Enumeration of almost Moore digraphs of diameter two

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Abstract

An almost Moore \((d, 2)\)-digraph is a regular directed graph of degree \(d\geq 1\), diameter \(k=2\) and order \(n\) one less than the (unattainable) Moore bound. Their enumeration is equivalent to the characterization of binary matrices \(A\) fulfilling the equation \(I+A+A^2=J+P\), where \(J\) denotes the all-one matrix and \(P\) is a permutation matrix that commutes with \(A\). In this paper we prove, using algebraic and graphical techniques, that if \(d\geq 2\) the previous equation has no solutions unless \(P=I\). This allows us to complete the classification of the almost Moore \((d, 2)\)-digraphs up to isomorphisms. Thus, we conclude that there is only one \((d, 2)\)-digraph, namely the line digraph \(LK_{d+1}\) of the complete digraph \(K_{d+1}\), apart from the particular case \(d=2\) for which there are two more digraphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The well-known degree/diameter problem for directed graphs can be formulated as an extremal graph theory problem in the following way: for a given positive integers \(d\) and \(k\), determine the minimum integer \(n(d,k)\) such that any strongly connected digraph of order greater than \(n(d,k)\) has maximum out-degree at most \(d\) and diameter at least \(k+1\). Since it has been proved that \(n(d,k) < 1 + d + \cdots + d^k = M(d,k)\), unless \(d=1\) or \(k=1\) (see [18] or [6]), the question of finding for which values of \(d>1\) and \(k>1\) we have \(n(d,k) = M(d,k) - 1\), where \(M(d,k)\) is known as the Moore bound, becomes an interesting problem. Any extremal digraph in this case — a digraph with maximum out-degree at most \(d>1\), diameter at most \(k>1\) and order \(n\) one less than the (unattainable) Moore bound — must have all vertices with out-degree \(d\) and its diameter must be equal to \(k\) (see [8]). Furthermore, Miller et al. [14] proved that, in fact, such extremal digraphs must be diregular, that is to say, their in-degrees are also

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constant (\(=d\)). From now on, diregular digraphs of degree \(d > 1\), diameter \(k > 1\) and order \(n = d + \cdots + d^k\) will be called almost Moore \((d,k)\)-digraphs (or \((d,k)\)-digraphs for short).

Every \((d,k)\)-digraph \(G\) has the characteristic property that for each vertex \(v \in V(G)\) there exists only one vertex, denoted by \(r(v)\) and called the repeat of \(v\), such that there are exactly two \(v \rightarrow r(v)\) walks of length at most \(k\) (one of them must be of length \(k\)). If \(r(v) = v\), which means that \(v\) is contained in exactly one \(k\)-cycle, \(v\) is called a self-repeat of \(G\). The map \(r\), which assigns to each vertex \(v \in V(G)\) the vertex \(r(v)\), is an automorphism of \(G\) (see [1]). Seeing it as a permutation, \(r\) has a cycle structure which corresponds to its unique decomposition in disjoint cycles. Such cycles will be called permutation cycles of \(G\). The number of permutation cycles of \(G\) of each length \(i \leq n\) will be denoted by \(m_i\) and the vector \((m_1, \ldots, m_n)\), which represents a partition of \(n\) with \(m_i\) parts equal to \(i\), will be referred to as the permutation cycle structure of \(G\) (see in Fig. 1 the corresponding permutation cycle structures of the \((2,2)\)-digraphs).

Using the basic properties of a \((d,k)\)-digraph \(G\) (see [3, Propositions 1, 2]), it can be easily seen that its adjacency matrix \(A\) fulfills the equation

\[
I + A + \cdots + A^k = J + P, \tag{1}
\]

where \(I\) is the \(n \times n\) identity matrix, \(J\) denotes the \(n \times n\) all-one matrix and \(P\) is the \((0,1)\)-matrix associated with the permutation \(r\) of \(V(G) = \{1, \ldots, n\}\), that is to say, its \((i,j)\) entry is 1 iff \(r(i) = j\). Therefore, \(AP = PA\) since \(AJ = JA\) (\(=dJ\)).

Several results have been obtained about the existence of \((d,k)\)-digraphs. Thus, fixing the degree and using graphical arguments, Miller and Fris [15] proved that the \((2,k)\)-digraphs do not exist for values of \(k > 2\) and, subsequently, Baskoro et al. [5] established the nonexistence of \((3,k)\)-digraphs unless \(k = 2\). On the other hand, Fiol et al. [7] showed that the \((d,2)\)-digraphs do exist for any degree. The digraph constructed is the line digraph \(LK_{d+1}\) of the complete digraph \(K_{d+1}\), which is in fact the Kautz digraph \(K(d,2)\), proposed in [11]. Concerning the enumeration of \((d,2)\)-digraphs it is known that there are exactly three non-isomorphic \((2,2)\)-digraphs [16] (see Fig. 1) while there is a unique \((d,2)\)-digraph for \(d = 3, 4\) (see [4,10], respectively). Moreover, in [9] it has been proved that \(LK_{d+1}\) is the only \((d,2)\)-digraph with all self-repeat
vertices \((m_1 = n)\), that is to say, the only digraph whose adjacency matrix satisfies the equation \(A + A^2 = J (P = I)\). In this paper, we will prove that if \(d > 2\) the more general equation
\[
I + A + A^2 = J + P
\] (2)
has no solutions unless \(P = I\), henceforth completing the classification of the almost Moore digraphs of diameter \(k = 2\).

