DISCRETE MATHEMATICS

ELSEVIER Discrete Mathematics 194 (1999) 195–203

Every cutset meets every fibre in certain poset products

Roy Maltby*,1

Department of Mathematics and Statistics, The University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada T2N 1N4

Received 27 June 1995; revised 17 July 1997; accepted 11 August 1997

Abstract

A *cutset* of a partially ordered set is a subset which meets every maximal chain, and a *fibre* of a partially ordered set is a subset which meets every maximal antichain. A poset is called *skeletal* if every cutset meets every fibre. $K_{1,n}$ stands for the linear sum of a singleton and an n-element antichain. Duffus et al. (1990) showed that any Boolean lattice $K_{1,1}\times\cdots\times K_{1,1}$ is skeletal. Gibson and Maltby (1993) showed that $K_{1,m}\times K_{1,n}$ is skeletal and asked if every $K_{1,n_1}\times\cdots\times K_{1,n_k}$ is skeletal. We prove that $K_{1,1}\times\cdots\times K_{1,1}\times K_{1,m}\times K_{1,n}$ and $K_{1,l}\times K_{1,m}\times K_{1,n}$ are skeletal.

AMS classification: 06A07

Keywords: Poset; Partial order; Fibre; Cutset; Chain; Antichain; Skeletal

A *cutset* of a poset is a subset which meets every maximal chain, and a *fibre* of a poset is a subset which meets every maximal antichain. Call a poset *skeletal* if it satisfies the following equivalent conditions:

(i) Every fibre meets every cutset.

(ii) Every red-blue colouring of the elements of the poset has a red maximal chain or a blue maximal antichain.

(iii) Every fibre contains a maximal chain.

(iv) Every cutset contains a maximal antichain.

The equivalence of (ii)–(iv) to each other is explained by Duffus et al. in [2], and their equivalence to (i) is explained by Gibson and Maltby in [3]. The main result of [2] is that finite Boolean lattices are skeletal. Gibson and Maltby [3] have several results concerning the skeletalness of posets, including an examination of certain poset operations preserving or destroying skeletalness. The main relevant result of [5], by Maltby and Williamson, is that the union of the kth and $(k+1)$st levels of the product of

* E-mail: maltby@cs.sfu.ca.

[1] These results are taken from the author's Master's thesis [4], during the writing of which the author was supported by the NSERC Operating Grants of Profs. Bill Sands, Richard Guy, and Karen Seyffarth.

PII S0012-365X(97)00254-9

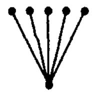

Fig. 1.

t copies of the whole numbers (i.e.$\{(x_1,\dots,x_t): \sum_{i=1}^{t} x_i = k\} \cup \{(x_1,\dots,x_t): \sum_{i=1}^{t} x_i = k+1\}$) is skeletal, unless $t=2$ and k is congruent to 2 modulo 3.

For any $n \in \mathbb{N}$, define $K_{1,n}$ to be the linear sum of a singleton and an n-element antichain. For instance, $K_{1,5}$ is shown in Fig. 1.

For any posets $P_1,\dots,P_k$, define the *direct product* $P_1 \times \cdots \times P_k$ to be the set of k-tuples $(p_1,\dots,p_k)$ having each $p_i \in P_i$ and ordered by: $(p_1,\dots,p_k) \leqslant (p'_1,\dots,p'_k)$ if and only if each $p_i \leqslant p'_i$ in P_i. Note that, up to isomorphism, direct product is a commutative and associative binary operation on posets.

If $P \times Q$ is skeletal, then each of P and Q is skeletal [3, Theorem 3.3], but counterexamples to the converse of this are so easy to find that it is perhaps surprising that Questions 1 and 2 remain unanswered. The question from [3] that we address in this paper is:

Question 1. Is $K_{1,m_1} \times \cdots \times K_{1,m_k}$ skeletal for every $k \in \mathbb{N}$ and all $m_1,\dots,m_k \in \mathbb{N}$?

In [3], Gibson and Maltby show that the answer is yes for $k \leqslant 2$. In [2], Duffus et al. show that the answer is yes when $m_1 = \cdots = m_k = 1$ (i.e. for finite Boolean lattices). In this paper, we prove positive answers in two more special cases: when $m_1 = \cdots = m_{k-2} = 1$ (Theorem 5) and when $k=3$ (Theorem 6).

