Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 163 (2011) 524–546
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

An extension of the associated rational functions on the unit circle

Karl Deckers[a,*], María José Cantero[b], Leandro Moral[b], Luis Velázquez[b]

[a] *Department of Computer Science, Katholieke Universiteit Leuven, Heverlee, Belgium*
[b] *Department of Applied Mathematics, University of Zaragoza, Zaragoza, Spain*

Received 28 August 2010; received in revised form 4 December 2010; accepted 17 January 2011
Available online 26 January 2011

Communicated by Leonid Golinskii

Dedicated to Professor Franz Peherstorfer, in memoriam.

Abstract

A special class of orthogonal rational functions (ORFs) is presented in this paper. Starting with a sequence of ORFs and the corresponding rational functions of the second kind, we define a new sequence as a linear combination of the previous ones, the coefficients of this linear combination being self-reciprocal rational functions. We show that, under very general conditions on the self-reciprocal coefficients, this new sequence satisfies orthogonality conditions as well as a recurrence relation. Further, we identify the Carathéodory function of the corresponding orthogonality measure in terms of such self-reciprocal coefficients.

The new class under study includes the associated rational functions as a particular case. As a consequence of the previous general analysis, we obtain explicit representations for the associated rational functions of arbitrary order, as well as for the related Carathéodory function. Such representations are used to find new properties of the associated rational functions.

Keywords: Orthogonal rational functions; Associated rational functions; Complex unit circle

1. Introduction

Since the fundamental work of Stieltjes and Chebyshev, among others, in the 19th century, orthogonal polynomials (OPs) have been an essential tool in the analysis of basic problems in

* Corresponding author.
E-mail address: karl.deckers@cs.kuleuven.be (K. Deckers).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2011.01.002

mathematics and engineering. For example, moment problems, numerical quadrature, rational and polynomial approximation and interpolation, linear algebra, and all the direct or indirect applications of these techniques in engineering are all indebted to the basic properties of OPs. Mostly orthogonality has been considered on the complex unit circle or on (a subset of) the real line.

Orthogonal rational functions (ORFs) were first introduced by Džrbašian in the 1960s. Most of his papers appeared in Russian literature, but an accessible survey in English can be found in [18,21]. These ORFs are a generalization of OPs in such a way that they are of increasing degree with a given sequence of poles, and the OPs result if all the poles are at infinity. During the last years, many classical results of OPs are extended to the case of ORFs.

Several generalizations for ORFs on the complex unit circle and the whole real line have been gathered in the book [4, Chapt. 2–10] (e.g. the recurrence relation and the Favard theorem, the Christoffel–Darboux relation, properties of the zeros, etc.). Other rational generalizations can be found in e.g. [11,35]. Further, we refer to [2,3,5] and to [34] for the use of these ORFs in respectively numerical quadrature and system identification, while several results about matrix-valued ORFs can be found in e.g. [19,20].

Of course, many of the classical OPs are not defined with respect to a measure on the whole unit circle or the whole real line. Several theoretical results for ORFs on a subset of the real line can be found in e.g. [4, Chapt. 11] and [12,8]. For the special case in which this subset is a real half-line or an interval, we refer to [6,7,13,15,16,31,26,25,27] respectively, while some computational aspects have been dealt with in e.g. [14,17,28,30,29,32,33].

By shifting the recurrence coefficients in the recurrence relation for OPs and ORFs, the so-called associated polynomials (APs) and associated rational functions (ARFs) respectively are obtained. ARFs on a subset of the real line have been studied in [9,10] as a rational generalization of APs (see e.g. [24]), while APs on the complex unit circle, on the other hand, have been studied in [22]. However, so far nothing is known about ARFs on the complex unit circle, and hence, the main purpose of this paper is to generalize [22] to the case of rational functions, following the ideas developed by Franz Peherstorfer.

The outline of this paper is as follows. After giving the necessary theoretical background in Section 2, in Section 3 we recall some basic properties of ORFs on the complex unit circle and their so-called functions of the second kind. Although these properties are basic, they are partially new in the sense that we prove them in a more general context. Next, in Section 4 we use these ORFs and their functions of the second kind to define a new class of ORFs on the complex unit circle. The ARFs on the complex unit circle will then turn out to be a special case of this new class of ORFs, and will be dealt with in Section 5. We conclude in Section 6 with some examples.

2. Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$, and for the real line we use the symbol $\mathbb{R}$. Let $a \in \mathbb{C}$, then $\Re\{a\}$ refers to the real part of a, while $\Im\{a\}$ refers to the imaginary part. Further, we denote the imaginary unit by $\mathbf{i}$. The unit circle and the open unit disc are denoted respectively by

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \quad \text{and} \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Whenever the value zero is omitted in the set $X \subseteq \mathbb{C}$, this will be represented by X_0; e.g., $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

For any complex function f, we define the involution operation or substar conjugate by $f_*(z) = \overline{f(1/\overline{z})}$. With $\mathcal{P}_n$ we denote the space of polynomials of degree not greater than n, while $\mathcal{P}$ represents the space of all polynomials. Further, the set of complex functions holomorphic on $X \subseteq \mathbb{C}$ is denoted by $H(X)$.

Let there be fixed a sequence of complex numbers $\mathcal{B} = \{\beta_0, \beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, the rational functions we then deal with, are of the form

$$f_k(z) = \frac{c_k z^k + c_{k-1} z^{k-1} + \cdots + c_0}{(1-\overline{\beta}_1 z)(1-\overline{\beta}_2 z)\cdots(1-\overline{\beta}_k z)}, \quad k = 1, 2, \ldots.$$

The first element β_0 has no influence on the poles of the rational functions, but it will play a role in the corresponding recurrence. The standard choice is $\beta_0 = 0$, but in this paper β_0 will be free. The reason is that, even if we choose $\beta_0 = 0$ for the orthogonal rational functions, the corresponding associated rational functions involve a shift in the poles so that the related sequence $\{\beta_N, \beta_{N+1}, \ldots\}$ starts at some β_N which is not necessarily zero.

We define the Blaschke[1] factors for $\mathcal{B}$ as

$$\zeta_k(z) = \eta_k \frac{\varpi_k^*(z)}{\varpi_k(z)}, \quad \eta_k = \begin{cases} \dfrac{\overline{\beta}_k}{|\beta_k|}, & \beta_k \neq 0 \\ 1, & \beta_k = 0, \end{cases} \quad k = 0, 1, 2, \ldots, \tag{1}$$

where

$$\varpi_k(z) = 1 - \overline{\beta}_k z, \qquad \varpi_k^*(z) = z\varpi_{k*}(z) = z - \beta_k,$$

and the corresponding Blaschke products for $\mathcal{B}$ as

$$B_{-1}(z) = \zeta_0^{-1}(z), \qquad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 0, 1, 2, \ldots. \tag{2}$$

These Blaschke products generate the spaces of rational functions with poles in $1/\overline{\beta}_k$, defined by

$$\mathcal{L}_{-1} = \{0\}, \qquad \mathcal{L}_0 = \mathbb{C}, \qquad \mathcal{L}_n := \mathcal{L}\{\beta_1, \ldots, \beta_n\} = \operatorname{span}\{B_0, \ldots, B_n\}, \quad n \geqslant 1,$$

and $\mathcal{L} = \cup_{n=0}^{\infty} \mathcal{L}_n$. Let

$$\pi_0(z) \equiv 1, \qquad \pi_k(z) = \prod_{j=1}^{k} \varpi_j(z), \quad k = 1, 2, \ldots,$$

then for $k \geqslant 1$ we may write equivalently

$$B_k(z) = \upsilon_k \frac{\pi_k^*(z)}{\pi_k(z)}, \quad \upsilon_k = \prod_{j=1}^{k} \eta_j \in \mathbb{T},$$

where $\pi_k^*(z) = z^k \pi_{k*}(z)$, and thus

$$\mathcal{L}_n = \{p_n/\pi_n : p_n \in \mathcal{P}_n\}, \quad n = 0, 1, 2, \ldots.$$

Note that $\mathcal{L}_n$ and $\mathcal{L}$ are rational generalizations of $\mathcal{P}_n$ and $\mathcal{P}$. Indeed, if $\beta_k = 0$ (or equivalently, $1/\overline{\beta}_k = \infty$) for every $k \geqslant 0$, the expression in (1) becomes $\zeta_k(z) = z$ and the expression in (2) becomes $B_k(z) = z^k$. With the definition of the substar conjugate we introduce $\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\}$.

[1] The factors and products are named after Wilhelm Blaschke, who introduced these for the first time in [1].

The superstar transformation of a complex function $f_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is defined as

$$f_n^*(z) = B_n(z) f_{n*}(z).$$

Note that the factor $B_n(z)$ merely replaces the polynomial with zeros $\{\beta_j\}_{j=1}^n$ in the denominator of $f_{n*}(z)$ by a polynomial with zeros $\{1/\overline{\beta}_j\}_{j=1}^n$ so that $\mathcal{L}_n^* := \{B_n f_* : f \in \mathcal{L}_n\} = \mathcal{L}_n$. Like in this identity, sometimes we will denote $f^* := B_n f_*$ when we only know that $f \in \mathcal{L}_n$, even if f could belong to $\mathcal{L}_k$ for some $k < n$. At any time, the meaning of the superstar transformation should be clear from the context.

A complex function F is called a C-function in $\mathbb{D}$ iff[2]

$$F \in H(\mathbb{D}) \quad \text{and} \quad \Re\{F(z)\} > 0, \quad z \in \mathbb{D}.$$

Important related functions are the Riesz–Herglotz kernel

$$D(t, z) = \frac{\zeta_0(t) + \zeta_0(z)}{\zeta_0(t) - \zeta_0(z)} = \frac{\varpi_0^*(t)\varpi_0(z) + \varpi_0^*(z)\varpi_0(t)}{\varpi_0(\beta_0)(t - z)},$$

and the Poisson kernel

$$P(t, z) = \frac{1}{2}\left(D(t, z) + D_*(t, z)\right) = \frac{\varpi_z(z)\varpi_0(t)\varpi_0^*(t)}{\varpi_0(\beta_0)\varpi_z(t)\varpi_z^*(t)}, \quad \varpi_z(t) = 1 - \overline{z}t,$$

where the substar conjugate is with respect to t. Note that $P_*(t, z) = P(t, z)$ and $P(t, z) = \Re\{D(t, z)\}$ for $t \in \mathbb{T}$. Further, with $P_n(t)$ we denote

$$P_n(t) = P(t, \beta_n). \tag{3}$$

To the C-function F we then associate a Hermitian $\left(\mathfrak{L}_F(t^{-k}) = \overline{\mathfrak{L}_F(t^k)}\right)$ linear functional $\mathfrak{L}_F$ on the set of formal power series $\sum_{k=-\infty}^{\infty} c_k t^k$ with complex coefficients, so that

$$F(z) = \mathfrak{L}_F\{D(t, z)\},$$

where we understand again that $\mathfrak{L}_F$ acts on t. In the remainder we will assume that $F(\beta_0) = 1$, and that the functional $\mathfrak{L}_F$ is positive definite. Thus, $\mathfrak{L}_F\{1\} = 1$, and for every $f \in \mathcal{L}$

$$\mathfrak{L}_F\{f_*\} = \overline{\mathfrak{L}_F\{f\}} \quad \text{and} \quad \mathfrak{L}_F\{f f_*\} > 0 \quad \text{for } f \neq 0.$$

This is equivalent to saying that

$$\mathfrak{L}_F\{f\} = \int_{\mathbb{T}} f(t)\, \mathrm{d}\mu(t)$$

for a positive Borel measure $\mathrm{d}\mu$ on the unit circle with $\int_{\mathbb{T}} \mathrm{d}\mu(t) = 1$.

