N}_{J_c}$. Hence, we have $\mathbb{H}_s^2(f) \subseteq \mathcal{N}_{J_c}$. To complete the proof of part (i), it is enough to show that

$$\overline{\operatorname{span}}\{\Gamma_\lambda \colon \lambda \in \mathbb{B}_f^<(\mathbb{C})\} = \mathcal{N}_{J_c}.$$

Assume that there is a vector $x := \sum_{\beta \in \mathbb{F}_n^+} c_\beta f_\beta \in \mathcal{N}_{J_c}$ and $x \perp \Gamma_\lambda$ for all $\lambda \in \mathbb{B}_f^<(\mathbb{C})$. Then

$$\left\langle \sum_{\beta \in \mathbb{F}_n^+} c_\beta f_\beta, \Omega_\lambda \right\rangle = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \big[f(\lambda)\big]_\beta = \sum_{\mathbf{k} \in \mathbb{N}_0^n} \left(\sum_{\beta \in \Lambda_{\mathbf{k}}} c_\beta \right) f(\lambda)^{\mathbf{k}} = 0$$

for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$. Since $\mathbb{B}_f^<(\mathbb{C})$ contains an open ball in $\mathbb{C}^n$, we deduce that

$$\sum_{\beta \in \Lambda_{\mathbf{k}}} c_\beta = 0 \quad \text{for all } \mathbf{k} \in \mathbb{N}_0^n. \tag{5.3}$$

Fix $\beta_0 \in \Lambda_{\mathbf{k}}$ and let $\beta \in \Lambda_{\mathbf{k}}$ be such that β is obtained from β_0 by transposing just two generators. So we can assume that $\beta_0 = \gamma g_j g_i \omega$ and $\beta = \gamma g_i g_j \omega$ for some $\gamma, \omega \in \mathbb{F}_n^+$ and $i \neq j$, $i, j = 1, \dots, n$. Since $x \in \mathcal{N}_{J_c} = \mathbb{H}^2(f) \ominus \overline{J_c(1)}$, we must have

$$\langle x, M_{f_\gamma}(M_{f_j}M_{f_i} - M_{f_i}M_{f_j})M_{f_\omega}(1)\rangle = 0,$$

which implies $c_{\beta_0} = c_\beta$. Since any element $\gamma \in \Lambda_{\mathbf{k}}$ can be obtained from β_0 by successive transpositions, repeating the above argument, we deduce that $c_{\beta_0} = c_\gamma$ for all $\gamma \in \Lambda_{\mathbf{k}}$. Now relation (5.3) implies $c_\gamma = 0$ for any $\gamma \in \Lambda_{\mathbf{k}}$ and $\mathbf{k} \in \mathbb{N}_0^n$, so $x = 0$. Consequently, we have $\overline{\operatorname{span}}\{\Gamma_\lambda \colon \lambda \in \mathbb{B}_f^<(\mathbb{C})\} = \mathcal{N}_{J_c}$.

Now, let us prove part (ii) of the theorem. Note that

$$\langle \omega^{(\mathbf{k})}, \Omega_\lambda \rangle = \frac{1}{\gamma_{\mathbf{k}}} \left\langle \sum_{\beta \in \Lambda_{\mathbf{k}}} f_\beta, \Omega_\lambda \right\rangle = \frac{1}{\gamma_{\mathbf{k}}} \sum_{\beta \in \Lambda_{\mathbf{k}}} \big[f(\lambda)\big]_\beta = f(\lambda)^{\mathbf{k}}$$

for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$ and $\mathbf{k} \in \mathbb{N}_0^n$. Hence, every element $\psi = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} \omega^{(\mathbf{k})}$ in $\mathbb{H}_s^2(f)$ has a functional representation on $\mathbb{B}_f^<(\mathbb{C})$ given by

$$\psi(\lambda) := \langle \psi, \Omega_\lambda \rangle = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} f(\lambda)^{\mathbf{k}}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_f^<(\mathbb{C}),$$

and

$$\left|\psi(\lambda)\right| \leqslant \|\psi\|_2 \|\Omega_\lambda\| = \frac{\|\psi\|_2}{\sqrt{1 - \sum_{i=1}^n |f_i(\lambda)|^2}}.$$

The identification of $\mathbb{H}_s^2(f)$ with $H^2(\mathbb{B}_f^<(\mathbb{C}))$ is now clear. If $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ are in $\mathbb{B}_f^<(\mathbb{C})$, then we have

$$\Lambda_f(\mu, \lambda) := \langle \Omega_\lambda, \Omega_\mu \rangle = \sum_{\beta \in \mathbb{F}_n^+} \left[f(\mu)\right]_\beta \overline{\left[f(\lambda)\right]}_\beta,$$

which implies item (iii). The proof is complete. □

If $A \in B(\mathcal{H})$ then we denote by Lat A the set of all invariant subspaces of A. When $\mathcal{U} \subset B(\mathcal{H})$, we define $\operatorname{Lat}\mathcal{U} = \bigcap_{A \in \mathcal{U}} \operatorname{Lat} A$. Given any collection $\mathcal{S}$ of subspaces of $\mathcal{H}$, then we set

$$\operatorname{Alg}\mathcal{S} := \left\{A \in B(\mathcal{H}):\ \mathcal{S} \subset \operatorname{Lat} A\right\}.$$

We recall that the algebra $\mathcal{U} \subset B(\mathcal{H})$ is reflexive if $\mathcal{U} = \operatorname{AlgLat}\mathcal{U}$.

Theorem 5.6. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. If $H^\infty(\mathcal{V}_{f,J_c})$ is the WOT-closed algebra generated by the operators*

$$L_i := P_{\mathbb{H}_s^2(f)} M_{Z_i}|_{\mathbb{H}_s^2(f)}, \quad i = 1, \ldots, n,$$

and the identity, then the following statements hold.

(i) *$H^\infty(\mathcal{V}_{f,J_c})$ can be identified with the algebra of all multipliers of the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$.*
(ii) *The algebra $H^\infty(\mathcal{V}_{f,J_c})$ is reflexive.*

Proof. According to the remarks following Theorem 5.4, we have $H^\infty(\mathcal{V}_{f,J_c}) = P_{\mathbb{H}_s^2(f)} H^\infty(\mathbb{B}_f)|_{\mathbb{H}_s^2(f)}$. Let $\varphi(M_Z) \in H^\infty(\mathbb{B}_f)$ and $\varphi(L) = P_{\mathbb{H}_s^2(f)} \varphi(M_Z)|_{\mathbb{H}_s^2(f)}$. Due to Theorem 5.5, since $\Omega_\lambda \in \mathbb{H}_s^2(f)$ for $\lambda \in \mathbb{B}_f^<(\mathbb{C})$, and $\varphi(M_Z)^* \Omega_\lambda = \overline{\varphi(\lambda)} \Omega_\lambda$ (see Theorem 5.2), we have

$$\begin{aligned}
\left[\varphi(L)\psi\right](\lambda) &= \left\langle \varphi(L)\psi, \Omega_\lambda \right\rangle = \left\langle \varphi(M_Z)\psi, \Omega_\lambda \right\rangle \\
&= \left\langle \psi, \varphi(M_Z)^* \Omega_\lambda \right\rangle = \left\langle \psi, \overline{\varphi(\lambda)} \Omega_\lambda \right\rangle \\
&= \varphi(\lambda)\psi(\lambda)
\end{aligned}$$

for any $\psi \in \mathbb{H}_s^2(f)$ and $\lambda \in \mathbb{B}_f^<(\mathbb{C})$. Therefore, the operators in $\mathbb{H}^\infty(\mathcal{V}_{f,J_c})$ are "analytic" multipliers of $\mathbb{H}_s^2(f)$. Moreover,

$$\left\|\varphi(L)\right\| = \sup\left\{\|\varphi\chi\|_2:\ \chi \in \mathbb{H}_s^2(f),\ \|\chi\| \leqslant 1\right\}.$$

Conversely, suppose that $\psi = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}}\omega^{(\mathbf{k})}$ is a bounded multiplier, i.e., $M_\psi \in B(\mathbb{H}_s^2(f))$. As in [9] (see Lemma 1.1), using Cesaro means, one can find a sequence $q_m = \sum c_{\mathbf{k}}^{(m)}\omega^{(\mathbf{k})}$ such that M_{q_m} converges to M_ψ in the strong operator topology and, consequently, in the w^*-topology. Since M_{q_m} is a polynomial in $L_1, \ldots, L_n$, we conclude that $M_\psi \in H^\infty(\mathcal{V}_{f,J_c})$. In particular L_i is the multiplier M_{λ_i} by the coordinate function.

Now, we prove part (ii). Let $Y \in B(\mathbb{H}_s^2(f))$ be an operator that leaves invariant all the invariant subspaces under each operator $L_1, \ldots, L_n$. According to Theorem 5.2, we have $L_i^*\Gamma_\lambda = \bar{\lambda}_i\Gamma_\lambda$ for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$ and $i = 1, \ldots, n$. Since Y^* leaves invariant all the invariant subspaces under $L_1^*, \ldots, L_n^*$, the vector Ω_λ must be an eigenvector for Y^*. Consequently, there is a function $\varphi : \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ such that $Y^*\Omega_\lambda = \overline{\varphi(\lambda)}\Omega_\lambda$ for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$. Due to Theorem 5.5, if $f \in H_s^2(f)$, then Yf has the functional representation

$$(Yf)(\lambda) = \langle Yf, \Omega_\lambda\rangle = \langle f, Y^*\Omega_\lambda\rangle = \varphi(\lambda)f(\lambda) \quad \text{for all } \lambda \in \mathbb{B}_f^<(\mathbb{C}).$$

In particular, if $f = 1$, then the functional representation of $Y(1)$ coincide with φ. Therefore, φ admits a representation $\sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}} f(\lambda)^{\mathbf{k}}$ on $\mathbb{B}_f^<(\mathbb{C})$ and can be identified with $X(1) \in \mathbb{H}_s^2(f)$. Moreover, the equality above shows that $\varphi f \in H^2(\mathbb{B}_f^<(\mathbb{C}))$ for any $f \in \mathbb{H}_s^2(f)$. Applying the first part of this theorem, we deduce that $Y = M_\varphi \in H^\infty(\mathcal{V}_{f,J_c})$. The proof is complete. □

We remark that, in the particular case when $f = (Z_1, \ldots, Z_n)$, we recover some of the results obtained by Arias and the author, Davidson and Pitts, and Arveson (see [24,1,2,7,9,3]).