A more general question in [3] is:

Question 2. If P_1, P_2, and P_3 are posets such that $P_1 \times P_2$, $P_1 \times P_3$, and $P_2 \times P_3$ are all skeletal, must $P_1 \times P_2 \times P_3$ also be skeletal?

Theorem 6 provides a positive answer when each P_i is some K_{1,n_i}. Gibson and Maltby [3] achieved a positive answer to this question in the special case of distributive lattices. We say that Question 2 is more general than Question 1 since if the answer to Question 2 is yes, then by induction the answer is yes for the direct product of any number of posets whose pairwise products are skeletal and, in particular, the answer to Question 1 would be yes as well.

We abbreviate $P \times \cdots \times P$ (r times) by P^r. We denote set difference by '$\setminus$'; that is, $A \setminus B = \{a \in A : a \notin B\}$. We will use the following notation adapted from Davey and Priestley's book [1]. For any X a subset of a poset P, define

$$X\uparrow = \{y \in P\colon\ y \geqslant x \text{ for some } x \in X\},$$

$$X\underset{\circ}{\uparrow} = \{y \in P\colon\ y > x \text{ for some } x \in X\} \setminus X,$$

$$X\downarrow = \{y \in P\colon\ y \leqslant x \text{ for some } x \in X\},$$

$$X\overset{\circ}{\downarrow} = \{y \in P\colon y < x \text{ for some } x \in X\} \setminus X,$$
$$X\updownarrow = X\uparrow \cup X\downarrow.$$

Actually, we will only use the symbols $\updownarrow$ and $\overset{\circ}{\updownarrow}$ with antichains in this paper, making the '$\setminus X$' parts of the definitions unnecessary. We will abbreviate this notation slightly for singletons by dropping the curly braces. For instance, $x\uparrow = \{x\}\uparrow$.

It will be very useful to refer to the construction in the following obvious lemma.

Lemma 3. *Let $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$. Let $E_1, \ldots, E_k$ be pairwise disjoint sets with $|E_i| = n_i$ for each i. Define a poset $\mathscr{P}$ by*

$$\mathscr{P} = \left\{ X \subseteq \bigcup_{i=1}^{k} E_i : |X \cap E_i| \leqslant 1 \text{ for } i = 1, \ldots, k \right\}$$

ordered by set containment. Then $\mathscr{P} \cong K_{1,n_1} \times \cdots \times K_{1,n_k}$.

The statement of Lemma 4 is less than elegant, but having this statement allows us to shorten the proofs of Theorems 5 and 6.

Lemma 4. *Let $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$. Suppose $K_{1,m_1} \times \cdots \times K_{1,m_k}$ is skeletal for all $(m_1, \ldots, m_k) < (n_1, \ldots, n_k)$ in $\mathbb{N}^k$. Construct $\mathscr{P} \cong K_{1,n_1} \times \cdots \times K_{1,n_k}$ as described in Lemma 3. If $\mathscr{F}$ is a fibre of $\mathscr{P}$ which contains no maximal chain of $\mathscr{P}$, then we have $\{x\} \in \mathscr{F}$ for every i such that $n_i > 1$ and every $x \in E_i$.*

Proof. Suppose $n_i > 1$, $x \in E_i$, and $\{x\} \notin \mathscr{F}$. Since $\{\emptyset\}$ is a maximal antichain of $\mathscr{P}$, we know that $\emptyset \in \mathscr{F}$. Let $\mathscr{P}' = \mathscr{P} \setminus \{x\}\uparrow$. So $\mathscr{P} \cong K_{1,n_1} \times \cdots \times K_{1,n_{i-1}} \times K_{1,n_i - 1} \times K_{1,n_{i+1}} \times \cdots \times K_{1,n_k}$ and, hence, $\mathscr{P}'$ is skeletal. If $\mathscr{A}$ is a maximal antichain of $\mathscr{P}'$ disjoint from $\mathscr{F}$, then $\emptyset \notin \mathscr{A}$, and thus it is easy to see that $\mathscr{A} \cup \{\{x\}\}$ is a maximal antichain of $\mathscr{P}$ disjoint from $\mathscr{F}$, contradicting $\mathscr{F}$ being a fibre. This tells us that $\mathscr{F} \cap \mathscr{P}'$ is a fibre of $\mathscr{P}'$. Hence, since $\mathscr{P}'$ is skeletal, there exists $\mathscr{C} \subseteq \mathscr{P}' \cap \mathscr{F}$ a maximal chain of $\mathscr{P}'$. But then $\mathscr{C} \subseteq \mathscr{F}$ is a maximal chain of $\mathscr{P}$, a contradiction. $\square$

