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On properties of the n-dimensional norm residue symbol in higher local class field theory

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Abstract

Let k be a finite extension of Q_p which contains the roots of unity μ. Here $|\mu| = q = p^n$, $p \neq 2$. We consider an n-dimensional local field given explicitly as a power series in $n-1$ variables by $X_n = k\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$. The norm residue symbol has been generalized by Vostokov for mixed characteristic local fields $X = \{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ of dimension n. It is a non-degenerate pairing given by

$$\frac{K_n(X)}{(K_n(X))^q} \times \frac{X^*}{(X^*)^q} \to \mu$$

where $K_n(X)$ is the nth Milnor K-group of $X = X_n$ and X^* is the multiplicative group of X. It is shown here that the Vostokov pairing on the n-dimensional local field $X_n = k\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ commutes with the Vostokov pairing on the $n-1$ dimensional local field $X_{n-1} = k\{\{t_1\}\}\dots\{\{t_{n-2}\}\}$. We achieve this by constructing a map M which projects the roots of unity from the pairing on $X = X_n$ onto the roots of unity from the pairing on X_{n-1}.

Introduction

The first results in higher dimensional local class field theory were obtained by Parshin in [4, 5]. Independently Kato [1–3] obtained similar results. For X, an n-dimensional local field of characteristic zero, Kato proves the existence of a homomorphism

$$/ : K_n(X) \to G(X^{\mathrm{ab}}/X)$$

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from $K_n(X)$, the nth Milnor K-group of X, into the Galois group of the maximal abelian extension X^{ab} over X. When $n = 1$, $K_1(X) = X^*$ and we recover local class field theory.

The norm residue symbol has been generalized to n-dimensional local fields by Vostokov [7, 8]. There the existence of a skew-symmetric pairing on the $n+1$-fold product of an n-dimensional local field X is shown. This pairing descends to give a non-degenerate pairing on

$$\frac{K_n(X)}{(K_n(X))^q} \times \frac{X^*}{(X^*)^q},$$

where here X^* is the multiplicative group of X and q is the order of the roots of unity contained in X.

This note will concern the norm residue symbol. Let X be a higher local field in the sense of Parshin [6], i.e., let k be a finite extension of Q_p and $k\{\{t\}\} = \sum_i a_i t^i$ where the a_i's are uniformly bounded and the limit of a_i as $i \to -\infty$ is zero. We consider the n-dimensional local field given by the power series $X = k\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}$. The valuation of $X = X_n$ is given in the following way. Observe any $x \in X$ can be explicitly written as a power series in t_i with coefficients $a \in k$. Let v_k be the standard valuation on k. We then define $\bar{v}_k$ on X as $\bar{v}_k(x) = \inf\{v_k(a) | a$ is a coefficient in the power series expansion of $X\}$. Then X is complete with respect to $\bar{v}_k$. In the case $k = Q_p$ we have $v_k = |\,|_p$ and the residue field of $X = X_n$ is $\bar{X}_n = Z/pZ\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}$. We aim to show the following:

Theorem. *Let $X = X_n$ be an n-dimensional higher local field containing the roots of unity μ of order $q = p^u$ (assume $p \neq 2$). Let $K_n(X_n)$ be the nth Milnor K-group of X_n.*

The diagram below

$$\begin{array}{ccccc}
\dfrac{K_n(X_n)}{(K_n(X_n))^q} & \times & \dfrac{X_n^*}{(X_n^*)^q} & \xrightarrow{(\,,\,)} & \mu \\
\Big\downarrow \partial & & \Big\uparrow i & & \Big\downarrow M \\
\dfrac{K_{n-1}(X_{n-1})}{(K_{n-1}(X_{n-1}))^q} & \times & \dfrac{X_{n-1}^*}{(X_{n-1}^*)^q} & \xrightarrow{(\,,\,)} & \mu
\end{array}$$

is commutative.