6. Characteristic functions and functional models

In this section, we introduce the characteristic function of an n-tuple $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$, present a model for pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ in terms of characteristic functions, and show that the characteristic function is a complete unitary invariant for pure n-tuples of operators in $\mathbb{B}_f(\mathcal{H})$.

Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. We introduce the characteristic function of an n-tuple $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ to be the multi-analytic operator, with respect to $M_{Z_1}, \ldots, M_{Z_n}$,

$$\Theta_{f,T} : \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$$

having the formal Fourier representation

$$\begin{aligned} &-I_{\mathbb{H}^2(f)} \otimes f(T) + (I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T})\left(I_{\mathbb{H}^2(f)\otimes\mathcal{H}} - \sum_{i=1}^n \Lambda_i \otimes f_i(T)^*\right)^{-1} \\ &\quad\times [\Lambda_1 \otimes I_{\mathcal{H}}, \ldots, \Lambda_n \otimes I_{\mathcal{H}}](I_{\mathbb{H}^2(f)} \otimes \Delta_{f,T^*}), \end{aligned}$$

where $\Lambda_1, \ldots, \Lambda_n$ are the right multiplication operators by the power series $f_1, \ldots, f_n$, respectively, on the Hardy space $\mathbb{H}^2(f)$, and the defect operators associated with $T := (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ are

$$\Delta_{f,T} := \left(I_{\mathcal{H}} - \sum_{i=1}^{n} f_i(T) f_i(T)^*\right)^{1/2} \in B(\mathcal{H}) \quad \text{and}$$

$$\Delta_{f,T^*} := \left(I - f(T)^* f(T)\right)^{1/2} \in B\left(\mathcal{H}^{(n)}\right),$$

while the defect spaces are $\mathcal{D}_{f,T} := \overline{\Delta_{f,T}\mathcal{H}}$ and $\mathcal{D}_{f,T^*} := \overline{\Delta_{f,T^*}\mathcal{H}^{(n)}}$, where $\mathcal{H}^{(n)}$ denotes the direct sum of n copies of $\mathcal{H}$. We remark that when $f = (f_1, \ldots, f_n) = (Z_1, \ldots, Z_n)$, we recover the characteristic function for row contractions. We recall that the characteristic function associated with an arbitrary row contraction $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, was introduced in [19] (see [37] for the classical case $n = 1$) and it was proved to be a complete unitary invariant for completely non-coisometric (c.n.c.) row contractions. Related to our setting, we remark that

$$\Theta_{f,T} = \left(U^* \otimes I_{\mathcal{D}_{f,T}}\right)\Theta_{f(T)}\left(U \otimes I_{\mathcal{D}_{f,T^*}}\right), \tag{6.1}$$

where $\Theta_{f(T)}$ is the characteristic function of the row contraction $f(T) = [f_1(T), \ldots, f_n(T)]$ and $U : \mathbb{H}^2(f) \to F^2(H_n)$ is the canonical unitary operator defined by $Uf_\alpha = e_\alpha$, $\alpha \in \mathbb{F}_n^+$. Consequently, due to Theorem 3.2 from [28], we deduce the following result.

Theorem 6.1. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(H)$. Then*

$$I_{\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}} - \Theta_{f,T}\Theta_{f,T}^* = K_{f,T}K_{f,T}^*, \tag{6.2}$$

where $\Theta_{f,T}$ is the characteristic function of T and $K_{f,T}$ is the corresponding Poisson kernel.

Now we present a model for pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ in terms of characteristic functions.

Theorem 6.2. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. If $T = (T_1, \ldots, T_n)$ is a pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$, then the characteristic function $\Theta_{f,T}$ is an isometry and T is unitarily equivalent to the n-tuple*

$$\left(P_{\mathbf{H}_{f,T}}(M_{Z_1} \otimes I_{\mathcal{D}_{f,T}})|_{\mathbf{H}_{f,T}}, \ldots, P_{\mathbf{H}_{f,T}}(M_{Z_n} \otimes I_{\mathcal{D}_{f,T}})|_{\mathbf{H}_{f,T}}\right), \tag{6.3}$$

where $P_{\mathbf{H}_{f,T}}$ is the orthogonal projection of $\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ on the Hilbert space

$$\mathbf{H}_{f,T} := \left(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}\right) \ominus \Theta_{f,T}\left(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\right).$$

Proof. According to Theorem 4.5, the noncommutative Poisson kernel $K_{f,T} : \mathcal{H} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ is an isometry, $K_{f,T}\mathcal{H}$ is a co-invariant subspace under $M_{Z_i} \otimes I_{\mathcal{D}_{f,T}}$, $i = 1, \ldots, n$, and

$$T_i = K_{f,T}^*(M_{Z_i} \otimes I_{\mathcal{D}_{f,T}})K_{f,T}, \quad i = 1, \ldots, n. \tag{6.4}$$

Hence, $K_{f,T}K_{f,T}^*$ is the orthogonal projection of $\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ onto $K_{f,T}\mathcal{H}$. Using relation (6.4), we deduce that $K_{f,T}K_{f,T}^*$ and $\Theta_{f,T}\Theta_{f,T}^*$ are mutually orthogonal projections such that

$$K_{f,T}K_{f,T}^* + \Theta_{f,T}\Theta_{f,T}^* = I_{\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}}.$$

This implies

$$K_{f,T}\mathcal{H} = \big(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}\big) \ominus \Theta_{f,T}\big(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\big).$$

Taking into account that $K_{f,T}$ is an isometry, we identify the Hilbert space $\mathcal{H}$ with $\mathbf{H}_{f,T} := K_{f,T}\mathcal{H}$. Using again relation (6.4), we deduce that T is unitarily equivalent to the n-tuple given by relation (6.3). That $\Theta_{f,T}$ is an isometry follows from relation (6.1) and the fact that the characteristic function of a pure row contraction is an isometry [19]. The proof is complete. $\square$

Let $\Phi : \mathbb{H}^2(f) \otimes \mathcal{K}_1 \to \mathbb{H}^2(f) \otimes \mathcal{K}_2$ and $\Phi' : \mathbb{H}^2(f) \otimes \mathcal{K}_1' \to \mathbb{H}^2(f) \otimes \mathcal{K}_2'$ be two multi-analytic operators with respect to $M_{Z_1}, \ldots, M_{Z_n}$. We say that Φ and Φ' coincide if there are two unitary multi-analytic operators $W_j : \mathbb{H}^2(f) \otimes \mathcal{K}_j \to \mathbb{H}^2(f) \otimes \mathcal{K}_j'$, $j = 1, 2$, with respect to $M_{Z_1}, \ldots, M_{Z_n}$ such that $\Phi' W_1 = W_2 \Phi$. Since $W_j(M_{Z_i} \otimes I_{\mathcal{K}_j}) = (M_{Z_i} \otimes I_{\mathcal{K}_j'})W_j$, $i = 1, \ldots, n$, we also have $W_j(M_{Z_i}^* \otimes I_{\mathcal{K}_j}) = (M_{Z_i}^* \otimes I_{\mathcal{K}_j'})W_j$, $i = 1, \ldots, n$. Taking into account that $C^*(M_{Z_1}, \ldots, M_{Z_n})$ is irreducible (see Theorem 3.3), we conclude that $W_j = I_{\mathbb{H}^2(f)} \otimes \tau_j$, $j = 1, 2$, for some unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}_j')$.

The next result shows that the characteristic function is a complete unitary invariant for pure n-tuple of operators in $\mathbb{B}_f(\mathcal{H})$.

Theorem 6.3. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ and $T' = (T_1', \ldots, T_n') \in \mathbb{B}_f(\mathcal{H}')$ be two pure n-tuples of operators. Then T and T' are unitarily equivalent if and only if their characteristic functions $\Theta_{f,T}$ and $\Theta_{f,T'}$ coincide.*

Proof. Assume that T and T' are unitarily equivalent and let $W : \mathcal{H} \to \mathcal{H}'$ be a unitary operator such that $T_i = W^* T_i' W$ for $i = 1, \ldots, n$. Note that

$$W\Delta_{f,T} = \Delta_{f,T'}W \quad \text{and} \quad \left(\bigoplus_{i=1}^n W\right)\Delta_{f,T^*} = \Delta_{f,T'^*}\left(\bigoplus_{i=1}^n W\right).$$

Consider the unitary operators τ and τ' defined by

$$\tau := W|_{\mathcal{D}_{f,T}} : \mathcal{D}_{f,T} \to \mathcal{D}_{f,T'} \quad \text{and} \quad \tau' := \left(\bigoplus_{i=1}^n W\right)\bigg|_{\mathcal{D}_{f,T^*}} : \mathcal{D}_{f,T*} \to \mathcal{D}_{f,T'*}.$$

Using the definition of the characteristic function, we deduce that $(I_{\mathbb{H}^2(f)} \otimes \tau)\Theta_{f,T} = \Theta_{f,T'}(I_{\mathbb{H}^2(f)} \otimes \tau')$.