Theorem 5. *Let r, m, and n be natural numbers. Then $(K_{1,1})^r \times K_{1,m} \times K_{1,n}$ is skeletal.*

Proof. Assume for a contradiction that the theorem is false. Then there is some $(r, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ for which the theorem fails. Pick (r, m, n) minimal in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ so that $(K_{1,1})^r \times K_{1,m} \times K_{1,n}$ is not skeletal. Since direct product is commutative, we may assume, without loss of generality, that $n \geqslant m$. Put $n_1 = \cdots = n_r = 1$, $n_{r+1} = m$, and $n_{r+2} = n$. Then construct $\mathscr{P}$ with $k = r + 2$ as in Lemma 3, so we have $\mathscr{P} \cong (K_{1,1})^r \times K_{1,m} \times K_{1,n}$.

Since $\mathscr{P}$ is not skeletal, it has a fibre $\mathscr{F}$ which contains no maximal chain. $\emptyset \in \mathscr{F}$ since $\{\emptyset\}$ is a maximal antichain. Furthermore, Lemma 4 tells us that if $m > 1$ then $\{s\} \in \mathscr{F}$ for each $s \in E_{r+1}$, and if $n > 1$ then $\{t\} \in \mathscr{F}$ for each $t \in E_{r+2}$.

We now proceed with the method of [1]. We define sets $X\updownarrow$ analogous to the 'lexical chains' used in [1]. Let $X \in \mathscr{P}$. Define

$$\begin{aligned} X\twoheaddownarrow &= \{X, X\setminus E_1, X\setminus(E_1\cup E_2), \dots, X\setminus(E_1\cup E_2\cup\cdots\cup E_{r+2}) = \emptyset\},\\ X\twoheaduparrow &= \{X, X\cup E_1, X\cup E_1\cup E_2, \dots, X\cup E_1\cup E_2\cup\cdots\cup E_r\}\\ &\quad \cup\{X\cup E_1\cup\cdots\cup E_r\cup\{s\} : s\in E_{r+1}\}\\ &\quad \cup\{X\cup E_1\cup\cdots\cup E_r\cup\{s\}\cup\{t\} : s\in E_{r+1}, t\in E_{r+2}\}. \end{aligned}$$

Notice that $X\twoheaddownarrow \subseteq X\downarrow$ and $X\twoheaduparrow \subseteq X\uparrow$. Put $X\updownarrow = X\twoheaduparrow \cup X\twoheaddownarrow$, so every $X\updownarrow$ is a union of maximal chains of $\mathscr{P}$. For all $\mathscr{X} \subseteq \mathscr{P}$, define $\mathscr{X}\twoheaduparrow = \bigcup_{X\in\mathscr{X}} X\twoheaduparrow$, $\mathscr{X}\twoheaddownarrow = \bigcup_{X\in\mathscr{X}} X\twoheaddownarrow$, and $\mathscr{X}\updownarrow = \mathscr{X}\twoheaduparrow \cup \mathscr{X}\twoheaddownarrow$.

For any $\mathscr{S} \subseteq \mathscr{F}$, call $\mathscr{S}$ *critical* if there do not exist $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}$ such that
(1) $\mathscr{A} \cup \mathscr{B}$ is an antichain disjoint from $\mathscr{F}$;
(2) $\mathscr{S} \subseteq \mathscr{A}\downarrow \cup \mathscr{B}\uparrow$;
(3) $\mathscr{A} \subseteq \mathscr{S}\twoheaduparrow$, $\mathscr{B} \subseteq \mathscr{S}\twoheaddownarrow$.