Firstly, we will obtain the factorization in \(\mathbb{Q}[x]\) of the characteristic polynomial \(\phi(G, x)\) of a \((d, 2)\)-digraph \(G\). For doing this, we use two fundamental facts: the known relations between the spectrum of \(G\) and its permutation cycle structure (see [10, Sections 2, 4]); the irreducibility in \(\mathbb{Q}[x]\) of the polynomials \(F_i(x) = \Phi_i(x^2 + x + 1)\), if \(i \neq 1, 4\), where \(\Phi_i(x)\) denotes the \(i\)th cyclotomic polynomial. Such irreducibility, which was conjectured in [10], has been proved by Lenstra and Poonen [13] (Section 2).

The spectrum of a \((d, 2)\)-digraph \(G\) is determined by its permutation cycle structure \((m_1, \ldots, m_n)\) and the values of the traces \(\text{tr} A\) and \(\text{tr} PA\), where \(A\) is the adjacency matrix of \(G\) and \(P\) is the \((0, 1)\)-matrix associated with the permutation \(\tau\) of repeats of \(G\). From the relation \(\text{tr} A = 0\) — a direct consequence of Eq. (2) — and taking into account the significance and the properties of \(\text{tr} PA\), as the cardinal of the following set
\[
R(G) = \{v \in V(G) | (\tau(v), v) \in E(G)\}
\]
(see [10, Section 3]), very restrictive conditions about the partition \((m_1, \ldots, m_n)\) of \(n = d + d^2\) will be derived. In particular, we will obtain that any \((d, 2)\)-digraph of degree \(d\) not ‘too small’ has a ‘relatively large’ number of self-repeats. Therefore, using some properties about the structure of a \((d, 2)\)-digraph with self-repeat vertices, given in [1, 2], we will deduce that if \(d > 5\) the only feasible partition is \(m_1 = n\). In the particular case \(d = 5\), there will be a second candidate for being the permutation cycle structure of a \((5, 2)\)-digraph, namely \(m_2 = m_4 = 5\) (Section 3).

The enumeration of \((d, 2)\)-digraphs will be reduced to the study of the equation \(A + A^2 = J\), once the nonexistence of a \((5, 2)\)-digraph with permutation cycle structure \(m_2 = m_4 = 5\) has been proved. This will be carried out in Section 4 by using graphical arguments in contrast with the algebraic (spectral) techniques used in the previous sections.

2. Spectrum

In [10, Section 1], we pointed out the connection between the problem of the factorization in \(\mathbb{Q}[x]\) of the characteristic polynomial \(\phi(G, x)\) of a \((d, k)\)-digraph \(G\) and the study of the irreducibility in \(\mathbb{Q}[x]\) of the polynomials \(\Phi_i(x^k + \cdots + x + 1)\). Thus, the following result is a particular case of Proposition 2 in [10].

Proposition 1. Let \((m_1, \ldots, m_n)\) be the permutation cycle structure of a \((d, 2)\)-digraph \(G\) and \(2 \leq i \leq n\). If \(F_i(x) = \Phi_i(x^2 + x + 1)\) is an irreducible polynomial in \(\mathbb{Q}[x]\), then
$F_i(x)$ is a factor of $\phi(G,x)$ and its multiplicity is $m(i)/2$, where $m(i) = \sum_{j|i} m_j$. In this case, $m(i)$ is even.

Moreover, a conjecture about the irreducibility of the polynomials $F_{i,k}(x) = \Phi_i(x^k + \cdots + x + 1)$ in $\mathbb{Q}[x]$ was formulated in [10]. For $k$ even and $i > 2$, it states that $F_{i,k}(x)$ is irreducible unless $i/(k+2)$. The case $k=2$ has been proved by Lenstra Jr. and Poonen who communicated to the author a detailed outline of their proof [13].

**Theorem 1.** The polynomial $F_n(x) = \Phi_n(x^2 + x + 1)$ is irreducible in $\mathbb{Q}[x]$ unless $n=1, 4$.

**Proof.** From the definition of the cyclotomic polynomial, we have that

$$F_n(x) = \Phi_n(x^2 + x + 1) = \prod_{\substack{1 \leq i \leq n \atop \gcd(i,n)=1}} (x^2 + x + 1 - \zeta_n^i),$$

where $\zeta_n = e^{2\pi i/n}$.