Conversely, assume that the characteristic functions of T and T' coincide. Then there exist unitary operators $\tau : \mathcal{D}_{f,T} \to \mathcal{D}_{f,T'}$ and $\tau_* : \mathcal{D}_{f,T^*} \to \mathcal{D}_{f,T'^*}$ such that

$$(I_{\mathbb{H}^2(f)} \otimes \tau)\Theta_{f,T} = \Theta_{f,T'}(I_{\mathbb{H}^2(f)} \otimes \tau_*). \tag{6.5}$$

Hence, we deduce that $V := (I_{\mathbb{H}^2(f)} \otimes \tau)|_{\mathbb{H}_{f,T}} : \mathbb{H}_{f,T} \to \mathbb{H}_{f,T'}$ is a unitary operator, where $\mathbb{H}_{f,T}$ and $\mathbb{H}_{f,T'}$ are the model spaces for the n-tuples T and T', respectively, as defined in Theorem 6.2. Since

$$\big(M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}}\big)\big(I_{\mathbb{H}^2(f)} \otimes \tau^*\big) = \big(I_{\mathbb{H}^2(f)} \otimes \tau^*\big)\big(M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T'}}\big), \quad i = 1, \ldots, n,$$

and $\mathbb{H}_{f,T}$ (resp. $\mathbb{H}_{f,T'}$) is a co-invariant subspace under $M_{Z_i} \otimes I_{\mathcal{D}_{f,T}}$ (resp. $M_{Z_i} \otimes I_{\mathcal{D}_{f,T'}}$), $i = 1, \ldots, n$, we deduce that

$$\left[\left(M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T}}\right)\big|_{\mathbb{H}_{f,T}}\right]V^* = V^*\left[\left(M_{Z_i}^* \otimes I_{\mathcal{D}_{f,T'}}\right)\big|_{\mathbb{H}_{f,T'}}\right], \quad i = 1, \ldots, n.$$

Consequently, we obtain

$$V\left[P_{\mathbb{H}_{f,T}}(M_{Z_i} \otimes I_{\mathcal{D}_{f,T}})|_{\mathbb{H}_{f,T}}\right] = \left[P_{\mathbb{H}_{f,T'}}(M_{Z_i} \otimes I_{\mathcal{D}_{f,T'}})|_{\mathbb{H}_{f,T'}}\right]V, \quad i = 1, \ldots, n.$$

Now, using Theorem 6.2, we conclude that T and T' are unitarily equivalent. The proof is complete. $\square$

Theorem 6.4. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Then the following statements hold.*

(i) *If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{H}^2(f)$ are invariant subspaces under the operators $M_{Z_1}, \ldots, M_{Z_n}$, then the n-tuple $(P_{\mathcal{M}_1^\perp} M_{Z_1}|\mathcal{M}_1^\perp, \ldots, P_{\mathcal{M}_1^\perp} M_{Z_n}|\mathcal{M}_1^\perp)$ is equivalent to $(P_{\mathcal{M}_2^\perp} M_{Z_1}|\mathcal{M}_2^\perp, \ldots, P_{\mathcal{M}_2^\perp} M_{Z_n}|\mathcal{M}_2^\perp)$ if and only if $\mathcal{M}_1 = \mathcal{M}_2$.*

(ii) *If $\mathcal{M} \subseteq \mathbb{H}^2(f)$ is an invariant subspace under $M_{Z_1}, \ldots, M_{Z_n}$, and*

$$T := (T_1, \ldots, T_n), \qquad T_i := P_{\mathcal{M}^\perp} M_{Z_i}|\mathcal{M}^\perp, \quad i = 1, \ldots, n,$$

then $\mathcal{M} = \Theta_{f,T}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*})$, where $\Theta_{f,T}$ is the characteristic function of T.

Proof. Assume the hypotheses of item (ii). Since $f = (f_1, \ldots, f_n)$ is an n-tuple of formal power series with the model property, $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$, where $(M_{Z_1}, \ldots, M_{Z_n})$ is either in the convergence set $\mathcal{C}_f^{SOT}(\mathbb{H}^2(f))$ or $\mathcal{C}_f^{rad}(\mathbb{H}^2(f))$. Since $\mathcal{M}^\perp$ is invariant under $M_{Z_1}^*, \ldots, M_{Z_n}^*$, we deduce that

$$\begin{aligned}\Delta_{f,T} &= I_{\mathcal{M}^\perp} - \sum_{i=1}^n f_i(T_1, \ldots, T_n) f_i(T_1, \ldots, T_n)^* \\ &= P_{\mathcal{M}^\perp}\left(I_{\mathbb{H}^2(f)} - M_{f_i} M_{f_i}^*\right)\big|_{\mathcal{M}^\perp} = P_{\mathcal{M}^\perp} P_{\mathbb{C}}|_{\mathcal{M}^\perp}.\end{aligned}$$

Hence, rank $\Delta_{f,T} \leqslant 1$. On the other hand, since $[M_{f_1}, \ldots, M_{f_n}]$ is a pure row contraction, so is $[f_1(T), \ldots, f_n(T)]$. Therefore, T is pure n-tuple in $\mathbb{B}_f(\mathcal{M}^\perp)$ and rank $\Delta_{f,T} \neq 0$, which implies rank $\Delta_{f,T} = 1$. Therefore, we can identify the subspace $\mathcal{D}_{f,T}$ with $\mathbb{C}$. The Poisson kernel $K_{f,T}$: $\mathcal{M}^\perp \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ can be identified with the injection of $\mathcal{M}^\perp$ into $\mathbb{H}^2(f)$, via a unitary operator from $\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ to $\mathbb{H}^2(f)$. Indeed, note that if $\sum_\alpha c_\alpha f_\alpha \in \mathcal{M}^\perp \subset \mathbb{H}^2(f)$, then, taking into account that $\Delta_{f,T} = P_{\mathcal{M}^\perp} P_{\mathbb{C}}|_{\mathcal{M}^\perp}$ and $[f(T)]_\alpha = P_{\mathcal{M}^\perp} M_{f_\alpha}|_{\mathcal{M}^\perp}$, we have

$$K_{f,T}\left(\sum_\alpha c_\alpha f_\alpha\right) = \sum_{\beta \in \mathbb{F}_n^+} f_\beta \otimes P_{\mathcal{M}^\perp} P_{\mathbb{C}}|_{\mathcal{M}^\perp} M_{f_\beta}^*\left(\sum_\alpha c_\alpha f_\alpha\right) = \sum_{\beta \in \mathbb{F}_n^+} c_\beta f_\beta \otimes P_{\mathcal{M}^\perp}(1),$$

which implies our assertion. As a consequence, we deduce that the n-tuple $(T_1, \dots, T_n)$ is unitarily equivalent to $(K_{f,T}^* M_{Z_1} K_{f,T}, \dots, K_{f,T}^* M_{Z_n} K_{f,T})$. Due to Theorem 4.5, the n-tuple $(M_{Z_1}, \dots, M_{Z_n})$ is the minimal dilation of $(T_1, \dots, T_n)$.

Now, using this result under the hypotheses of item (i) and the uniqueness of the minimal dilation (see Theorem 4.5), we obtain that the n-tuple $(P_{\mathcal{M}_1^\perp} M_{Z_1}|\mathcal{M}_1^\perp, \dots, P_{\mathcal{M}_1^\perp} M_{Z_n}|\mathcal{M}_1^\perp)$ is equivalent to $(P_{\mathcal{M}_2^\perp} M_{Z_1}|\mathcal{M}_2^\perp, \dots, P_{\mathcal{M}_2^\perp} M_{Z_n}|\mathcal{M}_2^\perp)$ if and only if there exists a unitary operator $W : \mathbb{H}^2(f) \to \mathbb{H}^2(f)$ such that $W M_{Z_i} = M_{Z_i} W$, $i = 1, \dots, n$, and $W(\mathcal{M}_1^\perp) = \mathcal{M}_2^\perp$. Hence we deduce that $W M_{Z_i}^* = M_{Z_i}^* W$, $i = 1, \dots, n$. Since $C^*(M_{Z_1}, \dots, M_{Z_n})$ is irreducible (see Theorem 3.3), W is a scalar multiple of the identity. Therefore, we must have $\mathcal{M}_1 = \mathcal{M}_2$, which proves part (i).

To prove part (ii), note that, due to Theorem 6.2, we have

$$\mathbf{H}_{f,T} = \mathbb{H}^2(f) \ominus \Theta_{f,T}\big(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*}\big)$$

and $T = (T_1, \dots, T_n)$ is unitarily equivalent to $(P_{\mathbf{H}_{f,T}} M_{Z_1}|_{\mathbf{H}_{f,T}}, \dots, P_{\mathbf{H}_{f,T}} M_{Z_n}|_{\mathbf{H}_{f,T}})$. Using part (i), we deduce that $\mathbf{H}_{f,T} = \mathcal{M}^\perp$ and therefore $\mathcal{M} = \Theta_{f,T}(\mathbb{H}^2(f) \otimes \mathcal{D}_{f,T^*})$. This completes the proof. $\square$

The commutative case. Assume that $f = (f_1, \dots, f_n)$ has the model property. According to Theorem 5.5 and Theorem 5.6, if J_c is the WOT-closed two-sided ideal of the Hardy algebra $H^\infty(\mathbb{B}_f)$ generated by the commutators

$$M_{Z_i} M_{Z_j} - M_{Z_j} M_{Z_i}, \quad i, j = 1, \dots, n,$$

then $\mathcal{N}_{J_c} = \mathbb{H}_s^2(f)$, the symmetric Hardy space associated with $\mathbb{B}_f$. Moreover, $\mathbb{H}_s^2(f)$ can be identified with the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$ of holomorphic functions on $\mathbb{B}_f^<(\mathbb{C})$, namely, the reproducing kernel Hilbert space with reproducing kernel $\Lambda_f : \mathbb{B}_f^<(\mathbb{C}) \times \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ defined by

$$\Lambda_f(\mu, \lambda) := \frac{1}{1 - \sum_{i=1}^n f_i(\mu)\overline{f_i(\lambda)}}, \quad \lambda, \mu \in \mathbb{B}_f^<(\mathbb{C}).$$

The algebra $P_{H_s^2(f)} \mathbb{H}^\infty(\mathbb{B}_f)|_{\mathbb{H}_s^2(f)}$ coincides with the WOT-closed algebra generated by the operators $L_i := P_{\mathbb{H}_s^2(f)} M_{Z_i}|_{\mathbb{H}_s^2(f)}$, $i = 1, \dots, n$, and can be identified with the algebra of all multipliers of the Hilbert space $H^2(\mathbb{B}_f^<(\mathbb{C}))$. Under this identification the operators $L_1, \dots, L_n$ become the multiplication operators $M_{z_1}, \dots, M_{z_n}$ by the coordinate functions $z_1, \dots, z_n$, respectively. Now, let $T := (T_1, \dots, T_n) \in \mathbb{B}_f(\mathcal{H})$ be such that $T_i T_j = T_j T_i$, $i, j = 1, \dots, n$. Under the above-mentioned identifications, we define the characteristic function of T to be the multiplier $\Theta_{f,J_c,T} : H^2(\mathbb{B}_f^<(\mathbb{C})) \otimes \mathcal{D}_{f,T^*} \to H^2(\mathbb{B}_f^<(\mathbb{C})) \otimes \mathcal{D}_{f,T}$ given by the operator-valued analytic function on $\mathbb{B}_f^<(\mathbb{C})$

$$\Theta_{f,J_c,T}(z) := -f(T) + \Delta_{f,T}\left(I - \sum_{i=1}^n f_i(z) f_i(T)^*\right)^{-1} \big[f_1(z) I_{\mathcal{H}}, \dots, f_n(z) I_{\mathcal{H}}\big] \Delta_{f,T^*}$$

for $z = (z_1, \dots, z_n) \in \mathbb{B}_f^<(\mathbb{C})$. All the results of this section can be written in this commutative setting.