Notice that $\mathscr{F}$ is critical since if there were $\mathscr{A}$ and $\mathscr{B}$ satisfying (1) and (2) for $\mathscr{S} = \mathscr{F}$, then any maximal antichain containing $\mathscr{A} \cup \mathscr{B}$ (of which there would have to be one) would be disjoint from $\mathscr{F}$, contradicting $\mathscr{F}$ being a fibre. Furthermore, since $\mathscr{F}$ is finite, $\mathscr{F}$ must contain a minimal critical set $\mathscr{M}$. That is, $\mathscr{M}$ is critical but no proper subset of $\mathscr{M}$ is critical. Notice that $\mathscr{M} \neq \emptyset$ since for $\mathscr{S} = \emptyset$, $\mathscr{A} = \mathscr{B} = \emptyset$ satisfy (1)–(3).

For each $X \in \mathscr{M}$ and each $Y \in X\twoheaduparrow\setminus\mathscr{F}$, define rank$(X, Y)$ to be the least i such that $Y \subseteq X \cup E_1 \cup \cdots \cup E_i$. For each $X \in \mathscr{M}$ and each $Y \in X\twoheaddownarrow\setminus\mathscr{F}$, define rank$(X, Y)$ to be the least i such that $Y = X\setminus(E_1 \cup E_2 \cup \cdots \cup E_i)$. For each $X \in \mathscr{M}$, define rank$(X) =$ min$\{$rank$(X, Y) : Y \in X\updownarrow\setminus\mathscr{F}\}$ — we know that $X\updownarrow\setminus\mathscr{F} \neq \emptyset$ since $X\updownarrow$ is a union of maximal chains of $\mathscr{P}$, and $\mathscr{F}$ contains no maximal chain of $\mathscr{P}$.

Let $M \in \mathscr{M}$ such that rank$(M) \leqslant$ rank(X) for every $X \in \mathscr{M}$. Let $M' \in M\updownarrow\setminus\mathscr{F}$ such that rank$(M, M') =$ rank(M). Since $\mathscr{M}\setminus\{M\}$ is not critical, we can pick $\mathscr{A}, \mathscr{B}$ satisfying conditions (1)–(3) for $\mathscr{S} = \mathscr{M}\setminus\{M\}$. Then $\mathscr{A}, \mathscr{B}$ satisfy (1) and (3) for $\mathscr{S} = \mathscr{M}$ also. $\mathscr{A}, \mathscr{B}$ cannot also satisfy (2) since $\mathscr{M}$ is critical, so $M \notin \mathscr{A}\downarrow \cup \mathscr{B}\uparrow$.

We have $M' \in M\updownarrow\setminus\{M\}$, so either $M' \in M\twoheaduparrow$ or $M' \in M\twoheaddownarrow$. We will examine each of these two cases separately and find that each of them leads to a contradiction.

First suppose $M' \in M\twoheaduparrow$. We will find $\mathscr{A}'$ so that $\mathscr{A}'$, $\mathscr{B}$ satisfy (1)–(3) for $\mathscr{S} = \mathscr{M}$, contradicting $\mathscr{M}$ being critical.

Let $\mathscr{A}' = (\mathscr{A}\setminus M'\downarrow) \cup \{M'\}$. We now derive a contradiction by showing that $\mathscr{A}', \mathscr{B}$ satisfy (1)–(3) for $\mathscr{S} = \mathscr{M}$. To see that (3) is satisfied (i.e. $\mathscr{A}' \subseteq \mathscr{M}\twoheaduparrow$ and $\mathscr{B} \subseteq \mathscr{M}\twoheaddownarrow$), observe the following. Since $M' \in \mathscr{M}\twoheaduparrow$ and $\mathscr{A}'\setminus\{M'\} \subseteq \mathscr{A} \subseteq \mathscr{M}\twoheaduparrow$, we know that $\mathscr{A}' \subseteq \mathscr{M}\twoheaduparrow$. And we already knew that $\mathscr{B} \subseteq \mathscr{M}\twoheaduparrow$, so (3) is satisfied.