We say that two rational functions $f, g \in \mathcal{L}$ are orthogonal with respect to $\mathfrak{L}_F$ $(f \perp_F g)$ if

$$\mathfrak{L}_F\{f g_*\} = 0.$$

The functions $\phi_n \in \mathcal{L}_n \setminus \{0\}$ of a sequence are called orthogonal rational functions (ORFs) if

$$\phi_n \perp_F \mathcal{L}_{n-1}$$

2 In other words, a C-function F is a Carathéodory function for which $\Re\{F(z)\} > 0,\ z \in \mathbb{D}$ (instead of $\Re\{F(z)\} \geqslant 0,\ z \in \mathbb{D}$).

and they are called orthonormal if at the same time

$$\mathfrak{L}_F\{\phi_n\phi_{n*}\} = 1.$$

The orthogonality $\phi_n \perp_F \mathcal{L}_{n-1}$ for a function $\phi_n \in \mathcal{L}_n \setminus \{0\}$ ensures that, in fact, $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$.

A sequence of functions $f_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is called para-orthogonal when $f_n \perp_F \mathcal{L}_{n-1}(\beta_n) = \{g \in \mathcal{L}_{n-1} : g(\beta_n) = 0\}$, $\mathfrak{L}_F\{f_n\} \neq 0$ and $\mathfrak{L}_F\{f_n^*\} \neq 0$. Further, a function $f_n \in \mathcal{L}_n$ is called k-invariant (or, self-reciprocal) iff $f_n^* = kf_n$, $k \in \mathbb{C}$. Let $\Phi_{n,\tau}$ be given by

$$\Phi_{n,\tau} = \phi_n + \tau\phi_n^*, \quad \tau \in \mathbb{T}. \tag{4}$$

Then, it is easily verified that a self-reciprocal rational function is para-orthogonal exactly when it is proportional to a function with the form (4). Furthermore, the following theorem has been proved in [4, Thm. 5.2.1].

Theorem 1. *The zeros of $\Phi_{n,\tau}$, given by (4), are on $\mathbb{T}$ and they are simple.*

3. Orthogonal rational functions and functions of the second kind

With the ORFs ϕ_n and para-orthogonal rational functions (para-ORFs) $\Phi_{n,\tau}$ we associate the so-called functions of the second kind:

$$\psi_n(z) = \mathfrak{L}_F\{D(t,z)[\phi_n(t) - \phi_n(z)]\} + \mathfrak{L}_F\{\phi_n(t)\}, \quad n \geqslant 0,$$

(where we understand that $\mathfrak{L}_F$ acts on t) and

$$\Psi_{n,\tau} = \psi_n - \tau\psi_n^*, \quad \tau \in \mathbb{T},$$

respectively. We now have the following two lemmas. The first one, which is partially stated in [4, Lem. 4.2.1], can be understood as a direct consequence of the recurrence relation appearing below. The second lemma has been proved in [4, Lem. 4.2.2] for $n > 0$ (the statement is obvious for $n = 0$).[3]

Lemma 2. *The functions ψ_n are in $\mathcal{L}_n \setminus \mathcal{L}_{n-1}$.*

Lemma 3. *For $n > 0$, it holds for every $f \in \mathcal{L}_{(n-1)*}$ and $g \in \zeta_{n*}\mathcal{L}_{(n-1)*}$ that*

$$(\psi_n f)(z) = \mathfrak{L}_F\{D(t,z)[(\phi_n f)(t) - (\phi_n f)(z)]\} + \mathfrak{L}_F\{(\phi_n f)(t)\},$$

and

$$-(\psi_n^* g)(z) = \mathfrak{L}_F\{D(t,z)[(\phi_n^* g)(t) - (\phi_n^* g)(z)]\} - \mathfrak{L}_F\{(\phi_n^* g)(t)\}.$$

The same holds true for $n = 0$, when $f, g \in \mathbb{C}$.

As in the polynomial case, a recurrence relation and a Favard-type theorem can be derived for ORFs and their functions of the second kind.

Theorem 4. *The following two statements are equivalent:*

(1) *$\phi_n \in \mathcal{L}_n \setminus \{0\}$ and $\phi_n \perp_F \mathcal{L}_{n-1}$, for a certain C-function F with $F(\beta_0) = 1$, and ψ_n is the rational function of the second kind of ϕ_n.*

[3] Although we use a slightly different definition of the Riesz–Herglotz kernel from the one in [4], the proofs in the reference remain valid.

(2) *ϕ_n and ψ_n satisfy a recurrence relation of the form*

$$\begin{pmatrix}\phi_n(z) & \psi_n(z)\\ \phi_n^*(z) & -\psi_n^*(z)\end{pmatrix} = u_n(z)\begin{pmatrix}1 & \overline{\lambda}_n\\ \lambda_n & 1\end{pmatrix}\begin{pmatrix}\zeta_{n-1}(z) & 0\\ 0 & 1\end{pmatrix} \times \begin{pmatrix}\phi_{n-1}(z) & \psi_{n-1}(z)\\ \phi_{n-1}^*(z) & -\psi_{n-1}^*(z)\end{pmatrix}, \quad n > 0, \tag{5}$$

where $\lambda_n \in \mathbb{D}$, and

$$u_n(z) = e_n \begin{pmatrix}\rho_n & 0\\ 0 & \overline{\rho}_n\overline{\eta}_{n-1}\eta_n\end{pmatrix}\frac{\varpi_{n-1}(z)}{\varpi_n(z)}, \quad |\rho_n| = 1,\ e_n \in \mathbb{R}_0, \tag{6}$$

and with initial conditions $\phi_0 = \psi_0 \in \mathbb{C}_0$.

In the special case of orthonormality, the initial conditions are

$$\phi_0 = \psi_0 = \varrho, \quad |\varrho| = 1,$$

and the constants e_n are given by

$$e_n^2 = \frac{\varpi_n(\beta_n)}{\varpi_{n-1}(\beta_{n-1})}\cdot\frac{1}{1-|\lambda_n|^2}. \tag{7}$$

Proof. (1) $\Rightarrow$ (2) has been proved in [4, Thm. 4.1.1] and [4, Thm. 4.2.4] for ϕ_n and ψ_n respectively, under the assumption $\beta_0 = 0$. Further, (2) $\Rightarrow$ (1) has been proved in [4, Thm. 8.1.4], again under the assumption $\beta_0 = 0$. It is easily verified that the proofs in [4, Thm. 4.1.1] and [4, Thm. 8.1.4] remain valid when $\beta_0 \neq 0$. Also the proof in [4, Thm. 4.2.4] where $n > 1$ remains valid under the assumption $\beta_0 \neq 0$. So, we only need to prove the recurrence relation for ψ_n when $n = 1$.

First, note that

$$\phi_1(t) = \frac{e_1\rho_1}{\varpi_1(t)}[\eta_0\varpi_0^*(t)\phi_0 + \overline{\lambda}_1\varpi_0(t)\phi_0^*]. \tag{8}$$

Thus, from the orthogonality of ϕ_1, it follows that

$$\eta_0\phi_0\mathfrak{L}_F\left\{\frac{\varpi_0^*(t)}{\varpi_1(t)}\right\} = -\overline{\lambda}_1\phi_0^*\mathfrak{L}_F\left\{\frac{\varpi_0(t)}{\varpi_1(t)}\right\}. \tag{9}$$

From (8) together with the definition of ψ_1 and $D(t,z)$, we obtain

$$\begin{aligned}\psi_1(z) &= \mathfrak{L}_F\{D(t,z)[\phi_1(t)-\phi_1(z)]\}\\ &= \frac{e_1\rho_1\left[\eta_0\varpi_1(\beta_0)\psi_0 + \overline{\lambda}_1\overline{\varpi_0^*(\beta_1)}\psi_0^*\right]}{\varpi_1(z)\varpi_0(\beta_0)}\mathfrak{L}_F\left\{\frac{\varpi_0^*(t)\varpi_0(z)+\varpi_0^*(z)\varpi_0(t)}{\varpi_1(t)}\right\}\\ &= \frac{e_1\rho_1[\eta_0\varpi_0^*(z)\psi_0 - \overline{\lambda}_1\varpi_0(z)\psi_0^*]}{\varpi_1(z)}\left[\frac{\varpi_1(\beta_0)}{\varpi_0(\beta_0)} + \frac{\overline{\lambda}_1\overline{\varpi_0^*(\beta_1)}\psi_0^*}{\eta_0\psi_0\varpi_0(\beta_0)}\right]\mathfrak{L}_F\left\{\frac{\varpi_0(t)}{\varpi_1(t)}\right\},\end{aligned}$$

where the last equality follows from (9). Further, we have that

$$\varpi_0(t) = \frac{\varpi_0(\beta_0)}{\varpi_1(\beta_0)}\varpi_1(t) + \frac{\overline{\varpi_0^*(\beta_1)}}{\varpi_1(\beta_0)}\varpi_0^*(t).$$

Consequently,

$$\mathfrak{L}_F\left\{\frac{\varpi_0(t)}{\varpi_1(t)}\right\} = \frac{\varpi_0(\beta_0)}{\varpi_1(\beta_0)} - \frac{\overline{\lambda}_1\overline{\varpi_0^*(\beta_1)}\psi_0^*}{\eta_0\varpi_1(\beta_0)\psi_0}\mathfrak{L}_F\left\{\frac{\varpi_0(t)}{\varpi_1(t)}\right\},$$

so that

$$\left[\frac{\varpi_1(\beta_0)}{\varpi_0(\beta_0)} + \frac{\overline{\lambda}_1\overline{\varpi_0^*(\beta_1)}\psi_0^*}{\eta_0\psi_0\varpi_0(\beta_0)}\right]\mathfrak{L}_F\left\{\frac{\varpi_0(t)}{\varpi_1(t)}\right\} = 1. \quad \square$$

By means of the recurrence relation in the previous theorem, we obtain the following determinant formula (a similar result has been proved in [4, Cor. 4.3.2.(2)] under the assumption $\beta_0 = 0$).

Theorem 5. *Suppose $\phi_n \in \mathcal{L}_n \backslash \{0\}$ and $\phi_n \perp_F \mathcal{L}_{n-1}$, for a certain C-function F with $F(\beta_0) = 1$, and let $\psi_n \in \mathcal{L}_n \setminus \{0\}$ be the rational function of the second kind of ϕ_n. Then,*

$$\left(\phi_n^*\psi_n + \phi_n\psi_n^*\right)(z) = d_n P_n(z) B_n(z), \quad d_n \in \mathbb{R}_0, \tag{10}$$

where $P_n(z)$ is defined as above in (3). In the special case of orthonormality, it holds that $d_n = 2$.

Proof. Since

$$\left(\phi_0^*\psi_0 + \phi_0\psi_0^*\right)(z) \equiv 2\left|\phi_0\right|^2,$$

the equality in (10) clearly holds for $n = 0$ and $d_0 = 2$ in the orthonormal case.

Suppose now that the equality in (10) holds true for $0 \leqslant k < n$ with $d_k = 2$ in the orthonormal case. We then continue by induction for $k = n$. From (5) it follows that

$$\begin{aligned}
&\left(\phi_n^*\psi_n + \phi_n\psi_n^*\right)(z) \\
&= e_n^2(1 - |\lambda_n|^2)\frac{\varpi_{n-1}^2(z)}{\varpi_n^2(z)}\overline{\eta}_{n-1}\eta_n\zeta_{n-1}(z)\left(\phi_{n-1}^*\psi_{n-1} + \phi_{n-1}\psi_{n-1}^*\right)(z) \\
&= e_n^2(1 - |\lambda_n|^2)\frac{\varpi_{n-1}^2(z)}{\varpi_n^2(z)}\frac{\overline{\eta}_{n-1}\zeta_{n-1}(z)}{\overline{\eta}_n\zeta_n(z)}\frac{P_{n-1}(z)}{P_n(z)}d_{n-1}P_n(z)B_n(z) \\
&= d_n P_n(z) B_n(z),
\end{aligned}$$

where

$$d_n = e_n^2\left[\frac{\varpi_n(\beta_n)}{\varpi_{n-1}(\beta_{n-1})}\frac{1}{1 - |\lambda_n|^2}\right]^{-1} d_{n-1}, \tag{11}$$

so that $d_n \in \mathbb{R}_0$ and in the orthonormal case $d_n = d_{n-1} = 2$, due to (7). $\square$

Finally, the following interpolation properties hold true for (para-)ORFs and their functions of the second kind.