7. Curvature invariant on $\mathbb{B}_f(\mathcal{H})$

In this section, we introduce a curvature invariant on the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ and show that it is a complete numerical invariant for the finite rank submodules of the free $\mathbb{B}_f$-Hilbert module $\mathbb{H}^2(f) \otimes \mathcal{K}$, where $\mathcal{K}$ is finite dimensional. We also provide an index type formula for the curvature in terms of the characteristic function.

Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Let $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ be such that

$$\operatorname{rank}_f(T) := \operatorname{rank}\left(I - \sum_{i=1}^{n} f_i(T) f_i(T)^*\right)^{1/2} < \infty.$$

We define the curvature of T by setting

$$\operatorname{curv}_f(T) := \lim_{m\to\infty} \frac{\operatorname{trace}[K_{f,T}^*(Q_{\leqslant m} \otimes I_{\mathcal{D}_{f,T}})K_{f,T}]}{\operatorname{trace}[K_{f,M_Z}^*(Q_{\leqslant m})K_{f,M_Z}]},$$

where $Q_{\leqslant m}$, $m = 0, 1, \ldots$, is the orthogonal projection of $\mathbb{H}^2(f)$ on the linear span of the formal power series f_α, $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \leqslant m$. In what follows, we show that the limit exists and we provide a formula for the curvature in terms of the characteristic function. We denote by Q_m, $m = 0, 1, \ldots$, the orthogonal projection of $\mathbb{H}^2(f)$ on the linear span of the formal power series f_α, $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = m$.

Theorem 7.1. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ be such that* $\operatorname{rank}_f(T) < \infty$. *Then*

$$\operatorname{curv}_f(T) = \operatorname{rank}_f(T) - \operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N\big],$$

where $\Theta_{f,T}$ is the characteristic function of T and

$$N := \sum_{k=0}^{\infty} \frac{1}{n^k} Q_k \otimes I_{\mathcal{D}_{f,T}}.$$

Proof. Since

$$\operatorname{trace}\big[K_{f,M_Z}^*(Q_{\leqslant m})K_{f,M_Z}\big] = \operatorname{trace}[Q_{\leqslant m}] = 1 + n + \cdots + n^m,$$

we can use Theorem 6.1 to deduce that

$$\begin{aligned}
\operatorname{curv}_f(T) &= \lim_{m\to\infty} \frac{\sum_{k=0}^{m} \operatorname{trace}[K_{f,T}^*(Q_k \otimes I_{\mathcal{D}_{f,T}})K_{f,T}]}{1 + n + \cdots + n^m} \\
&= \lim_{m\to\infty} \frac{\operatorname{trace}[K_{f,T}^*(Q_m \otimes I_{\mathcal{D}_{f,T}})K_{f,T}]}{n^m}
\end{aligned}$$

$$= \lim_{m\to\infty} \frac{\operatorname{trace}[(Q_m \otimes I_{\mathcal{D}_{f,T}})K_{f,T}K_{f,T}^*](Q_m \otimes I_{\mathcal{D}_{f,T}})}{n^m}$$
$$= \operatorname{rank}_f(T) - \lim_{m\to\infty} \frac{\operatorname{trace}[(Q_m \otimes I_{\mathcal{D}_{f,T}})\Theta_{f,T}\Theta_{f,T}^*(Q_m \otimes I_{\mathcal{D}_{f,T}})]}{n^m},$$

provided the latter limit exists, which we should prove now. Since $\Theta_{f,T}$ is a multi-analytic operator with respect to $M_{Z_1}, \ldots, M_{Z_n}$ and

$$\sum_{k=0}^{\infty}\sum_{|\alpha|=k} M_{f_\alpha} Q_0 M_{f_\alpha}^* = I_{\mathbb{H}^2(f)},$$

we deduce that

$$(Q_m \otimes I_{\mathcal{D}_{f,T}})\Theta_{f,T}\Theta_{f,T}^*(Q_m \otimes I_{\mathcal{D}_{f,T}})$$
$$= \sum_{k=0}^{m}\sum_{|\alpha|=k} (Q_m M_{f_\alpha} \otimes I)\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^*\big(M_{f_\alpha}^* Q_m \otimes I\big).$$

Hence, and taking into account that $\sum_{|\alpha|\leqslant m} M_{f_\alpha}^* Q_m M_{f_\alpha} = \sum_{k=0}^m n^k Q_{m-k}$, we obtain

$$\frac{\operatorname{trace}[(Q_m \otimes I_{\mathcal{D}_{f,T}})\Theta_{f,T}\Theta_{f,T}^*(Q_m \otimes I_{\mathcal{D}_{f,T}})]}{n^m}$$
$$= \frac{\operatorname{trace}[(\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^*)(\sum_{|\alpha|\leqslant m} M_{f_\alpha}^* Q_m M_{f_\alpha} \otimes I_{\mathcal{D}_{f,T}})]}{n^m}$$
$$= \operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N_m\big],$$

where $N_m := \sum_{k=0}^m \frac{1}{n^k} Q_k \otimes I_{\mathcal{D}_{f,T}}$. Consequently, we have

$$0 \leqslant \operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N_m\big] \leqslant \frac{\operatorname{trace}[(Q_m \otimes I_{\mathcal{D}_{f,T}})\Theta_{f,T}\Theta_{f,T}^*(Q_m \otimes I_{\mathcal{D}_{f,T}})]}{n^m}$$
$$\leqslant \|\Theta_{f,T}\|^2 \dim \mathcal{D}_{f,T} = \dim \mathcal{D}_{f,T} < \infty.$$

Since $\{N_m\}$ is an increasing sequence of positive operators convergent to N, we deduce that

$$\operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N\big] = \lim_{m\to\infty} \operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N_m\big].$$

Combining this result with the relations above, we complete the proof. □

We remark that the proof of Theorem 7.1 is simpler than that of the corresponding result from [27], in the particular case when $f = (Z_1, \ldots, Z_n)$.

Corollary 7.2. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series with the model property. If $T = (T_1, \dots, T_n) \in \mathbb{B}_f(\mathcal{H})$ and* $\operatorname{rank}_f(T) < \infty$*, then*

$$\operatorname{curv}_f(T) = \lim_{m\to\infty} \frac{\operatorname{trace}[I - \Phi_{f,T}^{m+1}(I)]}{1 + n + \cdots + n^m} = \operatorname{curv}\bigl(f(T)\bigr),$$

where the $\Phi_{f,T}(Y) := \sum_{i=1}^n f_i(T) Y f_i(T)^*$ and $\operatorname{curv}(f(T))$ *is the curvature of the row contraction $f(T)$.*

Proof. Due to the properties of the noncommutative Poisson Kernel $K_{f,T}$, we have

$$\begin{aligned} K_{f,T}^*\biggl(\sum_{|\alpha|=k} M_{f_\alpha} M_{f_\alpha}^* \otimes I\biggr) K_{f,T} &= \sum_{|\alpha|=k} \bigl[f(T)\bigr]_\alpha K_{f,T}^* K_{f,T} \bigl[f(T)\bigr]_\alpha^* \\ &= \sum_{|\alpha|=k} \bigl[f(T)\bigr]_\alpha \bigl[f(T)\bigr]_\alpha^* - \Phi_{f,T}^\infty(I), \end{aligned}$$

where $\Phi_{f,T}^\infty(I) := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$. Consequently, we obtain

$$K_{f,T}^*(Q_m \otimes I) K_{f,T} = \Phi_{f,T}^m\Biggl(I - \sum_{i=1}^n f_i(T) f_i(T)^*\Biggr).$$

Now, using the equalities from the proof of Theorem 7.1, the result follows. □

Theorem 7.3. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \dots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. If an n-tuple $T = (T_1, \dots, T_n) \in \mathbb{B}_f(H)$ is such that* $\operatorname{rank}_f(T) < \infty$*, then T is unitarily equivalent to the n-tuple $(M_{Z_1} \otimes I_\mathcal{K}, \dots, M_{Z_n} \otimes I_\mathcal{K})$ with* $\dim \mathcal{K} < \infty$ *if and only if T is pure and*

$$\operatorname{curv}_f(T) = \operatorname{rank}_f(T).$$

Proof. Assume that $T := (T_1, \dots, T_n) \in \mathbb{B}_f(H)$ is unitarily equivalent to $(M_{Z_1} \otimes I_\mathcal{K}, \dots, M_{Z_n} \otimes I_\mathcal{K})$, where $\dim \mathcal{K} < \infty$. Note that due to the fact that $f = (f_1, \dots, f_n)$ has the model property, we have

$$\begin{aligned} \operatorname{rank}_f(T) &= \operatorname{rank}\Biggl(I - \sum_{i=1}^n f_i(M_{Z_1} \otimes I_\mathcal{K}, \dots, M_{Z_n} \otimes I_\mathcal{K}) f_i(M_{Z_1} \otimes I_\mathcal{K}, \dots, M_{Z_n} \otimes I_\mathcal{K})^*\Biggr)^{1/2} \\ &= \operatorname{rank}\Biggl(I - \sum_{i=1}^n (M_{f_i} \otimes I_\mathcal{K})(M_{f_i} \otimes I_\mathcal{K})^*\Biggr) = \dim \mathcal{K}. \end{aligned}$$

On the other hand, according to the definition of the curvature, we have

$$\operatorname{curv}_f(T) = \lim_{m\to\infty} \frac{\operatorname{trace}[K_{f,M_Z\otimes I_\mathcal{K}}^*(Q_{\leqslant m} \otimes I_\mathcal{K}) K_{f,M_Z\otimes I_\mathcal{K}}]}{\operatorname{trace}[K_{f,M_Z}^*(Q_{\leqslant m}) K_{f,M_Z}]} = \dim \mathcal{K}.$$