To see that (2) is satisfied (i.e. $\mathscr{M} \subseteq \mathscr{A}'\downarrow \cup \mathscr{B}\uparrow$), observe that $\mathscr{A}'\downarrow \cup \mathscr{B}\uparrow = ((\mathscr{A}\setminus M'\downarrow) \cup \{M'\})\downarrow \cup \mathscr{B}\uparrow \supseteq \mathscr{A}\downarrow \cup \mathscr{B}\uparrow \supseteq \mathscr{M}\setminus\{M\}$, and $M \in M'\downarrow \subseteq \mathscr{A}'\downarrow$. So $\mathscr{M} \subseteq \mathscr{A}'\downarrow \cup \mathscr{B}\uparrow$. i.e., (2) is satisfied.

Now, we verify that (1) is satisfied (i.e. $\mathscr{A}' \cup \mathscr{B}$ is an antichain disjoint from $\mathscr{F}$), which takes longer than verifying the other two properties. It is obvious that

$\mathscr{A}' \cup \mathscr{B}$ is disjoint from $\mathscr{F}$. Since $\mathscr{A} \cup \mathscr{B}$ is an antichain, we only need to verify that $M' \notin (\mathscr{A}' \setminus \{M'\})\updownarrow$ and $M' \notin \mathscr{B}\updownarrow$ to show that (1) is satisfied. Since $M' \in M\uparrow$ and $M \notin \mathscr{A}\downarrow$, we know that $M' \notin \mathscr{A}\downarrow \supset (\mathscr{A}' \setminus \{M'\})\downarrow$. Since $\mathscr{A}' \setminus \{M'\} = \mathscr{A} \setminus \{M'\}\downarrow$, obviously $M' \notin (\mathscr{A}' \setminus \{M'\})\uparrow$. So $M' \notin (\mathscr{A}' \setminus \{M'\})\updownarrow$, which shows that $\mathscr{A}'$ is an antichain. Why is $M' \in \mathscr{B}\updownarrow$ impossible? Let $B \in \mathscr{B}$. $B \in \mathscr{M}\downarrow \setminus \mathscr{M}$ and so $B \cap E_1 = \emptyset$. But $M' \in M\uparrow \setminus \{M\}$ and so $M' \cap E_1 \neq \emptyset$, hence $M' \nsubseteq B$. Since there exists $B' \in \mathscr{M}$ such that $B \in B'\downarrow$ and $\operatorname{rank}(B') \geqslant \operatorname{rank}(M)$, we know that $B \cap (E_1 \cup \cdots \cup E_{\operatorname{rank}(M)}) = \emptyset$. Meanwhile, $M' \setminus M \subseteq E_1 \cup \cdots \cup E_{\operatorname{rank}(M)}$, and we know $B \setminus M \neq \emptyset$ since $M \notin \mathscr{B}\uparrow$. So clearly $\emptyset \neq B \setminus M \nsubseteq M' \setminus M$, and therefore $B \nsubseteq M'$. Hence $M' \notin \mathscr{B}\updownarrow$. But then $\mathscr{A}', \mathscr{B}$ satisfy (1)–(3) for $\mathscr{S} = \mathscr{M}$, a contradiction. So the case $M' \in M\uparrow$ cannot occur.