Here ∂ is the tame symbol defined as $\partial(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \pi) = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n-1})$ where π is the prime ideal $\pi = (t_{n-1})$ and the α_i are units. Here i is the injection of $X_{n-1} = k\{\{t_1\}\}\ldots\{\{t_{n-2}\}\}$ into $X_n = k\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}$ which sends $k \to k$ and $t_i \to t_i$ for $1 \leq i \leq n-2$. M is a map on the roots of unity μ which will be defined later. In essence we will show for $a \in K_n(X_n)/(K_n(X_n))^q$ and $b \in X_{n-1}^*/(X_{n-1}^*)^q$ we have $M(a, ib) = (\partial a, b)$ where here (,) is the Vostokov norm pairing.

Remark. The main ingredient in the proof is the construction of map M on the *qth*-roots of unity. Before stating M explicitly we review some facts concerning the n-dimensional norm residue symbol.

1. Preliminaries

Let k be a finite extension of Q_p and $X = k\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}$ be an n-dimensional local field as defined above. Again, we assume k contains the roots of unity μ of order $q = p^u$, $p \neq 2$. Let ζ be a generator of μ. Vostokov [7] has constructed the skew-symmetric map

$$\Gamma : X^* \times \cdots \times X^* \to \mu$$

$$\Gamma(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) = \zeta^{\mathrm{tr\ res}(\phi/s)}, \quad \alpha_i \in X^*,$$

with the property that $\alpha_i + \alpha_j = 1 \Leftrightarrow \Gamma(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) = 1$ for $i \neq j$. Here tr is the trace operator of the inertia subfield of k, s is determined in X by an expansion of ζ in X and $\phi(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ is given by an expansion of the α's in X. Here res is the residue of ϕ/s, i.e., the coefficient of $1/t_1 t_2 \cdots t_{n-1}$.

Vostokov goes on to show Γ then defines a non-degenerate pairing

$$\frac{K_n(X_n)}{(K_n(X_n))^q} \times \frac{X_n^*}{(X_n^*)^q} \xrightarrow{(\ ,\ )} \zeta^{\mathrm{tr\ res}(\phi/s)}$$

satisfying the norm property, i.e., $\Gamma(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) = 1 \Leftrightarrow \{\alpha_1, \ldots, \alpha_n\}$ in $K_n(X_n)$ is a norm in $K_n(X_n(q\sqrt{\alpha_{n+1}}))$. This property gives rise to its name as the n-dimensional norm residue symbol.

Now we give an explicit description of ϕ taken from Vostokov [7]. Recall Vostokov defines $\phi(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ as $\ell(\alpha_{n+1})D_{n+1} - \ell(\alpha_n)D_n \ldots (-1)^n \ell(\alpha_1)D_1$ where

$$D_i = \left\{ \begin{array}{c} \delta_1(\alpha_1)\delta_2(\alpha_1)\ldots\delta_n(\alpha_1) \\ \ldots \\ \delta_1(\alpha_{i-1})\delta_2(\alpha_{i-1})\ldots\delta_n(\alpha_{i-1}) \\ \eta_1(\alpha_{i+1})\eta_2(\alpha_{i+1})\ldots\eta_n(\alpha_{i+1}) \\ \eta_1(\alpha_{n+1})\eta_2(\alpha_{n+1})\ldots\eta_n(\alpha_{n+1}) \end{array} \right\}$$

such that

$$\ell(\alpha) = \frac{\log \alpha^{p-\triangle}}{p},$$

$$\delta_i(\alpha) = \frac{1}{\alpha}\frac{\partial \alpha}{\partial t_i}, \qquad 1 \leq i \leq n$$

and

$$\eta_i(\alpha) = \frac{1}{\alpha}\frac{\partial \alpha}{\partial t_i} - \frac{\partial \ell(\alpha)}{\partial t_i}.$$

Here $t_n = \mathscr{P}$, a uniformizer of k.

Here $\triangle$ is the Frobenius operator of the inertia field T of X. $\triangle$ acts by raising the t_i to the pth power and acts on coefficients via the usual Frobenius.

2. A lemma

We now prove a lemma which is the key step in proving the theorem.

Lemma. *Let* $X = X_n = k\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}$ *be an n-dimensional local field as above. Here* k *is a finite extension of* Q_p *which contains* μ *roots of unity of order* $q = p^u$, $p \neq 2$. *Let* μ *be generated by the element* ζ, *i.e.,* $\mu = \langle \zeta \rangle$.