Conversely, assume that T is pure and $\operatorname{curv}_f(T) = \operatorname{rank}_f(T)$. According to Theorem 6.1,

$$K_{f,T}K_{f,T}^* = I_{\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}} - \Theta_{f,T}\Theta_{f,T}^*$$

where $\Theta_{f,T}$ is the characteristic function associated with T. Since the noncommutative Poisson kernel $K_{f,T}$ is an isometry, $\Theta_{f,T}$ is an inner multi-analytic operator. On the other hand, Theorem 7.1 implies

$$\operatorname{curv}_f(T) = \operatorname{rank}_f(\mathcal{H}) - \operatorname{trace}\big[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N\big],$$

where N is the number operator. Therefore, $\operatorname{trace}[\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* N] = 0$. Since trace is faithful, we obtain $\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* Q_j = 0$ for any $j = 0, 1, \ldots.$ This implies $\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}})\Theta_{f,T}^* = 0$. Taking into account that $\Theta_{f,T}$ is an isometry, we infer that $\Theta_{f,T}(Q_0 \otimes I_{\mathcal{D}_{f,T^*}}) = 0$. Since $\Theta_{f,T}$ is multi-analytic with respect to $M_{Z_1}, \ldots M_{Z_n}$, and $\mathbb{C}[Z_1, \ldots, Z_n]$ is dense in $\mathbb{H}^2(f)$, we deduce $\Theta_{f,T} = 0$. Using again the fact that $K_{f,T}K_{f,T}^* + \Theta_{f,T}\Theta_{f,T}^* = I_{\mathbb{H}^2(f)\otimes\mathcal{D}_{f,T}}$, we deduce that $K_{f,T} : \mathcal{H} \to \mathbb{H}^2(f) \otimes \mathcal{D}_{f,T}$ is a unitary operator. According to the properties of the Poisson kernel, we have

$$K_{f,T}^*(M_{Z_i} \otimes I_{\mathcal{D}_{f,T}})K_{f,T} = T_i, \quad i = 1, \ldots, n.$$

This shows that the n-tuple $(T_1, \ldots, T_n)$ is unitarily equivalent to $(M_{Z_1} \otimes I_{\mathcal{D}_{f,T}}), \ldots, M_{Z_n} \otimes I_{\mathcal{D}_{f,T}})$ and $\dim \mathcal{D}_{f,T} < \infty$. This completes the proof. □

In what follows we show that the curvature on $\mathbb{B}_f(\mathcal{H})$ is a complete numerical invariant for the finite rank submodules of the $\mathbb{B}_f$-Hilbert module $\mathbb{H}^2(f) \otimes \mathcal{K}$, where $\mathcal{K}$ is finite dimensional.

Theorem 7.4. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $M_Z := (M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with the noncommutative domain $\mathbb{B}_f$. Given $\mathcal{M}, \mathcal{N} \subseteq \mathbb{H}^2(f)$ two invariant subspaces under $M_{Z_1}, \ldots, M_{Z_n}$, the following statements hold.*

(i) *If* $\operatorname{rank}_f(M_Z|_{\mathcal{M}}) < \infty$, *then* $\operatorname{curv}_f(M_Z|_{\mathcal{M}}) = \operatorname{rank}_f(M_Z|_{\mathcal{M}})$.
(ii) *If* $\operatorname{rank}_f(M_Z|_{\mathcal{M}}) < \infty$ *and* $\operatorname{rank}_f(M_Z|_{\mathcal{N}}) < \infty$, *then $M_Z|_{\mathcal{M}}$ is unitarily equivalent to $M_Z|_{\mathcal{N}}$ if and only if*

$$\operatorname{curv}_f(M_Z|_{\mathcal{M}}) = \operatorname{curv}_f(M_Z|_{\mathcal{N}}).$$

Proof. Let $g = (g_1, \ldots, g_n)$ be the inverse of $f = (f_1, \ldots, f_n)$ with respect to the composition of formal power series. Since f has the model property, we have

$$f_i(M_{Z_1}, \ldots, M_{Z_n}) = M_{f_i} \quad \text{and} \quad g_i(M_{f_1}, \ldots, M_{f_n}) = M_{Z_i}$$

for any $i = 1, \ldots, n$. Hence, we deduce that a subspace $\mathcal{M}$ is invariant under $M_{Z_1}, \ldots, M_{Z_n}$ if and only if it is invariant under $M_{f_1}, \ldots, M_{f_n}$. We recall that $M_{f_i} = U^{-1}S_iU$, $i = 1, \ldots, n$, where $U : \mathbb{H}^2(f) \to F^2(H_n)$ is the unitary operator defined by $U(f_\alpha) = e_\alpha$, $\alpha \in \mathbb{F}_n^+$, and $S_1, \ldots, S_n$ are the left creation operators. Now, one can easily see that $\mathcal{M}$ is an invariant subspace under

$M_{Z_1}, \ldots, M_{Z_n}$ if and only if $U\mathcal{M}$ is invariant under $S_1, \ldots, S_n$. Hence, using Corollary 7.2 and the fact that $UP_{\mathcal{M}}U^{-1} = P_{U\mathcal{M}}$, we have

$$\begin{aligned}\operatorname{rank}_f(M_Z|_{\mathcal{M}}) &= \operatorname{rank}\big(f_1(M_Z|_{\mathcal{M}}), \ldots, f_n(M_Z|_{\mathcal{M}})\big)\\ &= \operatorname{rank}\big(U^{-1}S_1U|_{\mathcal{M}}, \ldots, U^{-1}S_nU|_{\mathcal{M}}\big)\\ &= \operatorname{rank}(S_1|_{U\mathcal{M}}, \ldots, S_n|_{U\mathcal{M}})\end{aligned}$$

and

$$\begin{aligned}\operatorname{curv}_f(M_Z|_{\mathcal{M}}) &= \operatorname{curv}\big(f_1(M_Z|_{\mathcal{M}}), \ldots, f_n(M_Z|_{\mathcal{M}})\big)\\ &= \operatorname{curv}\big(U^{-1}S_1U|_{\mathcal{M}}, \ldots, U^{-1}S_nU|_{\mathcal{M}}\big)\\ &= \operatorname{curv}(S_1|_{U\mathcal{M}}, \ldots, S_n|_{U\mathcal{M}}).\end{aligned}$$

According to Theorem 3.2 from [27], we have

$$\operatorname{rank}(S_1|_{U\mathcal{M}}, \ldots, S_n|_{U\mathcal{M}}) = \operatorname{curv}(S_1|_{U\mathcal{M}}, \ldots, S_n|_{U\mathcal{M}}).$$

Combining the results above, we deduce item (i). To prove part (ii), note that the direct implication is due to the fact that, for any $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$ and $T' = (T_1', \ldots, T_n') \in \mathbb{B}_f(\mathcal{H}')$, if T is unitarily equivalent to T', then $\operatorname{curv}_f(T) = \operatorname{curv}_f(T')$. Conversely, assume that $\operatorname{curv}_f(M_Z|_{\mathcal{M}}) = \operatorname{curv}_f(M_Z|_{\mathcal{N}})$. As shown above, the latter equality is equivalent to

$$\operatorname{curv}(S_1|_{U\mathcal{M}}, \ldots, S_n|_{U\mathcal{M}}) = \operatorname{curv}(S_1|_{U\mathcal{N}}, \ldots, S_n|_{U\mathcal{N}}).$$

Applying again Theorem 3.2 from [27], we find a unitary operator $W : U\mathcal{M} \to U\mathcal{N}$ such that

$$W(S_i|_{U\mathcal{M}}) = (S_i|_{U\mathcal{N}})W, \quad i = 1, \ldots, n.$$

Consequently, we have

$$\big(U^{-1}WU|_{\mathcal{M}}\big)\big(U^{-1}S_iU|_{\mathcal{M}}\big) = \big(U^{-1}S_iU|_{\mathcal{N}}\big)\big(U^{-1}WU|_{\mathcal{M}}\big), \quad i = 1, \ldots, n,$$

which implies

$$\big(U^{-1}WU|_{\mathcal{M}}\big)(M_{f_i}|_{\mathcal{M}}) = (M_{f_i}|_{\mathcal{N}})\big(U^{-1}WU|_{\mathcal{M}}\big), \quad i = 1, \ldots, n.$$

Using now relation $g_i(M_{f_1}, \ldots, M_{f_n}) = M_{Z_i}$, $i = 1, \ldots, n$, we obtain

$$\big(U^{-1}WU|_{\mathcal{M}}\big)(M_{Z_i}|_{\mathcal{M}}) = (M_{Z_i}|_{\mathcal{N}})\big(U^{-1}WU|_{\mathcal{M}}\big), \quad i = 1, \ldots, n.$$

Since $U^{-1}WU|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}$ is a unitary operator, we conclude that the n-tuples $(M_{Z_1}|_{\mathcal{M}}, \ldots, M_{Z_n}|_{\mathcal{M}})$ and $(M_{Z_1}|_{\mathcal{N}}, \ldots, M_{Z_n}|_{\mathcal{N}})$ are unitarily equivalent. The proof is complete. □

We remark that all the results of this section have commutative versions when $T = (T_1, \ldots, T_n) \in \mathbb{B}_f(\mathcal{H})$, $T_iT_j = T_jT_i$, and the universal model $(M_{Z_1}, \ldots, M_{Z_n})$ is replaced by the n-tuple $(L_1, \ldots, L_n)$, where $L_i := P_{\mathbb{H}_s^2(f)} M_{Z_i}|_{\mathbb{H}_s^2(f)}$, $i = 1, \ldots, n$, and $\mathbb{H}_s^2(f)$ is the symmetric Hardy space associated with the noncommutative domain $\mathbb{B}_f$. In this case, we obtain analogues of Arveson's results [4] concerning the curvature for commuting row contractions, for the set of commuting n-tuples in the domain $\mathbb{B}_f(\mathcal{H})$.

8. Commutant lifting and interpolation

In this section, to provide a commutant lifting theorem for the pure n-tuples of operators in the noncommutative domain $\mathbb{B}_f(\mathcal{H})$ and solve the Nevanlinna Pick interpolation problem for the noncommutative Hardy algebra $H^\infty(\mathbb{B}_f)$.