Now that we have eliminated the case $M' \in M\uparrow$, suppose that $M' \in M\downarrow$. So $M' = M \setminus (E_1 \cup E_2 \cup \cdots \cup E_{\operatorname{rank}(M)})$. Let $\mathscr{B}' = (\mathscr{B} \setminus M'\uparrow) \cup \{M'\}$. Let $\mathscr{S} = \mathscr{M}$. Then, dually to the case $M' \in M\uparrow$, (2) and (3) are satisfied by $\mathscr{A}, \mathscr{B}'$. That $\mathscr{A} \cup \mathscr{B}'$ is disjoint from $\mathscr{F}$ and $\mathscr{B}'$ is an antichain are also dual to facts in the case $M' \in M\uparrow$. But to show that $\mathscr{A} \cap M'\updownarrow = \emptyset$ and therefore (1) is satisfied requires more work in this case. Let $A \in \mathscr{A}$. $A \in \mathscr{M}\uparrow \setminus \mathscr{M}$ and so $A \cap E_1 \neq \emptyset$. But $M' \in M\downarrow \setminus \{M\}$ and so $M' \cap E_1 = \emptyset$, hence $A \nsubseteq M'$. It remains only to show that $M' \nsubseteq A$. For this we will need the fact that $|E_1| = \cdots = |E_{\operatorname{rank}(M)}| = 1$ which we now prove. It is obvious that this fact holds if $\operatorname{rank}(M) \leqslant r$. It is also clear that $\operatorname{rank}(M) < r+2$ since if $\operatorname{rank}(M) = r+2$ then $M' = \emptyset$, contradicting $M' \notin \mathscr{F}$. So the only other possibility to consider is $\operatorname{rank}(M) = r+1$ while M' is a singleton subset of E_{r+2}. In this case, the fact that $M' \notin \mathscr{F}$ together with Lemma 4 tells us that $|E_{r+2}| = 1$. Considering our assumption that $|E_{r+1}| = m \leqslant n = |E_{r+2}|$, this tells us that $|E_{r+1}| = 1$. This shows that $|E_1| = \cdots = |E_{\operatorname{rank}(M)}| = 1$. Since there exists $A' \in \mathscr{M}$ such that $A \in A'\uparrow$ and $\operatorname{rank}(A') \geqslant \operatorname{rank}(M)$, we know that $1 = |A \cap E_1| = |A \cap E_2| = \cdots = |A \cap E_{\operatorname{rank}(M)}|$. So $M \setminus M' \subseteq E_1 \cup \cdots \cup E_{\operatorname{rank}(M)} \subseteq A$. We know that $M \notin \mathscr{A}\downarrow$ so $M \nsubseteq A$, or, equivalently, $M \setminus A \neq \emptyset$. Since $M \setminus A \neq \emptyset$ while $M \setminus M' \subseteq A$, we know that $(M \cap M') \setminus A \neq \emptyset$, i.e. $M' \setminus A \neq \emptyset$, so $M' \nsubseteq A$. Thus $M' \notin \mathscr{A}\updownarrow$. But then $\mathscr{A}, \mathscr{B}'$ satisfy (1)–(3) for $\mathscr{S} = \mathscr{M}$, a contradiction. So the case $M' \in M\downarrow$ cannot occur.

With this contradiction, we have proven that $\mathscr{F}$ contains a maximal chain. □

Theorem 6. *Let l, m, and n be natural numbers. Then $K_{1,l} \times K_{1,m} \times K_{1,n}$ is skeletal.*

Proof. Assume for a contradiction that the theorem is false. Then there is some $(l,m,n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ for which the theorem fails. Pick (l,m,n) minimal in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ so that $K_{1,l} \times K_{1,m} \times K_{1,n}$ is not skeletal.

Let $E_1 = \{1_1, 1_2, \ldots, 1_l\}$, $E_2 = \{2_1, 2_2, \ldots, 2_m\}$, and $E_3 = \{3_1, 3_2, \ldots, 3_n\}$. Let $\mathscr{P} = \{X \subseteq \bigcup_{i=1}^{3} E_i : |X \cap E_i| \leqslant 1 \text{ for } i = 1,2,3\}$. Order $\mathscr{P}$ by set containment. Then $\mathscr{P} \cong K_{1,l} \times K_{1,m} \times K_{1,n}$ by Lemma 1. We will abbreviate set notation by omitting commas and parentheses. For instance, $1_1 2_1$ will stand for $\{1_1, 2_1\}$. As an example of the construction, Fig. 2 shows $K_{12} \times K_{13} \times K_{15}$ with most of the points labelled.

Let $\mathscr{F}$ be a fibre of $\mathscr{P}$ which contains no maximal chain of $\mathscr{P}$. Then $\emptyset \in \mathscr{F}$ since $\{\emptyset\}$ is a maximal antichain of $\mathscr{P}$. We know that $l, m, n > 1$ since otherwise $K_{1,l} \times K_{1,m} \times K_{1,n}$