Thus, in particular, $F_n(x)$ has two roots $\varepsilon$ and $\varepsilon'$, which are solutions of the equation

$$x^2 + x + 1 - \zeta_n = 0$$

and, consequently, satisfy

$$\varepsilon + \varepsilon' = -1,$$

$$\varepsilon\varepsilon' = \varepsilon(-1-\varepsilon) = 1 - \zeta_n.$$  \hspace{1cm} (4)

Let us suppose that $F_n(x)$, $n > 1$, is reducible in $\mathbb{Q}[x]$. Then, using the following known properties about the degrees of the algebraic extensions $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(\varepsilon)$,

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) \quad \text{and} \quad [\mathbb{Q}(\varepsilon) : \mathbb{Q}] = [\mathbb{Q}(\varepsilon) : \mathbb{Q}(\zeta_n)] \cdot [\mathbb{Q}(\zeta_n) : \mathbb{Q}] < 2\phi(n)$$

(see [12]), where $\phi(n)$ stands for Euler’s function, we deduce that $\mathbb{Q}(\varepsilon) = \mathbb{Q}(\zeta_n)$. Hence, $\varepsilon$ and $\varepsilon' = -1 - \varepsilon$ belong to the ring of algebraic integers of $\mathbb{Q}(\zeta_n)$, which is $\mathbb{Z}[\zeta_n]$ (see [19, Theorem 2.6]). Moreover, at least one of the two elements in $\{\varepsilon, -1 - \varepsilon\}$ is a unit of $\mathbb{Z}[\zeta_n]$, since $\varepsilon(-1-\varepsilon) = 1 - \zeta_n$ and $1 - \zeta_n$ is either a prime element or a unit of $\mathbb{Z}[\zeta_n]$, if $n > 1$ (see [19, Lemma 1.4 and Proposition 2.8]). By symmetry, we can choose it to be $\varepsilon$. Then, using a known property of units in cyclotomic fields, we get that the (complex) conjugate of $\varepsilon$, denoted by $\bar{\varepsilon}$, can be expressed as $\bar{\varepsilon} = \bar{x}e$, where $x$ is a root of unity (see [19, Lemma 1.6]). Furthermore, since the only roots of unity

\[1\] In fact, this lemma says that if $x$ is an algebraic integer all of whose (algebraic) conjugates have modulus 1, then $x$ is a root of unity. In our case, $x = \bar{\varepsilon}/e \in \mathbb{Z}[\zeta_n]$, since $\varepsilon$ is a unit of $\mathbb{Z}[\zeta_n]$, and the modulus of any of its conjugates $x^e$ is

$$|x^e| = \frac{|x|}{|\varepsilon|} = \frac{|\varepsilon|}{|\bar{x}|} = 1,$$

since the complex conjugation $\zeta_n \rightarrow \bar{\zeta}_n = \zeta_n^{n-1}$ belongs to the Galois group of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$, which is abelian ($\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$ — see [19, Theorem 2.5]).
in \(\mathbb{Q}(\zeta_n)\) are of the form \(\pm \zeta_n^l\) and since \(x = \frac{e}{\varepsilon} \neq 1\) (Eq. (3) has no real solutions, if \(n > 1\)), it turns out\(^2\) that \(x\) is an integral power of \(\zeta_{2n} = e^{i\pi/n}\) such that

\[ x = \zeta_{2n}^l, \text{ where } l \in \mathbb{Z}_{2n} \setminus \{0\}, \text{ and } l \text{ even if } n \text{ is even.} \quad (5) \]

Now, we will see that \(x\) satisfies a certain polynomial equation which has no solutions of the required type (5) unless \(n = 4\). From this contradiction, the irreducibility of the polynomials \(F_n(x), n \neq 1, 4\), will follow.

First, let us find a relationship between \(\varepsilon\), \(x\) and \(\zeta_n\). Using the identities \(e(-1 - \varepsilon) = 1 - \zeta_n\) and \(\varepsilon = \zeta_n x\), we have that

\[ \frac{-1 - \varepsilon}{\varepsilon} = \frac{1 - 1/\zeta_n}{\varepsilon x_n} = \frac{1 + \varepsilon}{\zeta_n x} \quad \text{and} \quad \frac{-1 - \varepsilon}{\varepsilon} = -1 - x, \]

whence

\[ \varepsilon = -\frac{1 + \zeta_n x^2}{1 + \zeta_n x^2}. \quad (6) \]

Therefore, from (4) and (6), we deduce that \(x\) must satisfy the following polynomial equation with coefficients in \(\mathbb{Z}[\zeta_n]s\):

\[ (\zeta_n^2 - \zeta_n^2)x^4 + \zeta_n^2 x^3 + (\zeta_n^2 - \zeta_n)x^2 - \zeta_n x + \zeta_n - 1 = 0. \]

Since \(x\) must be an integral power of \(\zeta_{2n} = e^{i\pi/n}\), and taking into account that \(\zeta_n = \zeta_{2n}^2\), we have that

\[ (\zeta_{2n}^2 - \zeta_{2n}^2)x^4 + \zeta_{2n}^2 x^3 + (\zeta_{2n}^2 - \zeta_{2n})x^2 - \zeta_{2n} x + \zeta_{2n}^2 - 1 = 0. \]