The diagram below

$$\begin{array}{ccc}
(\alpha_1,\alpha_2,\ldots,\alpha_{n+1}) & \xrightarrow{\ \Gamma\ } & \zeta^{\mathrm{tr\ res}(\phi/s)} \\
\Big\downarrow {\scriptstyle \bmod t_{n-1} \circ pr} & & \Big\downarrow {\scriptstyle \bmod t_{n-1} \circ \bar{M} = M} \\
(\bar{\alpha}_1,\bar{\alpha}_2,\ldots,\bar{\alpha}_{n-1},\bar{\alpha}_{n+1}) & \xrightarrow{\ \Gamma\ } & \zeta^{\mathrm{tr\ res}(\phi/s)}
\end{array}$$

is commutative.

Here "mod t_{n-1}" means reducing $\alpha_i \in X$ by the generator t_{n-1}. pr is the projection map which kills α_n. $\bar{M}$ is a map defined on ϕ hence on $\zeta^{\mathrm{tr\ res}(\phi/s)}$.

We start our proof with the definition of $\bar{M}$.

Definition. $\bar{M}$ is defined on ϕ as follows:

(a) For $1 \leq i \leq n-1$ $\bar{M}D_i$ equals the negative of the $(n-1)\times(n-1)$ minor determinant corresponding to $\eta_{n-1}(\alpha_n)$. By taking this minor we eliminate all terms involving α_n and all derivatives with respect to t_{n-1}. $\bar{M}D_i$ is then the ith determinant involved in the expansion of ϕ for X_{n-1}.

(b) $\bar{M}D_n = 0$.

(c) $\bar{M}D_{n+1}$ equals the $(n-1)\times(n-1)$ minor corresponding to $\delta_{n-1}(\alpha_n)$. Again we have eliminated all terms involving α_n and all derivatives with respect to t_{n-1}. $\bar{M}D_{n+1}$ is then the nth determinant involved in the expansion of ϕ for X_{n-1}.

This definition of $\bar{M}$ gives

$$\begin{aligned}\bar{M}\phi &= \bar{M}(\ell(\alpha_{n+1})D_{n+1} - \ell(\alpha_n)D_n \ldots (-1)^n \ell(\alpha_1)D_1) \\ &= \ell(\alpha_{n+1})\bar{M}D_{n+1} - \ell(\alpha_{n-1})\bar{M}(D_{n-1})\ldots(-1)^n\ell(\alpha_1)\bar{M}(D_1).\end{aligned}$$

Proof of lemma. From the definition of $\overline{M}$ we see that $\ell(\alpha_i)\overline{M}(D_i(\alpha_1,\alpha_2,\dots,\alpha_{n+1}))$ equals $\ell(\bar{\alpha}_i)D_i(\bar{\alpha}_1,\bar{\alpha}_2,\dots,\bar{\alpha}_{n-1},\bar{\alpha}_{n+1})\mathrm{mod}(t_{n-1})$. From this it follows that $\overline{M}\phi(\alpha_1,\alpha_2,\dots,\alpha_{n+1}) \equiv \phi(\bar{\alpha}_1,\bar{\alpha}_2,\dots,\bar{\alpha}_{n-1},\bar{\alpha}_{n+1})\mathrm{mod}(t_{n-1})$ where $\phi(\bar{\alpha}_1,\bar{\alpha}_2,\dots,\bar{\alpha}_{n-1},\bar{\alpha}_{n+1})$ is the series defined by the Vostokov pairing on $k\{\{t_1\}\}\dots\{\{t_{n-2}\}\}$.