Theorem 6. *Suppose that F is a C-function, with $F(\beta_0) = 1$, and let ϕ_n and ψ_n be in $\mathcal{L}_n \setminus \{0\}$. Then the following two statements are equivalent:*

(1) *$\phi_n \perp_F \mathcal{L}_{n-1}$ and ψ_n is the rational function of the second kind of ϕ_n.*

(2) *ϕ_n, ψ_n satisfy*

$$\begin{cases}\left(\phi_n F + \psi_n\right)(z) = \zeta_0(z)B_{n-1}(z)g_n(z) \\ \left(\phi_n^* F - \psi_n^*\right)(z) = \zeta_0(z)B_n(z)h_n(z),\end{cases} \quad g_n, h_n \in H(\mathbb{D}). \tag{12}$$

Besides, the function g_n in (12) satisfies $g_n(\beta_n) \neq 0$.[4]

Proof. (1) $\Rightarrow$ (2) has been proved in [4, Thm. 6.1.1] under the assumption $\beta_0 = 0$. The proof in [4, Thm. 6.1.1] remains valid for $\beta_0 \neq 0$, when replacing t and z with $\zeta_0(t)$ and $\zeta_0(z)$

[4] From Lemma 13 it will in fact follow that $g_n(z) \neq 0$ for every $z \in \mathbb{D}$.

respectively. Thus, it remains to prove that the rational functions $\phi_n, \psi_n \in \mathcal{L}_n \setminus \{0\}$ in (12) are unique up to a common non-zero multiplicative factor, as well as the fact that $g_n(\beta_n) \neq 0$. We will prove both things simultaneously by induction on n.

First, consider the case in which $n = 0$. Clearly, $\phi_0, \psi_0 \in \mathbb{C}_0$ satisfy (12) iff $\phi_0 = \psi_0$. Furthermore, $g_0(\beta_0) \neq 0$ because, otherwise, evaluating (12) at β_0 would give

$$\phi_0 = -\psi_0, \qquad \phi_0 = \psi_0,$$

hence, $\phi_0 = \psi_0 = 0$, in contradiction with our assumption $\phi_0, \psi_0 \in \mathcal{L}_0 \setminus \{0\}$.

Next, suppose that for $0 \leqslant k < n$ the rational functions ϕ_k and ψ_k in (12) are unique up to a non-zero multiplicative factor, and that $g_k(\beta_k) \neq 0$. We then continue by induction to prove that the same holds true for $k = n$. Let $\tilde{\phi}_n, \tilde{\psi}_n \in \mathcal{L}_n \setminus \{0\}$, then $\tilde{\phi}_n = k_n\phi_n + a_{n-1}$ and $\tilde{\psi}_n = k_n\psi_n + b_n$, with $k_n \in \mathbb{C}$, $a_{n-1} \in \mathcal{L}_{n-1}$ and $b_n \in \mathcal{L}_n$. Assuming

$$\begin{cases} \left(\tilde{\phi}_n F + \tilde{\psi}_n\right)(z) = \zeta_0(z) B_{n-1}(z) \tilde{g}_n(z) \\ \left(\tilde{\phi}_n^* F - \tilde{\psi}_n^*\right)(z) = \zeta_0(z) B_n(z) \tilde{h}_n(z), \end{cases} \qquad \tilde{g}_n, \tilde{h}_n \in H(\mathbb{D}),$$

gives

$$\begin{cases} (a_{n-1} F + b_n)(z) = \zeta_0(z) B_{n-1}(z) \hat{g}_{n-1}(z) \\ \left(\zeta_n a_{n-1}^* F - b_n^*\right)(z) = \zeta_0(z) B_n(z) h_{n-1}(z), \end{cases} \qquad \hat{g}_{n-1}, h_{n-1} \in H(\mathbb{D}).$$

From the second equality it follows that b_n^* is of the form $\zeta_n b_{n-1}^*$, $b_{n-1} \in \mathcal{L}_{n-1}$, and hence, that $b_n = b_{n-1}$. Thus,

$$\begin{cases} (a_{n-1} F + b_{n-1})(z) = \zeta_0(z) B_{n-2}(z) g_{n-1}(z) \\ \left(a_{n-1}^* F - b_{n-1}^*\right)(z) = \zeta_0(z) B_{n-1}(z) h_{n-1}(z), \end{cases} \qquad g_{n-1}, h_{n-1} \in H(\mathbb{D}),$$

with $g_{n-1} = \zeta_{n-1}\hat{g}_{n-1}$. Therefore $a_{n-1}, b_{n-1} \in \mathcal{L}_{n-1}$ are solutions of (12) for $k = n - 1$, but with $g_{n-1}(\beta_{n-1}) = 0$. This contradicts the induction hypothesis that $g_{n-1}(\beta_{n-1}) \neq 0$ unless $a_{n-1} = b_{n-1} = 0$ which implies $\tilde{\phi}_n = k_n\phi_n$, $\tilde{\psi}_n = k_n\psi_n$.

Finally, let us prove that $g_n(\beta_n) \neq 0$. If $g_n(\beta_n) = 0$ then

$$\begin{cases} (\phi_n F + \psi_n)(z) = \zeta_0(z) B_n(z) \hat{g}_n(z) \\ \left(\phi_n^* F - \psi_n^*\right)(z) = \zeta_0(z) B_n(z) h_n(z), \end{cases} \qquad \hat{g}_n, h_n \in H(\mathbb{D}).$$

From (10) it then follows that

$$\begin{aligned} d_n P_n(z) B_n(z) &= \phi_n^*(z)\psi_n(z) + \phi_n(z)\psi_n^*(z) \\ &= \phi_n^*(z)\left(\phi_n(z)F(z) + \psi_n(z)\right) - \phi_n(z)\left(\phi_n^*(z)F(z) - \psi_n^*(z)\right) \\ &= \zeta_0(z) B_n(z) g(z), \quad g \in H(\mathbb{D}) \\ &= \zeta_0(z) B_n(z) \frac{\varpi_0(z) p_{n-1}(z)}{\pi_n(z)}, \quad p_{n-1} \in \mathcal{P}_{n-1}, \end{aligned}$$

where the last equality follows from the fact that $\left(\phi_n^*\psi_n + \phi_n\psi_n^*\right) \in \mathcal{L}_n \cdot \mathcal{L}_n$. Consequently,

$$\begin{aligned} \frac{\eta_0 \varpi_0^*(z) p_{n-1}(z)}{\varpi_n(z)\pi_{n-1}(z)} = \quad & d_n P_n(z) = \frac{\hat{d}_n \varpi_0(z)\varpi_0^*(z)}{\varpi_n(z)\varpi_n^*(z)}, \quad \hat{d}_n \in \mathbb{R}_0 \\ & \Longrightarrow p_{n-1}(z) = \tilde{d}_n \frac{\varpi_0(z)\pi_{n-1}(z)}{\varpi_n^*(z)} \notin \mathcal{P}_{n-1}, \quad \tilde{d}_n \in \mathbb{C}_0, \end{aligned}$$

which contradicts the assumption $p_{n-1} \in \mathcal{P}_{n-1}$. $\square$

The following theorem directly follows from Theorem 6, and the definition of $\Phi_{n,\tau}$ and $\Psi_{n,\tau}$.

Theorem 7. *The para-ORFs $\Phi_{n,\tau} \in \mathcal{L}_n \setminus \{0\}$ and their second kind ones $\Psi_{n,\tau} \in \mathcal{L}_n \setminus \{0\}$ satisfy*

$$\begin{cases} \left(\Phi_{n,\tau} F + \Psi_{n,\tau}\right)(z) = \zeta_0(z) B_{n-1}(z) g_n(z) \\ \left(\Phi^*_{n,\tau} F - \Psi^*_{n,\tau}\right)(z) = \overline{\tau}\zeta_0(z) B_{n-1}(z) g_n(z), \end{cases} \quad g_n \in H(\mathbb{D}), \tag{13}$$

with $g_n(z) \neq 0$ for every $z \in \mathbb{D}$.

Proof. The equalities in (13) have been proved in [4, Cor. 6.1.2] under the assumption $\beta_0 = 0$, but the proof remains valid for $\beta_0 \neq 0$. So, we only need to prove that $g_n(z) \neq 0$ for every $z \in \mathbb{D}$.

Suppose that there exists $\hat{\beta}_n \in \mathbb{D}$ such that $g_n(\hat{\beta}_n) = 0$. Let us then define $R_n, S_n \in \mathcal{L}\{\beta_1, \ldots, \beta_{n-1}, \hat{\beta}_n\} \setminus \{0\}$ as

$$R_n(z) = \frac{\varpi_n(z)}{\hat{\varpi}_n(z)} \Phi_{n,\tau}(z) \quad \text{and} \quad S_n(z) = \frac{\varpi_n(z)}{\hat{\varpi}_n(z)} \Psi_{n,\tau}(z),$$

where $\hat{\varpi}_n(z) = 1 - \overline{\hat{\beta}_n} z$. From the first equality in (13) we obtain that

$$\begin{aligned} (R_n F + S_n)(z) &= \zeta_0(z) B_{n-1}(z) \frac{\varpi_n(z)}{\hat{\varpi}_n(z)} g_n(z), \quad g_n \in H(\mathbb{D}) \\ &= \zeta_0(z) B_{n-1}(z) \hat{\zeta}_n(z) \varpi_n(z) \frac{g_n(z)}{\hat{\eta}_n \hat{\varpi}^*_n(z)}, \quad \hat{\zeta}_n(z) = \hat{\eta}_n \frac{\hat{\varpi}^*_n(z)}{\hat{\varpi}_n(z)} \\ &= \zeta_0(z) B_{n-1}(z) \hat{\zeta}_n(z) \tilde{g}_n(z), \quad \tilde{g}_n \in H(\mathbb{D}), \end{aligned}$$

where the last equality follows from the fact that $g_n(\hat{\beta}_n) = 0$. On the other hand,

$$\begin{aligned} R^*_n(z) &= B_{n-1}(z) \hat{\zeta}_n(z) R_{n*}(z) = B_n(z) \frac{\hat{\zeta}_n(z)}{\zeta_n(z)} \frac{\varpi_{n*}(z)}{\hat{\varpi}_{n*}(z)} \Phi_{(n,\tau)*}(z) \\ &= \frac{\hat{\zeta}_n(z)}{\zeta_n(z)} \frac{\varpi^*_n(z)}{\hat{\varpi}^*_n(z)} \Phi^*_{n,\tau}(z) = \frac{\hat{\eta}_n}{\eta_n} \frac{\varpi_n(z)}{\hat{\varpi}_n(z)} \overline{\tau} \Phi_{n,\tau}(z) = \frac{\hat{\eta}_n}{\eta_n} \overline{\tau} R_n(z), \end{aligned}$$

and similarly,

$$-S^*_n(z) = \frac{\hat{\eta}_n}{\eta_n} \overline{\tau} S_n(z).$$

Consequently,

$$\begin{cases} (R_n F + S_n)(z) = \zeta_0(z) B_{n-1}(z) \hat{\zeta}_n(z) \tilde{g}_n(z) \\ \left(R^*_n F - S^*_n\right)(z) = \dfrac{\hat{\eta}_n}{\eta_n} \overline{\tau} \zeta_0(z) B_{n-1}(z) \hat{\zeta}_n(z) \tilde{g}_n(z), \end{cases} \quad \tilde{g}_n \in H(\mathbb{D}). \tag{14}$$

Now, consider the ORF $\hat{\phi}_n \perp_F \mathcal{L}_{n-1}$, with $\hat{\phi}_n \in \mathcal{L}\{\beta_1, \ldots, \beta_{n-1}, \hat{\beta}_n\} \setminus \{0\}$, and let $\hat{\psi}_n \in \mathcal{L}\{\beta_1, \ldots, \beta_{n-1}, \hat{\beta}_n\} \setminus \{0\}$ denote the rational function of the second kind of $\hat{\phi}_n$. Theorem 6 states that $\hat{\phi}_n$ and $\hat{\psi}_n$ are (up to a multiplicative factor) the only non-zero rational functions in $\mathcal{L}\{\beta_1, \ldots, \beta_{n-1}, \hat{\beta}_n\}$ satisfying

$$\begin{cases} \left(\hat{\phi}_n F + \hat{\psi}_n\right)(z) = \zeta_0(z) B_{n-1}(z) \hat{g}_n(z) \\ \left(\hat{\phi}^*_n F - \hat{\psi}^*_n\right)(z) = \zeta_0(z) B_{n-1}(z) \hat{\zeta}_n(z) \hat{h}_n(z), \end{cases} \quad \hat{g}_n, \hat{h}_n \in H(\mathbb{D}).$$

Moreover, it holds that $\hat{g}_n(\hat{\beta}_n) \neq 0$ for this solution. Therefore, there cannot exist rational functions $R_n, S_n \in \mathcal{L}\{\beta_1, \ldots, \beta_{n-1}, \hat{\beta}_n\} \setminus \{0\}$ satisfying (14). $\square$

Theorem 6 is the main result of this section. It is the rational extension of [23, Thm. 2.1]. Its importance relies on the fact that it provides us with a characterization of ORFs and their second kind in terms of only the C-function F. Theorem 6 will be the key tool to study the associated ORFs and their extensions, analogously to a similar analysis of the polynomial case in [22].