Then, in order to make the symmetry apparent, set \(x = i\omega/\zeta_{2n}\) to get

\[ \zeta_{2n}^2 - 1 = \omega^2 - \omega^2 + 1 - (\omega^3 + \omega) = 0, \]

which is in fact a polynomial equation of degree 4 in \(\omega\) with real coefficients since

\[ \frac{\zeta_{2n}^2 - 1}{i\zeta_{2n}} = \frac{\zeta_{2n} - 1/\zeta_{2n}}{i} = \frac{\zeta_{2n} - \bar{\zeta}_{2n}}{i} = 2 \sin \frac{\pi}{n}. \]

Now, we will show that if \(n \neq 4\) the resulting equation

\[ 2 \sin \frac{\pi}{n}(\omega^4 - \omega^2 + 1) - (\omega^3 + \omega) = 0 \quad (7) \]

has no solutions of the form

\[ \omega = i\zeta_{2n}^l, \text{ where } l \in \mathbb{Z}_{2n} \setminus \{n + 1\}, \text{ and } l \text{ odd if } n \text{ is even}, \quad (8) \]

which contradicts the fact that \(x\) satisfies (5) and

\[ \omega = \frac{\zeta_{2n}^l}{i} = -i\zeta_{2n}^{l+1} = i\zeta_{2n}^{-l+1}. \]

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\(^2\)Let \(x \in \mathbb{Q}(\zeta_n)\) be a root of unity of order \(m = m'd\), where \(d = \gcd(m, n)\). Then, \(x^d \in \mathbb{Q}(\zeta_{n'}) \cap \mathbb{Q}(\zeta_{n'})\), where \(n = n'd\). By taking into account that \(\mathbb{Q}(\zeta_{n'}) \cap \mathbb{Q}(\zeta_{n'}) = \mathbb{Q}\), since \(\gcd(n', m') = 1\) (see [19, Proposition 2.4]), it follows that \(x^d = \pm 1\). Hence, \(x^n = (x^d)^{n'} = 1\), that is to say, \(x\) is a 2nth root of unity. Moreover, if \(n\) is even then \(x = \zeta_{2n}^l\) (otherwise, \(\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{2n}, x) = \mathbb{Q}(\zeta_{2n})\), which is impossible since \(\phi(n) < \phi(2n)\)).
To this end, we will locate the four roots of (7). Thus, since the real continuous function
\[ f(\omega) = 2\sin(\pi/n)(\omega^4 - \omega^2 + 1) - (\omega^3 + \omega) \]
satisfies
\[ f(0) = 2\sin\frac{\pi}{n} > 0 \quad \text{and} \quad f(1) = 2\left(\sin\frac{\pi}{n} - 1\right) \leq 0 \]
we deduce that \( f(\omega) \) has a real zero \( \omega_1 \in (0,1) \). Moreover, its inverse \( \omega_2 = 1/\omega_1 \) is also a zero of \( f(\omega) \) since \( f(1/\omega) = f(\omega)/\omega^3 \). Clearly, if \( n > 2 \) neither \( \omega_1 \) nor \( \omega_2 \) can be expressed as \( \omega = i\sqrt{2n} \), since \( |\omega_j| \neq 1, \ j = 1,2. \) In the particular case \( n = 2 \) we get the double root \( \omega = 1 = i\sqrt{2} \), which does not fulfill (8). Now, let us see that the two remaining roots of (7) are located on the unit circle \( S^1 \) but neither of them are of the required form \( \omega = i\sqrt{2n} \), unless \( n = 4 \). Indeed, let us consider the function
\[ g(\omega) = \frac{f(\omega)}{\omega^2} = 2\sin\frac{\pi}{n}\left(\omega^2 + \frac{1}{\omega^2} - 1\right) - \left(\omega + \frac{1}{\omega}\right), \]
which has the same zeros as \( f(\omega) \) and is real on \( S^1 \) since
\[ g(\omega) = 2\sin\frac{\pi}{n}(2\Re(\omega^2) - 1) - 2\Re(\omega), \quad \text{if } |\omega| = 1. \]
So, in particular,
\[
\begin{align*}
g(i\sqrt{2n}) &= -2\sin\frac{\pi}{n}\left(2\cos\frac{4\pi}{n} + 1\right) + 2\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\left(4\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n} - 1\right), \\
g(i\sqrt{3n}) &= -2\sin\frac{\pi}{n}\left(2\cos\frac{6\pi}{n} + 1\right) + 2\sin\frac{3\pi}{n} = 8\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{4\pi}{n}.
\end{align*}
\]
It can be easily checked that \( g(i\sqrt{2n}) < 0 \), if \( n \geq 12 \), and \( g(i\sqrt{3n}) > 0 \), if \( n > 4 \). So, if \( n \geq 12 \), then \( g(\omega) \) has a zero \( \omega_3 = e^{3}\pi \), where \( \pi/2 + 2\pi/n < \gamma < \pi/2 + 3\pi/n \) and, consequently, neither \( \omega_1 \) nor its inverse \( \omega_2 \) belong to the set of points \( \{i\sqrt{2n}, \ i \in \mathbb{Z}_{2n}\} \). Analogously, for the particular cases \( 1 < n < 12 \) it can be seen that \( g(i)g(i\sqrt{2n}) < 0 \), if \( n = 2,3, \ g(i\sqrt{2n}) = g(i\sqrt{3n}) = 0 \), if \( n = 4, \) and \( g(i\sqrt{2n})g(i\sqrt{3n}) < 0 \), otherwise. Therefore, Eq. (7) has solutions of the required form only when \( n = 4 \). Hence, \( F_n(x) \) is irreducible in \( \mathbb{Q}[x] \) unless \( n = 1,4 \), where
\[
\begin{align*}
\Phi_1(x^2 + x + 1) &= (x + 1)x, \\
\Phi_4(x^2 + x + 1) &= (x^2 + x + 1)^2 + 1 = (x^2 + 1)(x^2 + 2x + 2). & \square
\end{align*}
\]