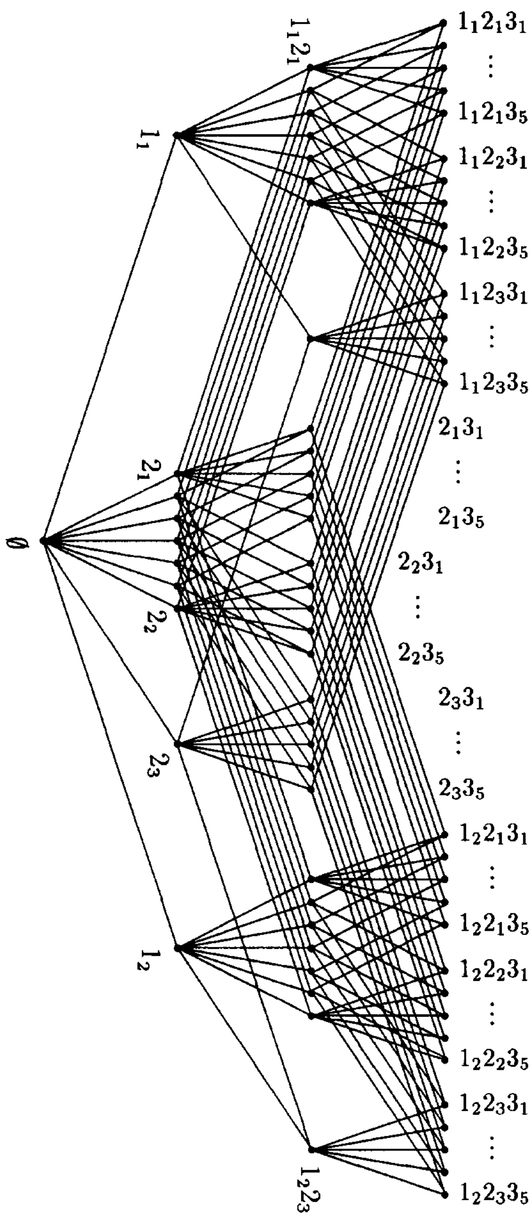

Fig. 2.

is skeletal by Theorem 5. So Lemma 4 tells us that $\mathscr{F}$ must include every singleton in $\mathscr{P}$.

Since the set of all doubletons in $\mathscr{P}$ is a maximal antichain, one of them must be in $\mathscr{F}$. Assume, without loss of generality, that 1_12_1 is in $\mathscr{F}$.

We will now construct $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ a maximal antichain of $\mathscr{P}$ disjoint from $\mathscr{F}$. Make the following definitions.

$$\mathscr{A}_1 = \{1_i 2_j 3_k : 1_i 2_j \in \mathscr{F},\ k = 1, \ldots, n\},\ \mathscr{P}_1 = \mathscr{A}_1 \downarrow,$$
$$\mathscr{B} = \{2_j : 2_j \notin \mathscr{A}_1 \downarrow\},\ \mathscr{P}_2 = \mathscr{B}\uparrow,$$
$$\mathscr{P}_3 = \mathscr{P} \setminus (\mathscr{P}_1 \cup \mathscr{P}_2).$$

It is easy to see that $\{\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3\}$ is a partition of $\mathscr{P}$. $\mathscr{P}_2 \cong (K_{1,l} \times K_{1,n}) \times \bar{b}$, where $\bar{b}$ is a $|\mathscr{B}|$-element antichain.

Obviously, $\mathscr{A}_1$ is an antichain and $\mathscr{P}_1 = \mathscr{A}_1 \updownarrow$. $\mathscr{A}_1$ is disjoint from $\mathscr{F}$ since $\emptyset$ and all the singletons are in $\mathscr{F}$, and $\mathscr{F}$ contains no maximal chain, so each $1_i 2_j \in \mathscr{F}$ implies $1_i 2_j 3_k \notin \mathscr{F}$ for $k = 1, \ldots, n$.