4. A new class of orthogonal rational functions

Analogously as has been done in [22], we will study a new class of ORFs generated by a given sequence of ORFs. The rational functions of the new class will satisfy a similar recurrence to that one of the initial ORFs, but starting at some index r and with shifted poles and (rotated) parameters. The associated rational functions will be a particular case when the starting index is $r = 0$ and there is no rotation of the parameters.

To introduce the new class, we need to consider spaces of rational functions based on different sequences of complex numbers.

Given the sequences of complex numbers $\mathcal{B} = \{\beta_0, \beta_1, \ldots, \beta_N, \beta_{N+1}, \ldots,\} \subset \mathbb{D}, \tilde{\mathcal{B}} = \{\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_r, \tilde{\beta}_{r+1}, \ldots\} \subset \mathbb{D}$ and $\hat{\mathcal{B}} = \{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \ldots\} \subset \mathbb{D}$, with $\beta_{N+k} = \hat{\beta}_k = \tilde{\beta}_{r+k}$ for $k = 1, 2, \ldots$, we define the spaces of rational functions

$$\begin{aligned}
&\mathcal{L}_N := \mathcal{L}\{\beta_1, \ldots, \beta_N\} = \operatorname{span}\{B_0(z), B_1(z), \ldots, B_N(z)\}, \quad \mathcal{L}_0 = \mathbb{C},\\
&\hat{\mathcal{L}}_n := \mathcal{L}\{\hat{\beta}_1, \ldots, \hat{\beta}_n\} = \operatorname{span}\{\hat{B}_0(z), \hat{B}_1(z), \ldots, \hat{B}_n(z)\}, \quad \hat{\mathcal{L}}_0 = \mathbb{C}\\
&\tilde{\mathcal{L}}_r := \mathcal{L}\{\tilde{\beta}_1, \ldots, \tilde{\beta}_r\} = \operatorname{span}\{\tilde{B}_0(z), \tilde{B}_1(z), \ldots, \tilde{B}_r(z)\}, \quad \tilde{\mathcal{L}}_0 = \mathbb{C},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_{N+n} := \mathcal{L}\{\beta_1, \ldots, \beta_N, \hat{\beta}_1, \ldots, \hat{\beta}_n\} = \mathcal{L}_N \cdot \hat{\mathcal{L}}_n, \quad N, n \geqslant 0,\\
&\tilde{\mathcal{L}}_{r+n} := \mathcal{L}\{\tilde{\beta}_1, \ldots, \tilde{\beta}_r, \hat{\beta}_1, \ldots, \hat{\beta}_n\} = \tilde{\mathcal{L}}_r \cdot \hat{\mathcal{L}}_n, \quad r, n \geqslant 0,
\end{aligned}$$

with the convention that $\mathcal{L}_n = \mathcal{L}_{0+n} = \hat{\mathcal{L}}_n = \tilde{\mathcal{L}}_{0+n} = \tilde{\mathcal{L}}_n$,

$$\mathcal{L}_{N+n-1} = \begin{cases} \mathcal{L}_{N+(n-1)} = \mathcal{L}_N \cdot \hat{\mathcal{L}}_{n-1}, & n > 0\\ \mathcal{L}_{N-1}, & n = 0, \end{cases}$$

and

$$\tilde{\mathcal{L}}_{r+n-1} = \begin{cases} \tilde{\mathcal{L}}_{r+(n-1)} = \tilde{\mathcal{L}}_r \cdot \hat{\mathcal{L}}_{n-1}, & n > 0\\ \tilde{\mathcal{L}}_{r-1}, & n = 0. \end{cases}$$

Further, we set $\hat{\beta}_0 = \beta_N$, and hence, $\hat{\zeta}_0(z) = \zeta_N(z)$ and $\hat{B}_{-1}(z) = 1/\zeta_N(z)$.

The main idea is, starting with ORFs whose poles are defined by

$$\beta_1, \beta_2, \ldots, \beta_N, \hat{\beta}_1, \ldots, \hat{\beta}_n,$$

to generate new rational functions with poles defined by

$$\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_r, \hat{\beta}_1, \ldots, \hat{\beta}_n.$$

This is the purpose of the following theorem.

Theorem 8. *For $N, n, r \geqslant 0$, suppose $\phi_{N+n} \in \mathcal{L}_{N+n} \setminus \{0\}$ and $\phi_{N+n} \perp_F \mathcal{L}_{N+n-1}$, and let ψ_{N+n} denote the rational function of the second kind of ϕ_{N+n}. Further, suppose that A, B, C and D*

are self-reciprocal rational functions in $\mathcal{L}_N \cdot \tilde{\mathcal{L}}_r$*, satisfying the following conditions:*

$$\tau_A := \frac{A^*(z)}{A(z)} = -\frac{B^*(z)}{B(z)} = -\frac{C^*(z)}{C(z)} = \frac{D^*(z)}{D(z)}, \quad \tau_A \in \mathbb{T}, \tag{15}$$

$$(A - BF)(z) = \zeta_0(z) B_{N-1}(z) g(z), \quad g \in H(\mathbb{D}), \tag{16}$$

and

$$(C - DF)(z) = \zeta_0(z) B_{N-1}(z) \hat{g}(z), \quad \hat{g} \in H(\mathbb{D}). \tag{17}$$

Then the rational functions G_{r+n}, H_{r+n}, J_{r+n} *and* K_{r+n}*, defined by*

$$\begin{pmatrix} G_{r+n}(z) & J_{r+n}(z) \\ H_{r+n}(z) & -K_{r+n}(z) \end{pmatrix} = \begin{pmatrix} \phi_{N+n}(z) & \psi_{N+n}(z) \\ \phi^*_{N+n}(z) & -\psi^*_{N+n}(z) \end{pmatrix} \begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix} \times \{c_n P_N(z) B_N(z)\}^{-1}, \quad c_n \in \mathbb{R}_0, \tag{18}$$

are all in $\tilde{\mathcal{L}}_{r+n}$*. Furthermore,* $G^*_{r+n}(z) = \tau_A H_{r+n}(z)$ *and* $J^*_{r+n}(z) = \tau_A K_{r+n}(z)$*.*

Proof. From (18) it follows that the rational functions G_{r+n} and H_{r+n} are given by

$$G_{r+n}(z) = \frac{\phi_{N+n}(z)A(z) + \psi_{N+n}(z)B(z)}{c_n P_N(z) B_N(z)},$$

and

$$H_{r+n}(z) = \frac{\phi^*_{N+n}(z)A(z) - \psi^*_{N+n}(z)B(z)}{c_n P_N(z) B_N(z)}.$$

Concerning the numerators of G_{r+n} and H_{r+n}, (12) and (16) give

$$\begin{aligned} \left(\psi^*_{N+n} B - \phi^*_{N+n} A\right)(z) &= \psi^*_{N+n}(z)B(z) - \phi^*_{N+n}(z)\{B(z)F(z) + \zeta_0(z)B_{N-1}(z)g(z)\} \\ &= -\{\phi^*_{N+n}(z)F(z) - \psi^*_{N+n}(z)\}B(z) - \zeta_0(z)B_{N-1}(z)g(z)\phi^*_{N+n}(z) \\ &= -\zeta_0(z)B_{N+n}(z)h_{N+n}(z)B(z) - \zeta_0(z)B_{N-1}(z)g(z)\phi^*_{N+n}(z) \\ &= \zeta_0(z)B_{N-1}(z)k_1(z), \quad k_1 \in H(\mathbb{D}), \end{aligned} \tag{19}$$

and

$$\begin{aligned} (\phi_{N+n} A + \psi_{N+n} B)(z) &= \phi_{N+n}(z)\{B(z)F(z) + \zeta_0(z)B_{N-1}(z)g(z)\} + \psi_{N+n}(z)B(z) \\ &= \{\phi_{N+n}(z)F(z) + \psi_{N+n}(z)\}B(z) + \zeta_0(z)B_{N-1}(z)g(z)\phi_{N+n}(z) \\ &= \zeta_0(z)B_{N+n-1}(z)g_{N+n}(z)B(z) + \zeta_0(z)B_{N-1}(z)g(z)\phi_{N+n}(z) \\ &= \zeta_0(z)B_{N-1}(z)k_2(z), \quad k_2 \in H(\mathbb{D}). \end{aligned} \tag{20}$$

Since the left-hand side of (19) and (20) is in $\mathcal{L}_N \cdot \mathcal{L}_N \cdot \tilde{\mathcal{L}}_{r+n}$, it follows that

$$k_1, k_2 \in \begin{cases} \mathcal{L}\{\beta_N\} \cdot \mathcal{L}_N \cdot \tilde{\mathcal{L}}_{r+n}, & N > 0 \\ \tilde{\mathcal{L}}_{r+n}, & N = 0. \end{cases}$$

On the other hand, taking the superstar conjugate of (20), and using the fact that A and B are self-reciprocal and satisfy (15), we obtain that

$$-\tau_A \left(\psi^*_{N+n} B - \phi^*_{N+n} A\right)(z) = \frac{\zeta_N(z)}{\zeta_0(z)} B_N(z) \hat{B}_n(z) \tilde{B}_r(z) k_{2*}(z),$$

and hence,

$$-\tau_A \zeta_0^2(z) k_1(z) = \zeta_N^2(z) \hat{B}_n(z) \tilde{B}_r(z) k_{2*}(z).$$

Consequently,

$$k_1, k_2 \in \begin{cases} \mathcal{L}\{\beta_N, \beta_N\} \cdot \tilde{\mathcal{L}}_{r+n}, & N > 0 \\ \tilde{\mathcal{L}}_{r+n}, & N = 0, \end{cases}$$

and

$$k_1(z) = \frac{\varpi_0^2(z) p_{r+n}(z)}{\varpi_N^2(z) \hat{\pi}_n(z) \tilde{\pi}_r(z)}, \qquad k_2(z) = \frac{\varpi_0^2(z) q_{r+n}(z)}{\varpi_N^2(z) \hat{\pi}_n(z) \tilde{\pi}_r(z)}, \qquad p_{r+n}, q_{r+n} \in \mathcal{P}_{r+n}.$$

Since

$$c_n P_N(z) B_N(z) = c_n \eta_N \overline{\eta}_0 \frac{\varpi_N(\beta_N)}{\varpi_0(\beta_0)} \cdot \frac{\varpi_0^2(z)}{\varpi_N^2(z)} \zeta_0(z) B_{N-1}(z),$$

it now follows that $G_{r+n}(z)$ and $H_{r+n}(z)$ are in $\tilde{\mathcal{L}}_{r+n}$. Further, we have that

$$\begin{aligned} G_{r+n}^*(z) &= \tilde{B}_r(z) \hat{B}_n(z) G_{(r+n)*}(z) = \frac{\tau_A \left(\phi_{N+n}^*(z) A(z) - \psi_{N+n}^*(z) B(z) \right)}{B_N^2(z) \cdot c_n P_{N*}(z) B_{N*}(z)} \\ &= \tau_A \frac{\phi_{N+n}^*(z) A(z) - \psi_{N+n}^*(z) B(z)}{c_n P_N(z) B_N(z)} = \tau_A H_{r+n}(z). \end{aligned}$$

Finally, proving the statement for $J_{r+n}(z)$ and $J_{r+n}^*(z) = \tau_A K_{r+n}(z)$ can be done in a similar way as before, under the condition that (17) holds true. □

As a consequence of the previous theorem and Theorem 4, we have the following corollary.