As a consequence of this result, we can almost obtain the complete factorization of the characteristic polynomial of a \((d,2)\)-digraph in terms of its permutation cycle structure, which represents an extension of Proposition 4 in [10]. The remaining unknown multiplicities will be determined from the values of \( \text{tr} A \) and \( \text{tr} PA \), as detailed in the next section.

**Corollary 1.** Let \( G \) be a \((d,2)\)-digraph and let \((m_1,\ldots,m_n)\) be the permutation cycle structure of \( G \). Then, the characteristic polynomial of \( G \) can be written as
\[
\phi(G,x) = (x - d)(x + 1)^{m_1}x^{2e_2}(x^2 + 1)^{b_1}(x^2 + 2x + 2)^{b_2} \prod_{2 \leq i < n, i \neq 4}(F_i(x))^{m(i)/2},
\]
where \(a_i, b_i\) are nonnegative integers such that \(a_1 + a_2 = m(1) - 1\) and \(b_1 + b_2 = m(4)\), where \(m(1) = \sum_{i=1}^{n} m_i\) represents the total number of permutation cycles. In addition, the numbers \(m_i\) fulfill the following conditions:

(i) \(m_1 = d_1 + (d_1)^2\), where \(0 \leq d_1 \leq d\),
(ii) \(2|m_i, \ \forall i \neq 2, 4\),
(iii) \(m_2\) and \(m_4\) have the same parity.

3. Permutation cycle structure

Let \(A\) be the adjacency matrix of a \((d, 2)\)-digraph \(G\) of order \(n\) and let \((m_1, \ldots, m_n)\) be its permutation cycle structure. Since \(I + A + A^2 = J + P\), it follows that

\[
\text{tr} A = 0, \quad \text{tr} A^2 = \text{tr} P = m_1 \quad \text{and} \quad \text{tr} PA = m_1 + \text{tr} A^3 - nd.
\]

By expressing these relations in terms of the eigenvalues of \(G\), several necessary conditions for its permutation cycle structure were derived in [10, Proposition 5], when \(d \leq 15\). Thus,

\[
\text{tr} A = d - a_1 - 2b_2 - \sum_{2 \leq i \leq n, i \neq 4} \frac{m(i)}{2} \phi(i) = 0, \quad (9)
\]

\[
\text{tr} A^2 = d^2 + a_1 - 2b_1 + \sum_{2 \leq i \leq n, i \neq 4} \frac{m(i)}{2} (2\mu(i) - \phi(i)) = m_1, \quad (10)
\]

\[
\text{tr} PA = 4b_2 - 2b_1 + \sum_{2 \leq i \leq n, i \neq 4} \frac{m(i)}{2} (\phi(i) - \mu(i)), \quad (11)
\]

where \(a_1, b_1, b_2\) must be nonnegative integers such that \(a_1 \leq m(1) - 1\) and \(b_1 + b_2 = m(4)\) and where \(\mu(i)\) stands for Möbius’s function. We remark that the constraint on the size of the degree was due to the fact that we made use of the irreducibility of the polynomials \(F_i(x)\), which we had only verified for \(i \leq 15 + 15^2\) and \(i \neq 1, 4\). So, Theorem 1 allows us to remove this restriction. Furthermore, we have noticed that the sums involved in (9)–(12) can be simplified by using the following well-known identities:

\[
\sum_{i | l} \phi(i) = l \quad \text{and} \quad \sum_{i | l} \mu(i) = \begin{cases} 1 & \text{if } l = 1, \\ 0 & \text{otherwise} \end{cases}
\]