Now we consider the expansion s. Here $s = Z^q - 1$ where Z is an expansion of the qth root of unity contained in X_n. The expansion of Z is of the form $1 + h_1\mathscr{P} + h_2\mathscr{P}^2 + \cdots$ where $h_i \in Z/qZ\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$. Reducing s $\mathrm{mod}(t_{n-1})$ reduces the coefficients h_i $\mathrm{mod}(t_{n-1})$. This gives an expansion of Z (and hence s) inside X_n $\mathrm{mod}(t_{n-1})$. Hence s $\mathrm{mod}(t_{n-1})$ gets mapped down to the $\mathscr{P}$ expansion of $\zeta^q - 1$ in $X_{n-1} = k\{\{t_1\}\}\dots\{\{t_{n-2}\}\}$. □

3. Proof of theorem

This is straightforward from the lemma. Commutativity follows from the fact that Γ induces the pairing on

$$\frac{K_n(X_n)}{(K_n(X_n))^q} \times \frac{X_n^*}{(X_n^*)^q}$$

combined with the fact that this pairing is nondegenerate.

Example. Let $X_3 = Q_p\{\{t_1\}\}\{\{t_2\}\}$ where $p \neq 2$. If we consider the prime ideal $P = (t_2)$ then X_3 $\mathrm{mod}P = Q_p\{\{t_1\}\} = X_2$. The series $\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ on $X_3^* \times X_3^* \times X_3^* \times X_3^*$ is given by

$$\begin{aligned}\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \ell(\alpha_4)D_4 - \ell(\alpha_3)D_3 + \ell(\alpha_2)D_2 - \ell(\alpha_1)D_1\\ &= \ell(\alpha_4)\begin{bmatrix}\delta_1(\alpha_1)\delta_2(\alpha_1)\delta_3(\alpha_1)\\ \delta_1(\alpha_2)\delta_2(\alpha_2)\delta_3(\alpha_2)\\ \delta_1(\alpha_3)\delta_2(\alpha_3)\delta_3(\alpha_3)\end{bmatrix} - \ell(\alpha_3)\begin{bmatrix}\delta_1(\alpha_1)\delta_2(\alpha_1)\delta_3(\alpha_1)\\ \delta_1(\alpha_2)\delta_2(\alpha_2)\delta_3(\alpha_2)\\ \eta_1(\alpha_4)\eta_2(\alpha_4)\eta_3(\alpha_4)\end{bmatrix}\\ &\quad + \ell(\alpha_2)\begin{bmatrix}\delta_1(\alpha_1)\delta_2(\alpha_1)\delta_3(\alpha_1)\\ \eta_1(\alpha_3)\eta_2(\alpha_3)\eta_3(\alpha_3)\\ \eta_1(\alpha_4)\eta_2(\alpha_4)\eta_3(\alpha_4)\end{bmatrix} - \ell(\alpha_1)\begin{bmatrix}\eta_1(\alpha_2)\eta_2(\alpha_2)\eta_3(\alpha_2)\\ \eta_1(\alpha_3)\eta_2(\alpha_3)\eta_3(\alpha_3)\\ \eta_1(\alpha_4)\eta_2(\alpha_4)\eta_3(\alpha_4)\end{bmatrix}.\end{aligned}$$

Applying $\overline{M}$ to the D_i gives minors corresponding to α_3. Applying $\overline{M}$ to $\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ we obtain

$$\begin{aligned}\overline{M}\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \ell(\alpha_4)\overline{M}(D_4) - \ell(\alpha_3)\overline{M}(D_3) + \ell(\alpha_2)\overline{M}(D_2) - \ell(\alpha_1)\overline{M}(D_1)\\ &= 0 + \ell(\alpha_4)\begin{bmatrix}\delta_1(\alpha_1)\delta_3(\alpha_1)\\ \delta_1(\alpha_2)\delta_3(\alpha_2)\end{bmatrix} - \ell(\alpha_2)\begin{bmatrix}\delta_1(\alpha_1)\delta_3(\alpha_1)\\ \eta_1(\alpha_4)\eta_3(\alpha_4)\end{bmatrix} + \ell(\alpha_1)\begin{bmatrix}\eta_1(\alpha_2)\eta_3(\alpha_2)\\ \eta_1(\alpha_4)\eta_3(\alpha_4)\end{bmatrix}.\end{aligned}$$

If we reduce the above $\mathrm{mod}P$ we obtain the expansion for $\phi(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_4)$ associated with the Vostokov pairing on X_3 $\mathrm{mod}P = Q_p\{\{t_1\}\}$.

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