Corollary 9. *The rational functions G_{r+n} and J_{r+n}, defined as before in Theorem 8 with fixed N and r, satisfy a recurrence relation of the form*

$$\begin{aligned} \begin{pmatrix} G_{r+n}(z) & J_{r+n}(z) \\ G_{r+n}^*(z) & -J_{r+n}^*(z) \end{pmatrix} &= v_{r+n}(z) \begin{pmatrix} 1 & \overline{\gamma}_{r+n} \\ \gamma_{r+n} & 1 \end{pmatrix} \begin{pmatrix} \zeta_{N+n-1}(z) & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} G_{r+n-1}(z) & J_{r+n-1}(z) \\ G_{r+n-1}^*(z) & -J_{r+n-1}^*(z) \end{pmatrix} \end{aligned}$$

for $n = 1, 2, \ldots$, where $\gamma_{r+n} = \overline{\tau}_A \lambda_{N+n}$,

$$v_{r+n}(z) = \frac{c_{n-1}}{c_n} u_{N+n}(z), \tag{21}$$

with $u_{N+n}(z)$ defined as above in (6), and (recall that $\hat{\beta}_0 = \beta_N$)

$$\zeta_{N+n-1}(z) = \begin{cases} \hat{\zeta}_{n-1}(z) = \tilde{\zeta}_{r+n-1}(z), & n > 1 \\ \hat{\zeta}_0(z), & n = 1, \end{cases}$$

and with initial conditions $G_r, J_r \in \tilde{\mathcal{L}}_r$. In the special case in which $\tilde{\beta}_r = \beta_N$, it holds that $\hat{\zeta}_{n-1}(z) = \tilde{\zeta}_{r+n-1}(z)$ for $n = 1$ too.

Theorem 8 provides us with a constructive method to generate a new class of rational functions starting with a given sequence of ORFs. As we pointed out before, the new rational functions have the same poles as the initial ORFs, except for the first N ones, which are substituted by r other poles. Besides, Corollary 9 states that these new rational functions satisfy a similar recurrence, but with different initial conditions G_r, J_r, and shifted and rotated parameters $\gamma_{r+n} = \bar{\tau}_A \lambda_{N+n}$. Nevertheless, this recurrence does not guarantee the orthogonality because it depends on the orthogonality of the initial conditions G_r, J_r. Our aim is to complete the hypothesis of Theorem 8 with a minimum number of conditions to ensure the orthogonality of the new rational functions. This is the purpose of the following theorem, which is our main result.

Theorem 10. *Let $G_{r+n}(z), J_{r+n}(z) \neq 0$ be defined as before in Theorem 8, and suppose $\tilde{\beta}_r = \beta_N$. Further, assume that the self-reciprocals A, B, C and D in $\mathcal{L}_N \cdot \tilde{\mathcal{L}}_r$ satisfy (15), together with the following conditions:*

$$(A - BF)(z) = \zeta_0(z) B_{N-1}(z) g(z), \quad g(z) \in H(\mathbb{D}), \tag{22}$$

with

$$g(\beta) \neq 0 \quad \text{for } \beta \in \{\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_r, \hat{\beta}_1, \ldots, \hat{\beta}_n\}, \tag{23}$$

$$(AD - BC)(z) = \zeta_0(z) B_{N-1}(z) \tilde{\zeta}_0(z) \tilde{B}_{r-1}(z) f(z), \quad f(z) \in H(\mathbb{D}) \setminus \{0\}, \tag{24}$$

and $\tilde{F}$, given by

$$\tilde{F}(z) = \frac{-C(z) + D(z)F(z)}{A(z) - B(z)F(z)}, \tag{25}$$

is a C-function, with $\tilde{F}(\tilde{\beta}_0) = 1$. Then $G_{r+n} \perp_{\tilde{F}} \tilde{\mathcal{L}}_{r+n-1}$ (respectively $J_{r+n} \perp_{1/\tilde{F}} \tilde{\mathcal{L}}_{r+n-1}$), and J_{r+n} (respectively G_{r+n}) is the function of the second kind of G_{r+n} with respect to $\tilde{F}$ (respectively, of J_{r+n} with respect to $1/\tilde{F}$).

Proof. First, note that (17) is satisfied due to (22) and (25), together with $\tilde{F} \in H(\mathbb{D})$. Theorem 8 implies that $G_{r+n}, J_{r+n} \in \tilde{\mathcal{L}}_{r+n} \setminus \{0\}$. From (18), (24), (25) and (12) it follows that

$$\begin{aligned}\left\{(A - BF)\left(\tilde{F} G_{r+n} + J_{r+n}\right)\right\}(z) &= \frac{\{(AD - BC)(F\phi_{N+n} + \psi_{N+n})\}(z)}{c_n P_N(z) B_N(z)} \\ &= \tilde{\zeta}_0(z) \tilde{B}_{r-1}(z) \zeta_0(z) B_N(z) \hat{B}_{n-1}(z) h(z), \quad h \in H(\mathbb{D}).\end{aligned}$$

This, together with (22) and the condition on the function g, gives

$$\begin{aligned}\left(\tilde{F} G_{r+n} + J_{r+n}\right)(z) &= \tilde{\zeta}_0(z) \tilde{B}_{r-1}(z) \zeta_N(z) \hat{B}_{n-1}(z) \hat{h}(z) \\ &= \tilde{\zeta}_0(z) \tilde{B}_{r+n-1}(z) \hat{h}(z), \quad \hat{h} \in H(\mathbb{D}).\end{aligned}$$

Next, assuming that $\tilde{F}$ is a C-function, we also obtain that

$$\left(G_{r+n} + \frac{1}{\tilde{F}} J_{r+n}\right)(z) = \tilde{\zeta}_0(z) \tilde{B}_{r+n-1}(z) \tilde{h}(z), \quad \tilde{h} \in H(\mathbb{D}).$$

Further, it follows from (18), (24), (25) and (12) that

$$\begin{aligned}\bar{\tau}_A \left\{(A - BF) \cdot \left(\tilde{F} G^*_{r+n} - J^*_{r+n}\right)\right\}(z) &= \frac{\left\{(AD - BC) \cdot \left(F\phi^*_{N+n} - \psi^*_{N+n}\right)\right\}(z)}{c_n P_N(z) B_N(z)} \\ &= \tilde{\zeta}_0(z) \tilde{B}_{r-1}(z) \zeta_0(z) B_N(z) \hat{B}_n(z) h(z), \quad h \in H(\mathbb{D}).\end{aligned}$$

This, together with (22), the condition on the function g, and the assumption that $\tilde{F}$ is a C-function, yields

$$\begin{aligned}\left(\tilde{F}G_{r+n}^{*}-J_{r+n}^{*}\right)(z) &= \tilde{\zeta}_0(z)\tilde{B}_{r-1}(z)\zeta_N(z)\hat{B}_n(z)\hat{h}(z)\\ &= \tilde{\zeta}_0(z)\tilde{B}_{r+n}(z)\hat{h}(z), \quad \hat{h}\in H(\mathbb{D}),\end{aligned}$$

and

$$\left(G_{r+n}^{*}-\frac{1}{\tilde{F}}J_{r+n}^{*}\right)(z) = \tilde{\zeta}_0(z)\tilde{B}_{r+n}(z)\tilde{h}(z), \quad \tilde{h}\in H(\mathbb{D}).$$

The orthogonality now follows from Theorem 6. □

The orthogonality properties $G_{r+n}\perp_{\tilde{F}}\tilde{\mathcal{L}}_{r+n-1}$ and $J_{r+n}\perp_{1/\tilde{F}}\tilde{\mathcal{L}}_{r+n-1}$ imply that the hypothesis of Theorem 10 ensures that, not only $G_{r+n}, J_{r+n}\in\tilde{\mathcal{L}}_{r+n}$, but $G_{r+n}, J_{r+n}\in\tilde{\mathcal{L}}_{r+n}\setminus\tilde{\mathcal{L}}_{r+n-1}$ too.

Remark 11. From Theorem 5 it follows that, under the same conditions as in Theorem 10, it should hold that

$$\left(G_{r+n}^{*}J_{r+n}+G_{r+n}J_{r+n}^{*}\right)(z) = \tilde{d}_n\tilde{P}_{r+n}(z)\tilde{B}_{r+n}(z), \quad \tilde{d}_n\in\mathbb{R}_0.$$

Indeed, taking the determinant on both sides of (18), we find for $n \geqslant 0$ that

$$\begin{aligned}\left(G_{r+n}^{*}J_{r+n}+G_{r+n}J_{r+n}^{*}\right)(z) &= \overline{\tau}_A\frac{\left\{(AD-BC)\cdot\left(\phi_{N+n}^{*}\psi_{N+n}+\phi_{N+n}\psi_{N+n}^{*}\right)\right\}(z)}{[c_nP_N(z)B_N(z)]^2}\\ &= \frac{\tilde{\zeta}_0(z)\tilde{B}_{r-1}(z)\cdot P_{N+n}(z)\hat{B}_n(z)}{P_N(z)\frac{\varpi_0^2(z)}{\varpi_N^2(z)}}\hat{f}(z), \quad \hat{f}\in H(\mathbb{D})\setminus\{0\}\\ &= \tilde{P}_{r+n}(z)\tilde{B}_{r+n}(z)\tilde{f}(z),\end{aligned} \tag{26}$$

where

$$\tilde{f}(z) = \frac{P_{N+n}(z)}{P_N(z)\tilde{P}_{r+n}(z)}\cdot\frac{\tilde{\zeta}_0(z)}{\tilde{\zeta}_r(z)}\cdot\frac{\varpi_N^2(z)}{\varpi_0^2(z)}\cdot\hat{f}(z)\in H(\mathbb{D})\setminus\{0\}.$$

Bearing in mind that the left-hand side of (26) is in $\tilde{\mathcal{L}}_{r+n}\cdot\tilde{\mathcal{L}}_{r+n}$, it follows that $\tilde{f}\in\left(\mathcal{L}\{\tilde{\beta}_0\}\cdot\tilde{\mathcal{L}}_{r+n-1}\right)\setminus\{0\}$ for $r+n>0$, respectively $\tilde{f}\in\mathbb{C}_0$ for $r+n=0$. Furthermore, taking the superstar conjugate of (26), we obtain that

$$\left(G_{r+n}^{*}J_{r+n}+G_{r+n}J_{r+n}^{*}\right)(z) = \tilde{P}_{r+n}(z)\tilde{B}_{r+n}(z)\tilde{f}_{*}(z),$$

and hence,

$$\tilde{f}(z) = \tilde{f}_{*}(z)\equiv\tilde{d}_n\in\mathbb{R}_0.$$

5. Associated rational functions

A special class of rational functions, the so-called associated rational functions (ARFs), is obtained when $\tau_A = 1$ and $r = 0$. ARFs orthogonal on a subset of the real line are investigated in detail in [9]. Analogously to the case of a subset of the real line, we define the ARFs on the unit circle as follows.