(see [17]). Thus, we have that

\[
\sum_{i=1}^{n} m(i) \phi(i) = \sum_{i=1}^{n} \left( \sum_{i | l} m_i \right) \phi(i) = \sum_{i=1}^{n} m_i \left( \sum_{i | l} \phi(i) \right) = n,
\]

\[
\sum_{i=1}^{n} m(i) \mu(i) = \sum_{i=1}^{n} m_i \left( \sum_{i | l} \mu(i) \right) = m_1,
\]
since \((m_1, \ldots, m_n)\) represents a partition of \(n\). Therefore, condition (9) can be reformulated as

\[
\text{tr} A = d - a_1 - 2b_2 - \frac{1}{2} \left( \sum_{i=1}^{n} m(i)\phi(i) - m(1)\phi(1) - m(4)\phi(4) \right)
\]

\[
= d - a_1 - 2b_2 - \frac{1}{2}(n - m(1) - 2m(4))
\]

\[
= -\frac{1}{2}(d^2 - d) - a_1 - 2b_2 + \frac{1}{2}m(1) + m(4) = 0,
\]

since \(n = d + d^2\). Analogously, relation (10) can be expressed as

\[
\text{tr} A^2 = d^2 + a_1 - 2b_1 + \sum_{i=1}^{n} m(i)\mu(i) - m(1)\mu(1) - m(4)\mu(4)
\]

\[
- \frac{1}{2}(n - m(1) - 2m(4))
\]

\[
= \frac{1}{2}(d^2 - d) + a_1 - 2b_1 - \frac{1}{2}m(1) + m(4) + m_1 = m_1,
\]

which turns out to be equivalent to (12), since \(b_1 = m(4) - b_2\).

In a similar manner, it follows that

\[
\text{tr} PA = 4b_2 - 2b_1 + \frac{1}{2}n - m(4) - \frac{1}{2}m_1.
\]

Hence, the properties about the permutation of repeats of a \((d,2)\)-digraph, derived in [10, Section 4], can be reformulated and extended in the following way:

**Proposition 2.** Let \(G\) be a \((d,2)\)-digraph of order \(n\) and let \((m_1, \ldots, m_n)\) be its permutation cycle structure. Then, there exist nonnegative integers \(a_1, b_2\), with \(a_1 \leq m(1) - 1\) and \(b_2 \leq m(4)\), fulfilling the following equations:

\[
d^2 - d + 2(a_1 + 2b_2) - m(1) - 2m(4) = 0,
\]

\[
\frac{1}{2}n - \frac{1}{2}m_1 - 3m(4) + 6b_2 = \text{tr} PA.
\]

The above conditions allow us to conclude that all the \((d,2)\)-digraphs of degree \(d > 5\) must have the same permutation cycle structure. Firstly, we will deduce that \(m_1 > 0\) and then, by using some known properties on the structure of a \((d,2)\)-digraph with selfrepeat vertices, we will conclude that \(m_1 = n\).

**Corollary 2.** Let \((m_1, m_2, \ldots, m_n)\) be the permutation cycle structure of a \((d,2)\)-digraph \(G\).

(i) If \(d > 5\), then \(m_1 = n\);

(ii) If \(d = 5\), then \(m_1 = n\) or \(m_2 = m_4 = 5\).

**Proof.** Let us suppose that \(m_1 = 0\). Then, from Eq. (13) and taking into account that the parameters \(a_1, b_2\) are nonnegative, we have that

\[
d^2 - d \leq m(1) + 2m(4) = m_2 + m_3 + 3m_4 + \cdots \leq \frac{3(d^2 + d)}{4}
\]
since, in the case $m_1 = 0$, the sum $m(1) + 2m(4)$ gets its maximum value when all the permutation cycles are of length 4, if $d \equiv 0, 3 \mod 4$, and $m_2 = 1$, $m_4 = (d^2 + d - 2)/4$, otherwise. Since inequality (15) is equivalent to $d(d - 7) \leq 0$, we deduce that if $d > 7$, then $m_1 > 0$. Now, we will analyse the particular cases $d = 5, 6, 7$. Thus, if $d = 7$, then $m_4 = n/4$. Therefore, from (13) we have that $b_2 = 0$ and, consequently, using (14) we obtain that $\text{tr} PA = -n/4$, which is impossible since $PA$ is a nonnegative matrix. If $d = 6$, according to condition (13), there are two possible permutation cycle structures: $m_4 = 10$, $m_2 = 1$ and $m_4 = 9$, $m_2 = 3$. While the first one is impossible since $m_2$ and $m_4$ must have the same parity (Corollary 1), the second one would imply a negative value for $\text{tr} PA$. The feasible permutation cycle structures of a $(5, 2)$-digraph can be found in a similar manner (see Table 1). Hence, if $d = 5$ and $m_1 = 0$, then $m_2 = m_4 = 5$. So, apart from this particular case, $m_1 > 0$ if $d \geq 5$.

In [2, Theorem 4], it is shown that $m_1 = n$, if $m_1 > 0$ and $d \leq 12$. Hence, we have already proved that all the vertices of a $(d, 2)$-digraph of degree $5 < d \leq 12$ are self-repeats and, moreover, this is also true if $d = 5$ and $m_1 > 0$. Let us now show the extended result.