First, we present a Sarason [35] type commutant lifting result in our setting.

Theorem 8.1. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property and let $(M_{Z_1}, \ldots, M_{Z_n})$ be the universal model associated with $\mathbb{B}_f$. Let $\mathcal{E}_j \subset \mathbb{H}^2(f) \otimes \mathcal{K}_j$, $j = 1, 2$, be a co-invariant subspace under each operator $M_{Z_i} \otimes I_{\mathcal{K}_j}$, $i = 1, \ldots, n$.*

If $X : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded operator such that

$$X\big[P_{\mathcal{E}_1}(M_{Z_i} \otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}\big] = \big[P_{\mathcal{E}_2}(M_{Z_i} \otimes I_{\mathcal{K}_2})|_{\mathcal{E}_2}\big]X, \quad i = 1, \ldots, n,$$

then there exists a bounded operator $Y : \mathbb{H}^2(f) \otimes \mathcal{K}_1 \to \mathbb{H}^2(f) \otimes \mathcal{K}_2$ with the property

$$Y(M_{Z_i} \otimes I_{\mathcal{K}_1}) = (M_{Z_i} \otimes I_{\mathcal{K}_2})Y, \quad i = 1, \ldots, n,$$

and such that $Y^*\mathcal{E}_2 \subseteq \mathcal{E}_1$, $Y^*|\mathcal{E}_2 = X^*$, and $\|Y\| = \|X\|$.

Proof. Setting $A_i := P_{\mathcal{E}_1}(M_{Z_i} \otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}$ and $B_i := P_{\mathcal{E}_2}(M_{Z_i} \otimes I_{\mathcal{K}_2})|_{\mathcal{E}_2}$, we have $XA_i = B_iX$, $i = 1, \ldots, n$. Since f has the model property and $\mathcal{E}_1$ is a co-invariant subspace under each operator $M_{Z_i} \otimes I_{\mathcal{K}_j}$, $i = 1, \ldots, n$, we deduce that $\mathcal{E}_1$ is a co-invariant subspace under $f_i(M_{Z_1}, \ldots, M_{Z_n}) \otimes I_{\mathcal{K}_1} = M_{f_i} \otimes I_{\mathcal{K}_1}$ and

$$f_i(A_1, \ldots, A_n) = P_{\mathcal{E}_1}\big[f_i(M_{Z_1}, \ldots, M_{Z_n}) \otimes I_{\mathcal{K}_1}\big]\big|_{\mathcal{E}_1} = P_{\mathcal{E}_1}(M_{f_i} \otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}, \quad i = 1, \ldots, n.$$

Similarly, $\mathcal{E}_2$ is a co-invariant subspace under $f_i(M_{Z_1}, \ldots, M_{Z_n}) \otimes I_{\mathcal{K}_2} = M_{f_i} \otimes I_{\mathcal{K}_2}$ and

$$f_i(B_1, \ldots, B_n) = P_{\mathcal{E}_2}\big[f_i(M_{Z_1}, \ldots, M_{Z_n}) \otimes I_{\mathcal{K}_2}\big]\big|_{\mathcal{E}_2} = P_{\mathcal{E}_2}(M_{f_i} \otimes I_{\mathcal{K}_2})|_{\mathcal{E}_2}, \quad i = 1, \ldots, n.$$

Using the canonical unitary operator $U : \mathbb{H}^2(f) \to F^2(H_n)$, defined by $U(f_\alpha) = e_\alpha$, $\alpha \in \mathbb{F}_n^+$, we have $M_{f_i} = U^*S_iU$ and the subspace $U(\mathcal{E}_1)$ is co-invariant under $S_1 \otimes I_{\mathcal{K}_1}, \ldots, S_n \otimes I_{\mathcal{K}_1}$, where $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$. Similarly, we have that $U(\mathcal{E}_2)$ is co-invariant under each operator $S_1 \otimes I_{\mathcal{K}_2}, \ldots, S_n \otimes I_{\mathcal{K}_2}$. Now, since $XA_i = B_iX$, we deduce that $Xf_i(A_1, \ldots, A_n) = f_i(B_1, \ldots, B_n)X$, $i = 1, \ldots, n$, which together with the considerations above imply

$$\widetilde{X}\big[P_{U(\mathcal{E}_1)}(S_i \otimes I_{\mathcal{K}_1})|_{U(\mathcal{E}_1)}\big] = \big[P_{U(\mathcal{E}_2)}(S_i \otimes I_{\mathcal{K}_2})|_{U(\mathcal{E}_2)}\big]\widetilde{X}, \quad i = 1, \ldots, n,$$

where $\widetilde{X} : U(\mathcal{E}_1) \to U(\mathcal{E}_2)$ is defined by $\widetilde{X} := UXU^*|_{U(\mathcal{E}_1)}$. Note that $[S_1 \otimes I_{\mathcal{K}_1}, \ldots, S_n \otimes I_{\mathcal{K}_1}]$ is an isometric dilation of the row contraction $[P_{U(\mathcal{E}_1)}(S_1 \otimes I_{\mathcal{K}_1})|_{U(\mathcal{E}_1)}, \ldots,$

$P_{U(\mathcal{E}_1)}(S_n \otimes I_{\mathcal{K}_1})|_{U(\mathcal{E}_1)}]$. Applying the noncommutative commutant lifting theorem from [18], we find a bounded operator $\widetilde{Y} : F^2(H_n) \otimes \mathcal{K}_1 \to F^2(H_n) \otimes \mathcal{K}_2$ with the properties $\widetilde{Y}(S_i \otimes I_{\mathcal{K}_1}) = (S_i \otimes I_{\mathcal{K}_2})\widetilde{Y}$ for $i = 1, \ldots, n$, $\widetilde{Y}^*(U(\mathcal{E}_2)) \subset U(\mathcal{E}_1)$, $\widetilde{Y}^*|_{U(\mathcal{E}_2)} = \widetilde{X}^*$, and $\|\widetilde{Y}\| = \|\widetilde{X}\|$. Now, setting $Y := U\widetilde{Y}U^*$, we deduce that $Y : \mathbb{H}^2(f) \otimes \mathcal{K}_1 \to \mathbb{H}^2(f) \otimes \mathcal{K}_2$ has the property

$$Y(M_{f_i} \otimes I_{\mathcal{K}_1}) = (M_{f_i} \otimes I_{\mathcal{K}_2})Y, \quad i = 1, \ldots, n,$$

and also satisfies the relations $Y^*\mathcal{E}_2 \subseteq \mathcal{E}_1$, $Y^*|\mathcal{E}_2 = X^*$, and $\|Y\| = \|X\|$. Once again, taking into account that f has the model property, we have $M_{Z_i} = g_i(M_{f_1}, \ldots, M_{f_n})$ for $i = 1, \ldots,$ where $g = (g_1, \ldots, g_n)$ is the inverse of $f = (f_1, \ldots, f_n)$ with respect to composition, and $g_i(M_{f_1}, \ldots, M_{f_n})$ is defined using the radial SOT-convergence. Consequently, the above-mentioned intertwining relation implies

$$Y(M_{Z_i} \otimes I_{\mathcal{K}_1}) = Y\big[g_i(M_{f_1}, \ldots, M_{f_n}) \otimes I_{\mathcal{K}_1}\big] = \big[g_i(M_{f_1}, \ldots, M_{f_n}) \otimes I_{\mathcal{K}_2}\big]Y = (M_{Z_i} \otimes I_{\mathcal{K}_2})Y$$

for $i = 1, \ldots, n$. The proof is complete. $\square$

Recall that, due to Theorem 5.5 and Theorem 5.2, we have $\mathbb{H}^2_s(f) = \overline{\operatorname{span}}\{\Gamma_\lambda\colon \lambda \in \mathbb{B}^<_f(\mathbb{C})\}$ and $M^*_{Z_i}\Gamma_\lambda = \bar{\lambda}_i\Gamma_\lambda$ far all $i = 1, \ldots, n$. This shows that $\mathbb{H}^2_s(f)$ is a co-invariant subspace under each operator $M_{Z_1}, \ldots, M_{Z_n}$. We remark that this observation can be used together with Theorem 8.1 to obtain a commutative version of the latter theorem, when $\mathcal{E}_j \subset \mathbb{H}^2_s(f) \otimes \mathcal{K}_j$, $j = 1, 2$, are co-invariant subspaces under each operator $L_i \otimes I_{\mathcal{K}_j}$, $i = 1, \ldots, n$.

Now we can obtain the following Nevanlinna and Pick [15] interpolation result in our setting.

Theorem 8.2. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of formal power series with the model property. If $\lambda_1, \ldots, \lambda_m$ are m distinct points in $\mathbb{B}^<_f(\mathbb{C})$ and $A_1, \ldots, A_m \in B(\mathcal{K})$, then there exists $\Phi \in H^\infty(\mathbb{B}_f)\bar{\otimes}B(\mathcal{K})$ such that*

$$\|\Phi\| \leqslant 1 \quad \text{and} \quad \Phi(\lambda_j) = A_j, \quad j = 1, \ldots, m,$$

if and only if the operator matrix

$$\left[\frac{I_{\mathcal{K}} - A_i A_j^*}{1 - \sum_{k=1}^n f_k(\lambda_i)\overline{f_k(\lambda_j)}}\right]_{m\times m}$$

is positive semidefinite.