Definition 12. Suppose that the rational functions $\{\phi_n\}$ and $\{\psi_n\}$, with poles among $\{1/\overline{\beta}_1, \ldots, 1/\overline{\beta}_n\}$, satisfy a recurrence relation of the form (5). Then, for a given $k \geqslant 0$, we call the rational functions $\phi^{(k)}_{n\backslash k}$ and $\psi^{(k)}_{n\backslash k}$ generated by the recurrence formula

$$\begin{pmatrix} \phi^{(k)}_{n\backslash k}(z) & \psi^{(k)}_{n\backslash k}(z) \\ \phi^{(k)*}_{n\backslash k}(z) & -\psi^{(k)*}_{n\backslash k}(z) \end{pmatrix} = u_n(z) \begin{pmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \phi^{(k)}_{(n-1)\backslash k}(z) & \psi^{(k)}_{(n-1)\backslash k}(z) \\ \phi^{(k)*}_{(n-1)\backslash k}(z) & -\psi^{(k)*}_{(n-1)\backslash k}(z) \end{pmatrix}, \quad n > k,$$

with initial conditions $\phi^{(k)}_{k\backslash k} = \psi^{(k)}_{k\backslash k} \in \mathbb{C}_0$, the ARFs of $\{\phi_n\}$ and $\{\psi_n\}$ of order k.

Note that the subscript '$\backslash k$' in the definition of the ARFs refers to the fact that the ARFs do not have poles among $\{1/\overline{\beta}_1, \ldots, 1/\overline{\beta}_k\}$. In other words, when shifting the recurrence coefficients, the poles are shifted too. Defining $\mathcal{L}_{n\backslash k}$ by

$$\mathcal{L}_{(k-1)\backslash k} = \{0\}, \qquad \mathcal{L}_{k\backslash k} = \mathbb{C}, \qquad \mathcal{L}_{n\backslash k} = \mathcal{L}\{\beta_{k+1}, \ldots, \beta_n\},$$

we have that $\phi^{(k)}_{n\backslash k}, \psi^{(k)}_{n\backslash k} \in \mathcal{L}_{n\backslash k} \setminus \mathcal{L}_{(n-1)\backslash k}$.

As an application of Theorems 8 and 10, we get an explicit representation of the ARFs and of the function to which they are orthogonal in Theorem 14. But first we need the following lemma.

Lemma 13. *Suppose that $\phi_k \in \mathcal{L}_k \setminus \{0\}$ such that $\phi_k \perp_F \mathcal{L}_{k-1}$, and let $\psi_k \in \mathcal{L}_k \setminus \{0\}$ denote the rational function of the second kind of ϕ_k. It then holds for every $z \in \mathbb{D}$ that*

$$\frac{|(\phi_k F + \psi_k)(z)|^2 - \left|(\phi_k^* F - \psi_k^*)(z)\right|^2}{\left|(\Phi_{k,-1} F + \Psi_{k,-1})(z)\right|^2} > 0.$$

Proof. First, note that Theorem 6 implies that

$$(\Phi_{k,\tau} F + \Psi_{k,\tau})(z) = \zeta_0(z) B_{k-1}(z)[g_k(z) + \tau \zeta_k(z) h_k(z)], \quad g_k + \tau \zeta_k h_k \in H(\mathbb{D}).$$

Moreover, from Theorem 7 it follows that

$$g_k(z) + \tau \zeta_k(z) h_k(z) \neq 0$$

for every $\tau \in \mathbb{T}$ and for every $z \in \mathbb{D}$. Therefore, we have that either

$$G(z) := \frac{|g_k(z)|^2 - |\zeta_k(z) h_k(z)|^2}{|g_k(z) - \zeta_k(z) h_k(z)|^2} > 0 \quad \text{for every } z \in \mathbb{D}, \tag{27}$$

or

$$G(z) < 0 \quad \text{for every } z \in \mathbb{D}.$$

However, the second option is not possible because for $z = \beta_k \in \mathbb{D}$ we get $G(\beta_k) = 1 > 0$.[5] The statement now follows by multiplying the numerator and denominator in (27) with $|\zeta_0(z) B_{k-1}(z)|^2$. $\square$

[5] From (27) it follows that $|g_k(z)|^2 > |\zeta_k(z) h_k(z)|^2 \geqslant 0$ for every $z \in \mathbb{D}$, which proves that in Theorem 6(2), $g_k(z) \neq 0$ for every $z \in \mathbb{D}$.

Theorem 14. *For $n \geqslant k \geqslant 0$, suppose $\phi_n \in \mathcal{L}_n \setminus \{0\}$ and $\phi_n \perp_F \mathcal{L}_{n-1}$, and let ψ_n denote the rational function of the second kind of ϕ_n. Then, there exist constants $c_{n,k} \in \mathbb{R}_0$ such that the ARFs $\phi_{n\backslash k}^{(k)}$ and $\psi_{n\backslash k}^{(k)}$ are given by*

$$\begin{pmatrix} \phi_{n\backslash k}^{(k)}(z) & \psi_{n\backslash k}^{(k)}(z) \\ \phi_{n\backslash k}^{(k)*}(z) & -\psi_{n\backslash k}^{(k)*}(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \psi_n(z) \\ \phi_n^*(z) & -\psi_n^*(z) \end{pmatrix} \begin{pmatrix} \Psi_{k,-1}(z) & -\Psi_{k,1}(z) \\ -\Phi_{k,-1}(z) & \Phi_{k,1}(z) \end{pmatrix} \times \left\{ c_{n,k} P_k(z) B_k(z) \right\}^{-1}. \tag{28}$$

Further, $\phi_{n\backslash k}^{(k)}$ (respectively $\psi_{n\backslash k}^{(k)}$) are orthogonal with respect to the C-function $F^{(k)}$ (respectively $1/F^{(k)}$), given by

$$F^{(k)}(z) = \frac{\Phi_{k,1}(z)F(z) + \Psi_{k,1}(z)}{\Phi_{k,-1}(z)F(z) + \Psi_{k,-1}(z)}, \tag{29}$$

with $F^{(k)}(\beta_k) = 1$. In the special case in which for every $n \geqslant 0$ it holds that

$$c_{n,k}^2 = d_k d_n, \tag{30}$$

where d_j is the constant defined in Theorem 5, then $\phi_{n\backslash k}^{(k)}$ and $\psi_{n\backslash k}^{(k)}$ are orthonormal.

Proof. First, put $\tilde{\beta}_0 = \beta_k$. Since $\Phi_{k,\tau}^* = \overline{\tau}\,\Phi_{k,\tau}$ and $\Psi_{k,\tau}^* = -\overline{\tau}\,\Psi_{k,\tau}$, condition (15) is satisfied by $A = \Psi_{k,-1}$, $B = -\Phi_{k,-1}$, $C = -\Psi_{k,1}$ and $D = \Phi_{k,1}$ with $\tau_A = 1$.

Next, Theorems 5 and 7 imply that

$$\begin{aligned} (A - BF)(z) &= \left(\Psi_{k,-1} + \Phi_{k,-1}F\right)(z) \\ &= \zeta_0(z)B_{k-1}(z)g_k(z), \quad g_k \in H(\mathbb{D}), \end{aligned} \tag{31}$$

$$\begin{aligned} (C - DF)(z) &= -\left(\Psi_{k,1} + \Phi_{k,1}F\right)(z) \\ &= \zeta_0(z)B_{k-1}(z)h_k(z), \quad h_k \in H(\mathbb{D}), \end{aligned} \tag{32}$$

with $g_k(z) \neq 0$ and $h_k(z) \neq 0$ for every $z \in \mathbb{D}$, and

$$\begin{aligned} \left(\Phi_{k,1}\Psi_{k,-1} - \Phi_{k,-1}\Psi_{k,1}\right)(z) &= \left\{(\phi_k + \phi_k^*)(\psi_k + \psi_k^*) - (\phi_k - \phi_k^*)(\psi_k - \psi_k^*)\right\}(z) \\ &= 2\left(\phi_k^*\psi_k + \phi_k\psi_k^*\right)(z) = \zeta_0(z)B_{k-1}(z)\frac{\hat{d}_k\varpi_0^2(z)}{\varpi_k^2(z)}, \quad \hat{d}_k \in \mathbb{R}_0. \end{aligned}$$

Hence, conditions (22)–(24) are satisfied too. Further, from (28) and Theorem 5,

$$\phi_{k\backslash k}^{(k)} = \psi_{k\backslash k}^{(k)} = \frac{d_k}{c_{k,k}} \neq 0.$$

Consequently, Corollary 9 and Definition 12 show that the rational functions $\phi_{n\backslash k}^{(k)}$ and $\psi_{n\backslash k}^{(k)}$, defined by (28), are the ARFs of order k of $\{\phi_n\}$ and $\{\psi_n\}$ respectively. Moreover, Theorem 4 ensures that $\phi_{n\backslash k}^{(k)}$ are orthogonal with respect to a C-function $F^{(k)}$, with

$$F^{(k)}(\beta_k) = \frac{\psi_{k\backslash k}^{(k)}}{\phi_{k\backslash k}^{(k)}} = 1.$$

Note that, for $\tilde{F}$ given by (25), we also have that

$$\tilde{F}(\beta_k) = \left\{ \frac{(\phi_k F + \psi_k) + (\phi_k^* F - \psi_k^*)}{(\phi_k F + \psi_k) - (\phi_k^* F - \psi_k^*)} \right\} (\beta_k) = 1,$$

where the last equality follows from the fact that $(\phi_k^* F - \psi_k^*)(\beta_k) = 0$ (see Theorem 6). Further, from (31) and (32) we get

$$\tilde{F}(z) = -\frac{h_k(z)}{g_k(z)} \in H(\mathbb{D}).$$

Furthermore,

$$\Re\{\tilde{F}(z)\} = \frac{|(\phi_k F + \psi_k)(z)|^2 - \left|(\phi_k^* F - \psi_k^*)(z)\right|^2}{\left|(\Phi_{k,-1} F + \Psi_{k,-1})(z)\right|^2} > 0, \quad z \in \mathbb{D},$$

due to Lemma 13. Therefore, $\tilde{F}$ is a C-function, and hence, the equality for $F^{(k)}$ in (29) follows from Theorem 10.

Finally, with $c_{n,k}$ given by (30), it holds for $n = k$ that $\phi_{k\backslash k}^{(k)} = \psi_{k\backslash k}^{(k)} = 1$, while, for $n > k$, we deduce from (6) and (21) that

$$\left[e_{n\backslash k}^{(k)}\right]^2 = \frac{c_{n-1,k}^2}{c_{n,k}^2} e_n^2 = \frac{d_{n-1}}{d_n} e_n^2 = \frac{\varpi_n(\beta_n)}{\varpi_{n-1}(\beta_{n-1})} \cdot \frac{1}{1 - |\lambda_n|^2},$$

where we have applied (11) in the last equality. Then, the orthonormality is a consequence of (7) in Theorem 4(2). □

Based on the previous theorem, the following relations between ARFs of different orders can be proved.

Corollary 15. *For $0 \leqslant j \leqslant k \leqslant n$, let $K_{n,k}^{(j)}$ be defined by*

$$K_{n,k}^{(j)} = \frac{d_{k\backslash j}^{(j)}}{c_{(n\backslash j),(k\backslash j)}^{(j)}}.$$

Then, the following relations hold:

$$\text{(a) } 2K_{n,k}^{(j)} \phi_{n\backslash j}^{(j)}(z) = \left[\left(\phi_{k\backslash j}^{(j)} + \phi_{k\backslash j}^{(j)*}\right) \phi_{n\backslash k}^{(k)} + \left(\phi_{k\backslash j}^{(j)} - \phi_{k\backslash j}^{(j)*}\right) \psi_{n\backslash k}^{(k)}\right](z) \tag{33}$$

$$\text{(b) } 2K_{n,k}^{(j)} \phi_{n\backslash j}^{(j)}(z) = \left[\left(\phi_{n\backslash k}^{(k)} + \psi_{n\backslash k}^{(k)}\right) \phi_{k\backslash j}^{(j)} + \left(\phi_{n\backslash k}^{(k)} - \psi_{n\backslash k}^{(k)}\right) \phi_{k\backslash j}^{(j)*}\right](z) \tag{34}$$

$$\begin{aligned} \text{(c) } & 2\frac{d_{n\backslash j}^{(j)}}{d_{k\backslash j}^{(j)}} K_{n,k}^{(j)} P_{n\backslash k}(z) B_{n\backslash k}(z) \phi_{k\backslash j}^{(j)}(z) \\ & = \left[\left(\psi_{n\backslash k}^{(k)*} + \phi_{n\backslash k}^{(k)*}\right) \phi_{n\backslash j}^{(j)} + \left(\psi_{n\backslash k}^{(k)} - \phi_{n\backslash k}^{(k)}\right) \phi_{n\backslash j}^{(j)*}\right](z), \end{aligned} \tag{35}$$

where $P_{n\backslash k}(z) = \frac{P_n(z)}{P_k(z)}$ and $B_{n\backslash k}(z) = \frac{B_n(z)}{B_k(z)}$.