Let $G$ be a $(d, 2)$-digraph of degree $d > 12$ and let $G_1$ be the induced subdigraph of $G$ by the (nonempty) set $V_1$ of all self-repeats of $G$. Since $G_1$ is either a 2-cycle or a $(d_1, 2)$-digraph with $2 \leq d_1 = d - t \leq d$ (see [1, Theorem 4]), we have that $m_1 = (d - t) + (d - t)^2$, where $0 \leq t \leq d - 1$. First, we will prove that $m_1 \geq (d - 4) + (d - 4)^2$. Thus, from (13) it follows that

$$d^2 - d \leq (d - t) + (d - t)^2 + \frac{3}{4}(d^2 + d - (d - t)^2) - (d - t),$$

which is equivalent to

$$2d(t - 4) \leq t(t - 1).$$

(16)

Such inequality, which is trivially satisfied when $0 \leq t \leq 4$, can be rewritten as $d \leq f(t)$, where $f(t) = t(t - 1)/(2(t - 4))$, if $t \geq 5$. Therefore, by studying the monotony of the function $f(t)$ in the interval $[5, d]$, it can be seen that

$$\max_{5 \leq t \leq d} f(t) \leq \begin{cases} f(5) = 10 & \text{if } d \leq 16, \\ f(d) = d(d - 1)/(2(d - 4)) & \text{otherwise}, \end{cases}$$

from where it turns out that condition (16) does not hold for any value of $t$, $5 \leq t \leq d$, if $d > 10$. Hence, the order of $G_1$ is $m_1 = (d - t) + (d - t)^2$, where $t \leq 4$. Let us suppose that $m_1 \neq n$ and let us consider a self-repeat vertex $v$ of $G$. Then, the set $N^+(v)$ of all its out-neighbours contains exactly $t$ vertices, where $1 \leq t \leq 4$, which are

<table>
<thead>
<tr>
<th>$m_4$</th>
<th>$m_5$</th>
<th>$m_2$</th>
<th>$\text{tr} A$</th>
<th>$\text{tr} PA$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0 ($a_1 = 1, b_2 = 0$)</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0 ($a_1 = 0, b_2 = 0$)</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>5</td>
<td>0 ($a_1 = 0, b_2 = 0$)</td>
<td>$0$</td>
</tr>
</tbody>
</table>
not self-repeats. Since \( r \) is an automorphism of \( G \), we have that these \( t \) vertices must be in permutation cycles of lengths 2 or 4 and, consequently, \( t = 2, 4 \). Furthermore, since the permutation cycle structure of any \((d, 2)\)-digraph with \( m_1 > 0 \) satisfies that \( m_2 = 0 \) (see [2, Theorem 2]), it turns out that \( t = 4 \). In such a case, the subdigraph of \( G \) induced by the subset of vertices of order \(^3\) a divisor of 4, which must also be a \((d', 2)\)-digraph of degree \( d' = |\{w \in N^+(v) \mid r^4(w) = w\}| \) (see [2, Theorem 1]), coincides with \( G \) since \( d' = d \). Therefore, \( m_4 = ((d^2 + d) - (d - 4)^2 - (d - 4))/4 = 2d - 3 \), since \( m_2 = 0 \). But this is impossible since \( m_2 \) and \( m_4 \) must have the same parity. Hence, \( m_1 = d + d^2 = n \), if \( d > 12 \). \( \square \)

4. Enumeration

From Corollary 2, we know that the problem of the enumeration of \((d, 2)\)-digraphs of degree \( d \geq 5 \) will be reduced to the study of the equation \( A + A^2 = J \) (\( P = I \), when \( m_1 = n \)) if the nonexistence of a \((5, 2)\)-digraph with permutation cycle structure \( m_2 = m_4 = 5 \) is shown. Next, we will present a proof of this by using graphical arguments on the local structure of such an object.

**Lemma 1.** There is no \((d, 2)\)-digraph of degree \( d = 5 \) with permutation cycle structure \( m_2 = m_4 = 5 \).

**Proof.** Let \( G \) be a \((d, 2)\)-digraph of degree \( d = 5 \) with permutation cycle structure \( m_2 = m_4 = 5 \), that is to say \( G \) has 10 vertices of order 2 and the remaining 20 have order 4. In particular,

(P1) \( G \) does not contain any 2-cycle since \( m_1 = 0 \).

(P2) Each arc \((u, v) \in E(G)\) is included in exactly one 3-cycle since \( \text{tr} PA = 0 \).

From these remarks and from the essential fact that \( r \) is an automorphism of \( G \), other properties about its graphical structure can be derived:

(P3) If \( v \in V(G) \) has order 2, then \( r(v) \notin N^+(v) \) \( \{r(v) \notin N^-(v)\} \) (otherwise, \( G \) would contain the 2-cycle: \( v, r(v), v \));

(P4) If \( v_1, v_2, v_3 \) is a walk of \( G \), where \( v_1, v_3 \) are vertices of order 2 and \( v_2 \) of order 4, then \( v_3 = r(v_1) \) (since \( v_1, r^2(v_2), v_3 \) is a second \( v_1 \rightarrow v_3 \) walk of length 2);

(P5) A 3-cycle of \( G \) cannot contain two vertices of order 2 and one vertex of order 4 (a direct consequence of (P4) and (P3));

(P6) If \( v \in V(G) \) has order 2, then \( N^+(v) \cap N^-(v) \) contains an even number of vertices of order 4 (since if \( w \in N^+(v) \) had order 4, then \( r^2(w) \in N^+(v) \));

(P7) If \( v \in V(G) \) has order 4, then \( N^+(v) \cap \{r(v), r^2(v), r^3(v)\} \subseteq \{r(v)\} \) (for instance, if \( (v, r^3(v)) \in E(G) \), then \( (r^3(v), r^3(v)) \in E(G) \), which is impossible since \( \text{tr} PA = 0 \)).