Proof. Let $\lambda_j := (\lambda_{j1}, \ldots, \lambda_{jn})$, $j = 1, \ldots, m$, be m distinct points in $\mathbb{B}^<_f(\mathbb{C})$. Consider the formal power series

$$\Gamma_{\lambda_j} := \left(1 - \sum_{i=1}^n \big|f_i(\lambda_j)\big|^2\right)^{1/2} \sum_{\alpha\in\mathbb{F}_n} \big[\overline{f(\lambda_j)}\big]_\alpha f_\alpha, \quad j = 1, \ldots, m,$$

and set $\Omega_{\lambda_j} := (1 - \sum_{i=1}^n |f_i(\lambda_j))^{-1/2}\Gamma_{\lambda_j}$. According to Theorem 5.2, they satisfy the equations

$$M^*_{Z_i}\Gamma_{\lambda_j} = \bar{\lambda}_{ji}\Gamma_{\lambda_j}, \quad i = 1, \ldots, n. \tag{8.1}$$

Note that the subspace $\mathcal{M} := \operatorname{span}\{\Gamma_{\lambda_j}\colon\ j = 1, \ldots, m\} \subset \mathbb{H}^2(f)$ is invariant under $M_{Z_i}^*$ for any $i = 1, \ldots, n$. Define the operators $X_i \in B(\mathcal{M} \otimes \mathcal{K})$ by setting $X_i := P_{\mathcal{M}} M_{Z_i}|_{\mathcal{M}} \otimes I_{\mathcal{K}}$, $i = 1, \ldots, n$. Since f is one-to-one on $\mathbb{B}_f^<(\mathbb{C})$, we deduce that $f(\lambda_1), \ldots, f(\lambda_m)$ are distinct points in $\mathbb{B}_n$. Consequently, the formal power series $\Gamma_{\lambda_1}, \ldots, \Gamma_{\lambda_m}$ are linearly independent and we can define an operator $T \in B(\mathcal{M} \otimes \mathcal{K})$ by setting

$$T^*(\Gamma_{\lambda_j} \otimes h) = \Gamma_{\lambda_j} \otimes A_j^* h \tag{8.2}$$

for any $h \in \mathcal{K}$ and $j = 1, \ldots, k$. A simple calculation using relations (8.1) and (8.2) shows that $TX_i = X_i T$ for $i = 1, \ldots, n$. Since $\mathcal{M}$ is a co-invariant subspace under each operator M_{Z_i}, $i = 1, \ldots, n$, we can apply Theorem 8.1 and find a bounded operator $Y : \mathbb{H}^2(f) \otimes \mathcal{K} \to \mathbb{H}^2(f) \otimes \mathcal{K}$ with the property

$$Y(M_{Z_i} \otimes I_{\mathcal{K}}) = (M_{Z_i} \otimes I_{\mathcal{K}})Y, \quad i = 1, \ldots, n, \tag{8.3}$$

and such that

$$Y^*(\mathcal{M} \otimes \mathcal{K}) \subset \mathcal{M} \otimes \mathcal{K}, \qquad Y^*\big|\mathcal{M} \otimes \mathcal{K} = T^*, \tag{8.4}$$

and $\|Y\| = \|T\|$. Due to relation (8.3) and the fact that $M_{f_i} = f_i(M_{Z_1}, \ldots, M_{Z_n})$, we deduce that $Y(M_{f_i} \otimes I_{\mathcal{K}}) = (M_{f_i} \otimes I_{\mathcal{K}})Y$, $i = 1, \ldots, n$, which implies

$$(U \otimes I_{\mathcal{K}})Y\big(U^* \otimes I_{\mathcal{K}}\big)(S_i \otimes I_{\mathcal{K}}) = (S_i \otimes I_{\mathcal{K}})(U \otimes I_{\mathcal{K}})Y\big(U^* \otimes I_{\mathcal{K}}\big), \quad i = 1, \ldots, n,$$

where $U : \mathbb{H}^2(f) \to F^2(H_n)$ is the canonical unitary operator defined by $U(f_\alpha) := e_\alpha$. Using the characterization of the commutant of $\{S_i \otimes I_{\mathcal{K}}\}_{i=1}^n$ (see [23]), we deduce that $(U \otimes I_{\mathcal{K}})Y \times (U^* \otimes I_{\mathcal{K}}) \in \mathcal{R}_n^\infty \bar{\otimes} B(\mathcal{K})$ and has a unique Fourier representation $\sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes C_{(\alpha)}$, $C_{(\alpha)} \in B(\mathcal{K})$, that is,

$$(U \otimes I_{\mathcal{K}})Y\big(U^* \otimes I_{\mathcal{K}}\big) = \text{SOT-}\lim_{r \to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes C_{(\alpha)}.$$

Using the flipping unitary operator $W : F^2(H_n) \to F^2(H_n)$, defined by $W(e_\alpha) := e_{\widetilde{\alpha}}$, where $\widetilde{\alpha}$ is the reverse of $\alpha \in \mathbb{F}_n^+$, we define $\Phi(M_Z) \in H^\infty(\mathbb{B}_f) \bar{\otimes} B(\mathcal{K})$ by setting

$$\Phi(M_Z) := \big(U^* W^* U \otimes I_{\mathcal{K}}\big)Y\big(U^* W U \otimes I_{\mathcal{K}}\big). \tag{8.5}$$

Note that

$$\Phi(M_Z) = \text{SOT-}\lim_{r \to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|} \big[f(M_Z)\big]_\alpha \otimes C_{(\alpha)}.$$

Hence, and using the equations $M_{Z_i}^* \Gamma_{\lambda_j} = \bar{\lambda}_{ji} \Gamma_{\lambda_j}$, $i = 1, \ldots, n$, we deduce that

$$
\begin{aligned}
\langle \Phi(M_Z)^*(\Omega_\lambda \otimes h), y \otimes h' \rangle &= \left\langle \Omega_\lambda \otimes h, \lim_{r\to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \big[f(M_Z) \big]_\alpha y \otimes C_{(\alpha)} h' \right\rangle \\
&= \lim_{r\to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \langle \Omega_\lambda \otimes h, \big[f(M_Z) \big]_\alpha y \otimes C_{(\alpha)} h' \rangle \\
&= \lim_{r\to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \overline{\big[f(\lambda) \big]}_\alpha \langle \Omega_\lambda, y \rangle \langle h, C_{(\alpha)} h' \rangle \\
&= \langle \Omega_\lambda, y \rangle \left\langle h, \lim_{r\to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \overline{\big[f(\lambda) \big]}_\alpha C_{(\alpha)} h' \right\rangle \\
&= \langle \Omega_\lambda, y \rangle \langle h, \Phi(\lambda) h' \rangle = \langle \Omega_\lambda \otimes \Phi(\lambda)^* h, y \otimes h' \rangle
\end{aligned}
$$

for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$, $y \in \mathbb{H}^2(f)$, and $h, h' \in \mathcal{K}$. Therefore,

$$\Phi(M_Z)^*(\Omega_\lambda \otimes h) = \Omega_\lambda \otimes \Phi(\lambda)^* h. \tag{8.6}$$

Hence, and using relation (8.5), we can show that

$$Y^*(\Gamma_\lambda \otimes h) = \Gamma_\lambda \otimes \Phi(\lambda)^* h, \quad \lambda \in \mathbb{B}_f^<(\mathbb{C}),\ h, h' \in \mathcal{K}. \tag{8.7}$$

Now, we prove that $\Phi(\lambda_j) = A_j$, $j = 1, \dots, k$, if and only if

$$P_{\mathcal{M}\otimes\mathcal{K}} Y|_{\mathcal{M}\otimes\mathcal{K}} = T.$$

Indeed, due to relation (8.7), we have

$$
\begin{aligned}
\langle Y^*(\Gamma_{\lambda_j} \otimes x), \Gamma_{\lambda_j} \otimes y \rangle &= \langle \Phi(M_Z)^*(\Gamma_{\lambda_j} \otimes x), \Gamma_{\lambda_j} \otimes y \rangle \\
&= \langle \Gamma_{\lambda_j} \otimes \Phi(\lambda_j)^* x, \Gamma_{\lambda_j} \otimes y \rangle \\
&= \langle \Gamma_{\lambda_j}, \Gamma_{\lambda_j} \rangle \langle \Phi(\lambda_j)^* x, y \rangle.
\end{aligned}
$$

On the other hand, relation (8.2) implies

$$\langle T^*(\Gamma_{\lambda_j} \otimes x), \Gamma_{\lambda_j} \otimes y \rangle = \langle \Gamma_{\lambda_j}, \Gamma_{\lambda_j} \rangle \langle A_j^* x, y \rangle.$$

Due to Theorem 5.2, we have

$$\langle \Omega_{\lambda_j}, \Omega_{\lambda_j} \rangle = \Lambda_f(\lambda_j, \lambda_i) = \frac{1}{1 - \sum_{k=1}^n f_k(\lambda_i) \overline{f_k(\lambda_j)}} \neq 0$$

for any $j = 1, \dots, k$. Consequently, the above relations imply our assertion.

Now, since $\|Y\| = \|T\|$, it is clear that $\|Y\| \leqslant 1$ if and only if $TT^* \leqslant I_{\mathcal{M}}$. Note that, for any $h_1, \dots, h_k \in \mathcal{K}$, we have

$$\begin{aligned}
&\left\langle \sum_{j=1}^{k} \Omega_{\lambda_j} \otimes h_j, \sum_{j=1}^{k} \Omega_{\lambda_j} \otimes h_j \right\rangle - \left\langle T^*\left(\sum_{j=1}^{k} \Omega_{\lambda_j} \otimes h_j \right), T^*\left(\sum_{j=1}^{k} \Omega_{\lambda_j} \otimes h_j \right) \right\rangle \\
&\quad = \sum_{i,j=1}^{k} \langle \Omega_{\lambda_i}, \Omega_{\lambda_j} \rangle \big\langle \big(I_{\mathcal{K}} - A_j A_i^*\big) h_i, h_j \big\rangle \\
&\quad = \sum_{i,j=1}^{k} \Lambda_f(\lambda_j, \lambda_i) \big\langle \big(I_{\mathcal{K}} - A_j A_i^*\big) h_i, h_j \big\rangle.
\end{aligned}$$

Consequently, we have $\|Y\| \leqslant 1$ if and only if the matrix $[\frac{I_{\mathcal{K}} - A_i A_j^*}{1 - \sum_{k=1}^{n} f_k(\lambda_i)\overline{f_k(\lambda_j)}}]_{m \times m}$ is positive semidefinite. This completes the proof. □

Corollary 8.3. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series with the model property and let $\lambda_1, \dots, \lambda_m$ be m distinct points in $\mathbb{B}_f^<(\mathbb{C})$. Given $A_1, \dots, A_m \in B(\mathcal{K})$, the following statements are equivalent*:

(i) *there exists $\Psi \in H^\infty(\mathbb{B}_f) \bar{\otimes} B(\mathcal{K})$ such that*

$$\|\Psi\| \leqslant 1 \quad \text{and} \quad \Psi(\lambda_j) = A_j, \quad j = 1, \dots, m;$$

(ii) *there exists $\Phi \in H^\infty(\mathbb{B}_f^<(\mathbb{C})) \bar{\otimes} B(\mathcal{K})$ such that*

$$\|\Phi\| \leqslant 1 \quad \text{and} \quad \Phi(\lambda_j) = A_j, \quad j = 1, \dots, m,$$

where $H^\infty(\mathbb{B}_f^<(\mathbb{C}))$ is the algebra of multipliers of $H^2(\mathbb{B}_f^<(\mathbb{C}))$;

(iii) *the operator matrix*

$$\left[\frac{I_{\mathcal{K}} - A_i A_j^*}{1 - \sum_{k=1}^{n} f_k(\lambda_i)\overline{f_k(\lambda_j)}} \right]_{m \times m}$$

is positive semidefinite.