In the special case in which all the involved ARFs are orthonormal, it holds that $K_{n,k}^{(j)} = 1$. Further, the corresponding relations for $\psi_{n\backslash j}^{(j)}$ are obtained by replacing ϕ with ψ in (33)–(35), and vice versa.

Proof. It suffices to prove the relations for $j = 0$. Relation (33) follows immediately from (28), with the help of the identity

$$\begin{aligned}\left[(\Phi_{k,1}\Psi_{k,-1} - \Phi_{k,-1}\Psi_{k,1})\right](z) &= \left[(\phi_k + \phi_k^*)(\psi_k + \psi_k^*) - (\phi_k - \phi_k^*)(\psi_k - \psi_k^*)\right](z)\\ &= 2d_k P_k(z) B_k(z).\end{aligned}$$

Next, note that (34) is just a reformulation of (33).

Finally, from (28) and (10), we get

$$\left(\phi_{n\backslash k}^{(k)}\phi_n^* - \phi_{n\backslash k}^{(k)*}\phi_n\right)(z) = \frac{d_n}{c_{n,k}} P_{n\backslash k}(z) B_{n\backslash k}(z)\left[\phi_k^*(z) - \phi_k(z)\right]$$

$$\left(\psi_{n\backslash k}^{(k)}\phi_n^* + \psi_{n\backslash k}^{(k)*}\phi_n\right)(z) = \frac{d_n}{c_{n,k}} P_{n\backslash k}(z) B_{n\backslash k}(z)\left[\phi_k(z) + \phi_k^*(z)\right].$$

Relation (35) now follows immediately by subtraction. □

6. Examples

In this section we will illustrate the preceding results with some examples. We will consider the orthonormal rational functions with respect to the Lebesgue measure

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{2\pi\mathbf{i}z} = \frac{\mathrm{d}\theta}{2\pi}, \quad z = \mathrm{e}^{\mathbf{i}\theta},$$

and poles defined by $\beta_0, \beta_1, \beta_2, \beta_3, \ldots$ with $\beta_0 = 0$. It is very well known that in this case the parameters λ_n of recurrence (5) are zero for all $n > 0$, so that

$$\phi_n(z) = \sqrt{\varpi_n(\beta_n)}\frac{z}{\varpi_n^*(z)} B_n(z) = \psi_n(z), \qquad \phi_n^*(z) = \sqrt{\varpi_n(\beta_n)}\frac{1}{\varpi_n(z)} = \psi_n^*(z),$$

where we have used the notation of the previous sections. The corresponding Carathéodory function is

$$F(z) = \int_{\mathbb{T}} \frac{\zeta_0(t) + \zeta_0(z)}{\zeta_0(t) - \zeta_0(z)}\,\mathrm{d}\mu(t) = \int_{\mathbb{T}} \frac{t+z}{t-z}\,\mathrm{d}\mu(t) = 1.$$

Example 16. The ARFs of order 1 are obtained when $r = 0$, $\tau_A = 1$, $N = 1$, and $\tilde{\beta}_0 = \beta_1$, thus the related poles are defined by $\beta_1, \beta_2, \beta_3, \ldots$. By Theorem 14,

$$\begin{aligned}A(z) &= \psi_1^*(z) + \psi_1(z) = \phi_1^*(z) + \phi_1(z) = D(z),\\ B(z) &= \phi_1^*(z) - \phi_1(z) = \psi_1^*(z) - \psi_1(z) = C(z),\end{aligned}$$

so that the orthonormal ARFs of order 1 and the functions of the second kind are given by

$$\begin{aligned}\phi_{n\backslash 1}^{(1)}(z) = \psi_{n\backslash 1}^{(1)}(z) &= \frac{1}{2P_1(z)B_1(z)}\left[A(z)\phi_n(z) + B(z)\psi_n(z)\right]\\ &= \frac{1}{2P_1(z)B_1(z)}\left[\left(\phi_1^*(z) + \phi_1(z)\right)\phi_n(z) + \left(\phi_1^*(z) - \phi_1(z)\right)\phi_n(z)\right]\\ &= \sqrt{\frac{\varpi_n(\beta_n)}{\varpi_1(\beta_1)}}\frac{\varpi_1^*(z)}{\varpi_n^*(z)} B_{n\backslash 1}(z).\end{aligned}$$

The corresponding Carathéodory function is again

$$F^{(1)}(z) = \frac{-C(z) + D(z)F(z)}{A(z) - B(z)F(z)} = 1,$$

but the orthogonality measure $\mathrm{d}\mu^{(1)}$ is not the Lebesgue measure because it must satisfy now

$$\int_{\mathbb{T}} \frac{\zeta_1(t) + \zeta_1(z)}{\zeta_1(t) - \zeta_1(z)} \, \mathrm{d}\mu^{(1)}(t) = F^{(1)}(z) = 1.$$

Taking into account that

$$\frac{\zeta_1(t) + \zeta_1(z)}{\zeta_1(t) - \zeta_1(z)} = \frac{\varpi_1^*(t)\varpi_1(z) + \varpi_1^*(z)\varpi_1(t)}{\varpi_1(\beta_1)(t - z)}$$

it is easy to see that

$$\mathrm{d}\mu^{(1)}(z) = \frac{\varpi_1(\beta_1)}{\varpi_1(z)\varpi_1^*(z)} \frac{\mathrm{d}z}{2\pi \mathbf{i}} = \frac{\varpi_1(\beta_1)}{|\mathrm{e}^{\mathbf{i}\theta} - \beta_1|^2} \frac{\mathrm{d}\theta}{2\pi}, \quad z = \mathrm{e}^{\mathbf{i}\theta},$$

which is a rational modification of the Lebesgue measure.

Next, consider the sequences of complex numbers $\mathcal{B} = \{\beta_0, \beta_1, \beta_2, \ldots,\} \subset \mathbb{D}$ and $\tilde{\mathcal{B}} = \{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \ldots\} \subset \mathbb{D}$, with $\tilde{\beta}_{r+k} = \beta_{N+k}$ for fixed $N, r \geq 0$ and for $k = 0, 1, 2, \ldots$. Let $\Phi_{k,\tau}$ and $\Psi_{k,\tau}$ denote the para-ORFs and their functions of the second kind with respect to the C-function F and sequence $\mathcal{B}$. Similarly, let $\tilde{\Phi}_{k,\tau}$ and $\tilde{\Psi}_{k,\tau}$ denote the para-ORFs and their functions of the second kind with respect to the C-function $\tilde{F}$ and sequence $\tilde{\mathcal{B}}$. From Theorem 14 it then follows that

$$F^{(N)}(z) = \frac{\Phi_{N,1}(z)F(z) + \Psi_{N,1}(z)}{\Phi_{N,-1}(z)F(z) + \Psi_{N,-1}(z)}, \quad \text{and} \quad \tilde{F}^{(r)}(z) = \frac{\tilde{\Phi}_{r,1}(z)\tilde{F}(z) + \tilde{\Psi}_{r,1}(z)}{\tilde{\Phi}_{r,-1}(z)\tilde{F}(z) + \tilde{\Psi}_{r,-1}(z)}.$$

Supposing that F is known, as well as the corresponding (para-)ORFs and functions of the second kind for the sequence $\mathcal{B}$, our goal is to find $\tilde{F}$ and the corresponding ORFs and functions of the second kind for the sequence $\tilde{\mathcal{B}}$ such that the ARFs of order $K \geqslant N$ for F and the ARFs of order $L \geqslant r$ for $\tilde{F}$ are the same (because they have the same recurrence coefficients and initial values in the recurrence relation (5)). Hence, it should hold that $F^{(N)}(z) \equiv \tilde{F}^{(r)}(z)$. This can only be the case for

$$\tilde{F}(z) = \frac{(-C + DF)(z)}{(A - BF)(z)},$$

with

$$\begin{aligned} A(z) &= k \left\{ \tilde{\Phi}_{r,1} \Psi_{N,-1} - \tilde{\Phi}_{r,-1} \Psi_{N,1} \right\}(z) \\ B(z) &= k \left\{ \tilde{\Phi}_{r,-1} \Phi_{N,1} - \tilde{\Phi}_{r,1} \Phi_{N,-1} \right\}(z) \\ C(z) &= k \left\{ \tilde{\Psi}_{r,1} \Psi_{N,-1} - \tilde{\Psi}_{r,-1} \Psi_{N,1} \right\}(z) \\ D(z) &= k \left\{ \tilde{\Psi}_{r,-1} \Phi_{N,1} - \tilde{\Psi}_{r,1} \Phi_{N,-1} \right\}(z), \end{aligned} \tag{36}$$

and $k \in \mathbb{C}_0$. Note that for $r = 0$, it holds that $\tilde{\Phi}_{r,1}(z) = \tilde{\Psi}_{r,-1}(z) \equiv 2$ and $\tilde{\Phi}_{r,-1}(z) = \tilde{\Psi}_{r,1}(z) \equiv 0$, so that $F^{(N)}(z) = \tilde{F}^{(0)}(z) = \tilde{F}(z)$, and the ARFs of order N are indeed obtained for $k = \frac{1}{2}$.

Example 17. Let $\phi_j(z)$ and $\psi_j(z)$ denote the ORFs and functions of the second kind corresponding to the sequence $\mathcal{B}$ and measure $\frac{\varpi_0(\beta_0)}{|e^{i\theta}-\beta_0|^2}\frac{d\theta}{2\pi}$, i.e.,

$$\phi_j(z) = \psi_j(z) = \sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{\varpi_0^*(z)B_j(z)}{\varpi_j^*(z)}, \qquad \phi_j^*(z) = \psi_j^*(z) = \sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{\varpi_0(z)}{\varpi_j(z)},$$

and let $\tilde{\varphi}_j(z)$ and $\tilde{\chi}_j(z)$ denote the ORFs and functions of the second kind corresponding to the sequence $\tilde{\mathcal{B}}$ and measure $\frac{\tilde{\varpi}_0(\tilde{\beta}_0)}{|e^{i\theta}-\tilde{\beta}_0|^2}\frac{d\theta}{2\pi}$, i.e.,

$$\tilde{\varphi}_j(z) = \tilde{\chi}_j(z) = \sqrt{\frac{\tilde{\varpi}_j(\tilde{\beta}_j)}{\tilde{\varpi}_0(\tilde{\beta}_0)}}\frac{\tilde{\varpi}_0^*(z)\tilde{B}_j(z)}{\tilde{\varpi}_j^*(z)}, \qquad \tilde{\varphi}_j^*(z) = \tilde{\chi}_j^*(z) = \sqrt{\frac{\tilde{\varpi}_j(\tilde{\beta}_j)}{\tilde{\varpi}_0(\tilde{\beta}_0)}}\frac{\tilde{\varpi}_0(z)}{\tilde{\varpi}_j(z)}.$$

From (18) and (36), with $k = \frac{1}{2}$, we then obtain the following relation between both families of ORFs:

$$\tilde{\varphi}_{r+n}(z) = \tilde{\chi}_{r+n}(z) = \frac{A(z)\phi_{N+n}(z) + B(z)\psi_{N+n}(z)}{2P_N(z)B_N(z)},$$

with

$$A(z) = \tilde{\varphi}_r(z)\psi_N^*(z) + \tilde{\varphi}_r^*(z)\psi_N(z) = \tilde{\chi}_r(z)\phi_N^*(z) + \tilde{\chi}_r^*(z)\phi_N(z) = D(z),$$

and

$$B(z) = \tilde{\varphi}_r(z)\phi_N^*(z) - \tilde{\varphi}_r^*(z)\phi_N(z) = \tilde{\chi}_r(z)\psi_N^*(z) - \tilde{\chi}_r^*(z)\psi_N(z) = C(z).$$