Moreover, if \( (v, r(v)) \in E(G) \), then \( v, r(v), r^2(v), r^3(v) \), \( v \) is a 4-cycle of \( G \).

\(^3\)It refers to its order as an element of the permutation of repeats \( r \).
Let \( v \in V(G) \) be a vertex of order 2. From (P3) we know that there are two \( v \to r(v) \) walks of \( G \) of length 2. Let \( w_1, w_2 \) be their intermediate vertices. Then, we will distinguish two situations depending on the order of such vertices, which must be equal. Thus, if \( w_1 \) has order 4, then \( w_2 = r^2(w_1) \) (same reasoning used in (P4)). Otherwise, \( w_1 \) and \( w_2 \) must have order 2 and, moreover, \( w_2 \neq r(w_1) \) (since if \( w_2 = r(w_1) \), then there would be a 2-cycle: \( w_2, r(v), w_2 \)). Let us take this last assumption, that is, let us suppose that \( G \) contains the subdigraph shown in Fig. 2, where all its vertices have order 2. Then, it can be seen that there does not exist any adjacency between vertices of the set \( \{w_1, r(w_1), w_2, r(w_2)\} \). So, taking into account (P2) and (P5), we can deduce that \( G \) must contain the subdigraph shown in Fig. 3. We notice that all vertices of \( G \) of order 2 belong to this subdigraph. Then, using (P6), it follows that the missing vertex \( z \in N^-(v) \) must have order 2. Moreover, since \( G \) does not contain any 2-cycle, we deduce that \( z \in \{r(w_1'), r(w_2')\} \). But this is impossible since it would imply that \( v \in \{r^2(w_1), r^2(w_2)\} \), where \( r^2(w_i) = w_i \).

Now, we have to consider the case where \( w_1 \) and \( w_2 \) are vertices of order 4 with \( w_2 = r^2(w_1) \). Then, from (P6) we have to consider two more subcases:

- \( N^+(v) = \{v_1, v_2, v_3, w_1, r^2(w_1)\} \), where each vertex \( v_i \) has order 2;
- \( N^+(v) = \{w_1, r^2(w_1), w_3, r^2(w_3), v_1\} \), where only the vertex \( v_1 \) has order 2.
Let us deal with the first subcase. Then, taking into account that $\text{tr} PA = 0$ and using properties (P5) and (P7), we can deduce that $G$ must contain the subdigraph $G'$ shown in Fig. 4. So, each of the sets $N^+(v)$ and $N^+(r(v))$ [$N^-(v)$ and $N^-(r(v))$] has just two vertices of order 4. Therefore, it can be shown that $r\{v_1, v_2, v_3\} \cap \{v'_1, v'_2, v'_3\} = \emptyset$, whence we conclude that there is a permutation cycle of length 2 formed by two vertices of the set $\{v_1, v_2, v_3\}$ and, consequently, the repeat of the third vertex does not belong to $G'$. Let us suppose, without loss of generality, that $r(v_1) = v_2$. Thus, $G$ must contain the subdigraph shown in Fig. 5. Then, in order to reach $r(v_3)$ from vertex $v$ it is necessary, according to (P4), that $\{v_1, r(v_1), v_3\} \cap N^-(r(v_3)) \neq \emptyset$. Since $v_3 \notin N^-(r(v_3))$, there must be an arc from $v_1$ [or $r(v_1)$] to $r(v_3)$, which is impossible since it would imply that $v = r(v_3)$.

Finally, we have to study the case $N^+(v) = \{w_1, r^2(w_1), w_3, r^2(w_3), v_1\}$, where $v_1$ has order 2. Then, $G$ must contain the subdigraph shown in Fig. 6. Therefore, the unique $v \rightarrow r(v_1)$ walk of length 2 has to go through vertex $v_1$ (a direct consequence of (P4)), which is impossible since $\{v_1, r(v_1)\} \notin E(G)$. □

Hence, the adjacency matrix of a $(d,2)$-digraph of degree $d \geq 5$ must fulfill the equation $A + A^2 = J$, as it happens when $d = 3, 4$ (see [4,10]). Since its solutions have been characterized (see [9, Theorem 1]), the enumeration of $(d,2)$-digraphs can be concluded.
**Theorem 2.** There is exactly one \((d, 2)\)-digraph of degree \(d > 2\), namely \(LK_{d+1}\).

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**References**