Using this corollary, we can obtain the following result.

Corollary 8.4. *Let $f = (f_1, \dots, f_n)$ be an n-tuple of formal power series with the model property and let φ be a complex-valued function defined on $\mathbb{B}_f^<(\mathbb{C}) \subset \mathbb{C}^n$. Then there exists $F \in H^\infty(\mathbb{B}_f)$ with $\|F\| \leqslant 1$ such that*

$$\varphi(z_1, \dots, z_n) = F(z_1, \dots, z_n) \quad \text{for all } (z_1, \dots, z_n) \in \mathbb{B}_f^<(\mathbb{C}),$$

if and only if for each m-tuple of distinct points $\lambda_1, \dots, \lambda_m \in \mathbb{B}_f^<(\mathbb{C})$, the matrix

$$\left[\frac{1 - \varphi(\lambda_i)\overline{\varphi(\lambda_j)}}{1 - \sum_{k=1}^{n} f_k(\lambda_i)\overline{f_k(\lambda_j)}} \right]_{m \times m}$$

is positive semidefinite. In this case, φ is a bounded analytic function on $\mathbb{B}_f^<(\mathbb{C})$.

Proof. One implication follows from Corollary 8.3. Conversely, assume that $\varphi : \mathbb{B}_f^<(\mathbb{C}) \to \mathbb{C}$ is such that the matrix above is positive semidefinite for any m-tuple of distinct points $\lambda_1, \dots, \lambda_m \in \mathbb{B}_f^<(\mathbb{C})$. Let $\{\lambda_j\}_{j=1}^\infty$ be a countable dense set in $\mathbb{B}_f^<(\mathbb{C})$. Applying Theorem 8.2, for each $m \in \mathbb{N}$, we find $F_m \in H^\infty(\mathbb{B}_f)$ such that $\|F_m\| \leqslant 1$ and

$$F_m(\lambda_j) = \varphi(\lambda_j) \quad \text{for } j = 1, \dots, m. \tag{8.8}$$

Since the Hardy algebra $H^\infty(\mathbb{B}_f)$ is w^*-closed subalgebra in $B(\mathbb{H}^2(f))$ and $\|F_k\| \leqslant 1$ for any $m \in \mathbb{N}$, we can use Alaoglu's theorem to find a subsequence $\{F_{k_m}\}_{m=1}^\infty$ and $F \in H^\infty(\mathbb{B}_f)$ such that $F_{k_m} \to F$, as $m \to \infty$, in the w^*-topology. Since $\lambda_j := (\lambda_{j1}, \dots, \lambda_{jn}) \in \mathbb{B}_f^<(\mathbb{C})$, the n-tuple is also of class $C_{\cdot 0}$. Due to Theorem 3.1 and Theorem 4.5, the $H^\infty(\mathbb{B}_f)$-functional calculus for pure n-tuples of operators in $\mathbb{B}_f(\mathcal{H})$ is WOT-continuous on bounded sets. Consequently, we deduce that $F_{k_m}(\lambda_j) \to F(\lambda_j)$, as $m \to \infty$, for any $j \in \mathbb{N}$. Hence, and using relation (8.8), we obtain $\varphi(\lambda_j) = F(\lambda_j)$ for $j \in \mathbb{N}$. Given an arbitrary element $z \in \mathbb{B}_f^<(\mathbb{C})$, we can apply again the above argument to find $G \in H^\infty(\mathbb{B}_f)$, $\|G\| \leqslant 1$ such that

$$G(z) = \varphi(z) \quad \text{and} \quad G(\lambda_j) = \varphi(\lambda_j), \quad j \in \mathbb{N}.$$

Due to Theorem 5.2, the maps $\lambda \mapsto G(\lambda)$ and $\lambda \mapsto F(\lambda)$ are analytic on $\mathbb{B}_f^<(\mathbb{C})$. Since they coincide on the set $\{\lambda_j\}_{j=1}^\infty$, which is dense in $\mathbb{B}_f^<(\mathbb{C})$), we deduce that $G(\lambda) = F(\lambda)$ for any $\lambda \in \mathbb{B}_f^<(\mathbb{C})$. In particular, we have $F(z) = \varphi(z)$. Since z is an arbitrary element in $\mathbb{B}_f^<(\mathbb{C})$, the proof is complete. □

We remark that, in the particular case when $f = (Z_1, \dots, Z_n)$, we recover some of the results obtained by Arias and the author and Davidson and Pitts (see [25,2,7]).

References

[1] A. Arias, G. Popescu, Factorization and reflexivity on Fock spaces, Integral Equations Operator Theory 23 (1995) 268–286.
[2] A. Arias, G. Popescu, Noncommutative interpolation and Poisson transforms, Israel J. Math. 115 (2000) 205–234.
[3] W.B. Arveson, Subalgebras of C^*-algebras III: Multivariable operator theory, Acta Math. 181 (1998) 159–228.
[4] W.B. Arveson, The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \dots, z_n]$, J. Reine Angew. Math. 522 (2000) 173–236.
[5] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1948) 239–251.
[6] H. Cartan, Elementary Theory of Analytic Functions of One or Several variables, Editions Scientifiques Herman, Paris, 1963.
[7] K.R. Davidson, D. Pitts, Nevanlinna–Pick interpolation for noncommutative analytic Toeplitz algebras, Integral Equations Operator Theory 31 (1998) 321–337.
[8] K.R. Davidson, D. Pitts, The algebraic structure of non-commutative analytic Toeplitz algebras, Math. Ann. 311 (1998) 275–303.
[9] K.R. Davidson, D. Pitts, Invariant subspaces and hyper-reflexivity for free semigroup algebras, Proc. Lond. Math. Soc. 78 (1999) 401–430.
[10] J.W. Helton, S.A. McCullough, V. Vinnikov, Noncommutative convexity arises from linear matrix inequalities, J. Funct. Anal. 240 (1) (2006) 105–191.
[11] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.
[12] P.S. Muhly, B. Solel, Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes, J. Funct. Anal. 158 (1998) 389–457.
[13] P.S. Muhly, B. Solel, Hardy algebras, W^*-correspondences and interpolation theory, Math. Ann. 330 (2004) 353–415.

[14] P.S. Muhly, B. Solel, Schur class operator functions and automorphisms of Hardy algebras, Doc. Math. 13 (2008) 365–411.
[15] R. Nevanlinna, Über beschränkte Functionen, die in gegebenen Punkten vorgeschribene Werte annehmen, Ann. Acad. Sci. Fenn. Ser A 13 (1919) 7–23.
[16] V.I. Paulsen, Every completely polynomially bounded operator is similar to a contraction, J. Funct. Anal. 55 (1984) 1–17.
[17] V.I. Paulsen, Completely Bounded Maps and Dilations, Pitman Res. Notes Math., vol. 146, John Wiley and Sons, New York, 1986.
[18] G. Popescu, Isometric dilations for infinite sequences of noncommuting operators, Trans. Amer. Math. Soc. 316 (1989) 523–536.
[19] G. Popescu, Characteristic functions for infinite sequences of noncommuting operators, J. Operator Theory 22 (1989) 51–71.
[20] G. Popescu, Multi-analytic operators and some factorization theorems, Indiana Univ. Math. J. 38 (1989) 693–710.
[21] G. Popescu, Von Neumann inequality for $(B(H)^n)_1$, Math. Scand. 68 (1991) 292–304.
[22] G. Popescu, Functional calculus for noncommuting operators, Michigan Math. J. 42 (1995) 345–356.
[23] G. Popescu, Multi-analytic operators on Fock spaces, Math. Ann. 303 (1995) 31–46.
[24] G. Popescu, Noncommutative disc algebras and their representations, Proc. Amer. Math. Soc. 124 (1996) 2137–2148.
[25] G. Popescu, Interpolation problems in several variables, J. Math. Anal. Appl. 227 (1998) 227–250.
[26] G. Popescu, Poisson transforms on some C^*-algebras generated by isometries, J. Funct. Anal. 161 (1999) 27–61.
[27] G. Popescu, Curvature invariant for Hilbert modules over free semigroup algebras, Adv. Math. 158 (2001) 264–309.
[28] G. Popescu, Operator theory on noncommutative varieties, Indiana Univ. Math. J. 55 (2) (2006) 389–442.
[29] G. Popescu, Entropy and multivariable interpolation, Mem. Amer. Math. Soc. 184 (868) (2006), vi+83 pp.
[30] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$, J. Funct. Anal. 241 (2006) 268–333.
[31] G. Popescu, Unitary invariants in multivariable operator theory, Mem. Amer. Math. Soc. 200 (941) (2009), vi+91 pp.
[32] G. Popescu, Noncommutative transforms and free pluriharmonic functions, Adv. Math. 220 (2009) 831–893.
[33] G. Popescu, Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$, J. Reine Angew. Math. 638 (2010) 119–168.
[34] G. Popescu, Free holomorphic functions on the unit ball of $B(H)^n$, II, J. Funct. Anal. 258 (5) (2010) 1513–1578.
[35] D. Sarason, Generalized interpolation in H^∞, Trans. Amer. Math. Soc. 127 (1967) 179–203.
[36] W.F. Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (1955) 211–216.
[37] B.Sz.- Nagy, C. Foiaş, Harmonic Analysis of Operators on Hilbert Space, North-Holland, New York, 1970.
[38] J. von Neumann, Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes, Math. Nachr. 4 (1951) 258–281.