Example 18. Consider the case in which $r = 1$ and $N = 0$. Let $\phi_j(z)$ and $\psi_j(z)$ be defined as above in the previous example, with $\beta_0 = \tilde{\beta}_1$, and let $\tilde{\varphi}_j(z)$ and $\tilde{\chi}_j(z)$ denote the ORFs and functions of the second kind corresponding to the C-function

$$\tilde{F}(z) = \frac{1 + \overline{a}z}{1 - \overline{a}z}, \quad a \in \mathbb{D},$$

and sequence $\tilde{\mathcal{B}}$, with $\tilde{\beta}_0 = 0$. From (36) it then follows that $\tilde{F}(z) = \frac{\tilde{\chi}_1^*(z)}{\tilde{\varphi}_1^*(z)}$; hence,

$$\tilde{\varphi}_1(z) = K\frac{(z-a)\tilde{\zeta}_1(z)}{\tilde{\varpi}_1^*(z)} \quad \text{and} \quad \tilde{\chi}_1^*(z) = K\frac{(z+a)\tilde{\zeta}_1(z)}{\tilde{\varpi}_1^*(z)}, \quad K \in \mathbb{C}_0.$$

Next, by means of (18) and (36), with $k = \frac{1}{2}$, we find for $n \geqslant 0$ that

$$\begin{aligned}
\tilde{\varphi}_{1+n}(z) &= \frac{1}{2}\left[(\tilde{\varphi}_1 + \tilde{\varphi}_1^*)\phi_n + (\tilde{\varphi}_1 - \tilde{\varphi}_1^*)\psi_n\right](z) \\
&= \tilde{\varphi}_1\phi_n = K\frac{(z-a)\tilde{\zeta}_1(z)}{\tilde{\varpi}_1^*(z)}\sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{\varpi_0^*(z)B_j(z)}{\varpi_j^*(z)} \\
&= K\sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{(z-a)\zeta_0(z)B_j(z)}{\varpi_j^*(z)},
\end{aligned}$$

and

$$\begin{aligned}\tilde{\chi}_{1+n}(z) &= \frac{1}{2}\left[(\tilde{\chi}_1 - \tilde{\chi}_1^*)\phi_n + (\tilde{\chi}_1 + \tilde{\chi}_1^*)\psi_n\right](z) \\ &= \tilde{\chi}_1\phi_n = K\frac{(z+a)\tilde{\zeta}_1(z)}{\tilde{\varpi}_1^*(z)}\sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{\varpi_0^*(z)B_j(z)}{\varpi_j^*(z)} \\ &= K\sqrt{\frac{\varpi_j(\beta_j)}{\varpi_0(\beta_0)}}\frac{(z+a)\zeta_0(z)B_j(z)}{\varpi_j^*(z)}.\end{aligned}$$

Finally, with $K = \sqrt{\frac{\varpi_0(\beta_0)}{(1-|a|^2)}}$, it follows from the previous examples that the rational functions $\tilde{\varphi}_j$ and $\tilde{\chi}_j$ are ORFs corresponding to the sequence $\{a, \beta_0, \beta_1, \ldots\}$ and measures $\frac{1-|a|^2}{|e^{\mathbf{i}\theta}-a|^2}\frac{d\theta}{2\pi}$ and $\frac{1-|a|^2}{|e^{\mathbf{i}\theta}+a|^2}\frac{d\theta}{2\pi}$ respectively. Indeed, it holds that

$$\int_{\mathbb{T}}\frac{t+z}{t-z}\frac{(1-|a|^2)}{(t-a)(1-\overline{a}t)}\frac{dt}{2\pi\mathbf{i}} = \frac{1+\overline{a}z}{1-\overline{a}z} = \tilde{F}(z)$$

and

$$\int_{\mathbb{T}}\frac{t+z}{t-z}\frac{(1-|a|^2)}{(t+a)(1+\overline{a}t)}\frac{dt}{2\pi\mathbf{i}} = \frac{1-\overline{a}z}{1+\overline{a}z} = 1/\tilde{F}(z).$$

Acknowledgments

The first author is a Postdoctoral Fellow of the Research Foundation—Flanders (FWO). The work of this author is partially supported by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors.

Most of this work was done during a recent stay of the first author at the University of Zaragoza, Spain. This author is very grateful to Professor María José Cantero, Professor Leandro Moral, and Professor Luis Velázquez, for their hospitality at the University of Zaragoza, and for the valuable discussion during the development of this work.

The work of the second, third and fourth authors is partially supported by a research grant from the Ministry of Science and Innovation of Spain, project code MTM20008-06689-C02, and by Project E-64 of Diputacíon General de Aragón (Spain).

References

[1] W. Blaschke, Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen, Berichteüber die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse 67 (1915) 194–200.

[2] A. Bultheel, M.J. Cantero, A matrical computation of rational quadrature formulas on the unit circle, Numerical Algorithms 52 (1) (2009) 47–68.

[3] A. Bultheel, R. Cruz-Barroso, K. Deckers, P. González-Vera, Rational Szegő quadratures associated with Chebyshev weight functions, Mathematics of Computation 78 (2009) 1031–1059.

[4] A. Bultheel, P. González Vera, E. Hendriksen, O. Njåstad, Orthogonal Rational Functions, in: Cambridge Monographs on Applied and Computational Mathematics, vol. 5, Cambridge University Press, Cambridge, 1999.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Quadrature and orthogonal rational functions, Journal of Computational and Applied Mathematics 127 (1–2) (2001) 67–91. Invited Paper.

[6] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions and quadrature on the real half line, Journal of Complexity 19 (3) (2003) 212–230.
[7] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions on the real half line with poles in $[-\infty, 0]$, Journal of Computational and Applied Mathematics 179 (1–2) (2005) 121–155.
[8] K. Deckers, A. Bultheel, Rational Krylov sequences and orthogonal rational functions, Technical Report TW499, Department of Computer Science, K.U. Leuven, August 2007.
[9] K. Deckers, A. Bultheel, Associated rational functions based on a three-term recurrence relation for orthogonal rational functions, IAENG International Journal of Applied Mathematics 38 (4) (2008) 214–222.
[10] K. Deckers, A. Bultheel, Orthogonal rational functions, associated rational functions and functions of the second kind, in: S.I. Ao, L. Gelman, D.W.L. Hukins, A. Hunter, A.M. Korsunsky (Eds.), Proceedings of the World Congress on Engineering 2008, in: Lecture Notes in Engineering and Computer Science, vol. 2, Newswood Limited, International Association of Engineers, London, 2008, pp. 838–843.
[11] K. Deckers, A. Bultheel, Orthogonal rational functions and rational modifications of a measure on the unit circle, Journal of Approximation Theory 157 (1) (2009) 1–18.
[12] K. Deckers, A. Bultheel, Orthogonal rational functions with complex poles: the Favard theorem, Journal of Mathematical Analysis and Applications 356 (2) (2009) 764–768.
[13] K. Deckers, A. Bultheel, Recurrence and asymptotics for orthogonal rational functions on an interval, IMA Journal of Numerical Analysis 29 (1) (2009) 1–23.
[14] K. Deckers, J. Van Deun, A. Bultheel, Computing rational Gauss–Chebyshev quadrature formulas with complex poles, in: B.H.V. Topping, G. Montero, R. Montenegro (Eds.), Proceedings of the Fifth International Conference on Engineering Computational Technology, Civil-Comp. Press, Stirlingshire, 2006, Paper 30.
[15] K. Deckers, J. Van Deun, A. Bultheel, An extended relation between orthogonal rational functions on the unit circle and the interval $[-1, 1]$, Journal of Mathematical Analysis and Applications 334 (2) (2007) 1260–1275.
[16] K. Deckers, J. Van Deun, A. Bultheel, Rational Gauss–Chebyshev quadrature formulas for complex poles outside $[-1, 1]$, Mathematics of Computation 77 (262) (2008) 967–983.
[17] K. Deckers, J. Van Deun, A. Bultheel, Computing rational Gauss–Chebyshev quadrature formulas with complex poles: the algorithm, Advances in Engineering Software 40 (8) (2009) 707–717.
[18] M.M. Džrbašian, A survey on the theory of orthogonal systems and some open problems, in: P. Nevai (Ed.), Orthogonal Polynomials: Theory and Practice, in: NATO-ASI Series C: Mathematical and Physical Sciences, vol. 294, Kluwer Academic Publishers, Boston, 1990, pp. 135–146.
[19] B. Fritzsche, B. Kirstein, A. Lasarow, Orthogonal rational matrix-valued functions on the unit circle, Mathematische Nachrichten 278 (5) (2005) 525–553.
[20] B. Fritzsche, B. Kirstein, A. Lasarow, Orthogonal rational matrix-valued functions on the unit circle: recurrence relations and a Favard-type theorem, Mathematische Nachrichten 279 (5–6) (2006) 513–542.
[21] K. Müller, A. Bultheel, Translation of the Russian paper "Orthogonal systems of rational functions on the unit circle" by M.M. Džrbašian, Technical Report TW253, Department of Computer Science, K.U. Leuven, February 1997.
[22] F. Peherstorfer, A special class of polynomials orthogonal on the unit circle including the associated polynomials, Constructive Approximation 12 (1996) 161–185.
[23] F. Peherstorfer, R. Steinbauer, Characterization of orthogonal polynomials with respect to a functional, in: Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions, Delft, 1994, J. Comput. Appl. Math. 65 (1–3) (1995) 339–355.
[24] W. Van Assche, Orthogonal polynomials, associated polynomials and functions of the second kind, Journal of Computational and Applied Mathematics 37 (1–3) (1991) 237–249.
[25] J. Van Deun, A. Bultheel, Orthogonal rational functions on an interval, Technical Report TW322, Department of Computer Science, K.U. Leuven, March 2001.
[26] J. Van Deun, A. Bultheel, Orthogonal rational functions and quadrature on an interval, Journal of Computational and Applied Mathematics 153 (1–2) (2003) 487–495.
[27] J. Van Deun, A. Bultheel, Ratio asymptotics for orthogonal rational functions on the interval $[-1, 1]$, Journal of Approximation Theory 123 (2) (2003) 162–172.
[28] J. Van Deun, A. Bultheel, An interpolation algorithm for orthogonal rational functions, Journal of Computational and Applied Mathematics 164–165 (2004) 749–762.
[29] J. Van Deun, A. Bultheel, Modified moments and orthogonal rational functions, Applied Numerical Analysis and Computational Mathematics 1 (3) (2004) 455–468.
[30] J. Van Deun, A. Bultheel, Computing orthonormal rational functions on a halfline, in: Rendiconti del Circolo di Palermo Serie II, Proceedings of the 5th International Conference on Functional Analysis and Approximation Theory, FAAT, vol. 76, Maratea, 16–22 June, 2005, pp. 621–634.

[31] J. Van Deun, A. Bultheel, On the construction of a rational Fejér rule, Technical Report TW438, Department of Computer Science, K.U. Leuven, October 2005.

[32] J. Van Deun, A. Bultheel, P. González-Vera, Computing orthogonal rational functions with poles near the boundary, Computers & Mathematics with Applications 53 (9) (2007) 1421–1428.

[33] J. Van Deun, K. Deckers, A. Bultheel, J.A.C. Weideman, Algorithm 882: near best fixed pole rational interpolation with applications in spectral methods, ACM Transactions on Mathematical Software 35 (2) (2008) 14:1–14:21.

[34] P. Van gucht, A. Bultheel, Using orthogonal rational functions for system identification, IEEE Transactions on Automatic Control 48 (4) (2003) 705–709.

[35] L. Velázquez, Spectral methods for orthogonal rational functions, Journal of Functional Analysis 254 (4) (2008) 954–986.