Next, we find an antichain $\mathscr{A}_2 \subseteq \mathscr{P}_2$ disjoint from $\mathscr{F}$ such that $\mathscr{P}_2 \subseteq \mathscr{A}_2 \updownarrow$. To do this, we shall break down $\mathscr{P}_2$ into smaller pieces. For each $2_j \in \mathscr{P}_2$, define $\mathscr{P}_{2,j} = 2_j \uparrow$. Then $\mathscr{P}_2 = \bigcup_j \mathscr{P}_{2,j}$. Choose a particular $\mathscr{P}_{2,j}$. We want to find an antichain $\mathscr{A}_{2,j} \subset \mathscr{P}_{2,j}$ such that $\mathscr{P}_{2,j} \subseteq \mathscr{A}_{2,j} \updownarrow$ and $\mathscr{A}_{2,j}$ is disjoint from $\mathscr{F}$. $\{2_j\}$ is not a satisfactory choice for $\mathscr{A}_{2,j}$ since $2_j \in \mathscr{F}$ (remember that all singletons are in $\mathscr{F}$). The next obvious choice to check is the set of all doubletons in $\mathscr{P}_{2,j}$. We know that each $1_i 2_j \notin \mathscr{F}$ since otherwise we would have $2_j \in \mathscr{P}_1$. Unfortunately, there is no guarantee that every $2_j 3_k \notin \mathscr{F}$. But we will make this choice whenever possible; i.e., if $\{2_j 3_k : k = 1, \ldots, n\} \cap \mathscr{F} = \emptyset$ then let

$$\mathscr{A}_{2,j} = \{1_i 2_j : i = 1, \ldots, l\} \cup \{2_j 3_k : k = 1, \ldots, n\}.$$

When $\{2_j 3_k : k = 1, \ldots, n\} \cap \mathscr{F} \neq \emptyset$ we will choose $\mathscr{A}_{2,j}$ as close as possible to the choice just described. We will modify the choice by replacing $2_j 3_k$ by $1_1 2_j 3_k$ for each $2_j 3_k \in \mathscr{F}$. Since $\mathscr{F}$ contains no maximal chain of $\mathscr{P}$, we know that $1_1 2_j 3_k \notin \mathscr{F}$ whenever $2_j 3_k \in \mathscr{F}$. This choice necessitates dropping $1_1 2_j$ from $\mathscr{A}_{2,j}$ to keep it an antichain. To put this in the proper notation, if $\{2_j 3_k : k = 1, \ldots, n\} \cap \mathscr{F} \neq \emptyset$ then let

$$\begin{aligned} \mathscr{A}_{2,j} = {} & \{1_1 2_j 3_k : 2_j 3_k \in \mathscr{F}, k = 1, \ldots, n\} \\ & \cup (\{2_j 3_k : k = 1, \ldots, n\} \setminus \mathscr{F}) \\ & \cup \{1_i 2_j : i = 2, \ldots, l\}. \end{aligned}$$

By either definition, $\mathscr{A}_{2,j}$ is an antichain disjoint from $\mathscr{F}$, and $2_j \uparrow \subseteq \mathscr{A}_{2,j} \updownarrow$. We have just described the choice of a particular $\mathscr{A}_{2,j}$. Apply the same method for every j for which $\mathscr{P}_{2,j}$ is defined. Then let $\mathscr{A}_2$ be the union of the $\mathscr{A}_{2,j}$'s. $\mathscr{A}_2$ is an antichain since every element of any $\mathscr{A}_{2,j}$ includes 2_j and no $2_{j'}$ for any $j' \neq j$. Thus $\mathscr{A}_2$ is an antichain disjoint from $\mathscr{F}$ and $\mathscr{P}_2 \subseteq \mathscr{A}_2 \updownarrow$. In fact, $\mathscr{A}_1 \cup \mathscr{A}_2$ is an antichain since $\mathscr{A}_1 \subset \max \mathscr{P}$ and each element of $\mathscr{A}_2$ includes a 2_j such that $2_j \notin \mathscr{A}_1 \downarrow$. So $\mathscr{A}_1 \cup \mathscr{A}_2$ is an antichain disjoint from $\mathscr{F}$ and $\mathscr{P}_1 \cup \mathscr{P}_2 \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2) \updownarrow$.

Another fact we will need is that $\mathscr{A}_2 \updownarrow \cap \mathscr{P}_3 \subset \mathscr{F}$. $\mathscr{A}_2 \uparrow \subset \mathscr{P}_2 \uparrow = \mathscr{P}_2$, leaving just $\mathscr{A}_2 \downarrow \cap \mathscr{P}_3 \subset \mathscr{F}$ to be verified. Since $\emptyset$ and all singletons are in $\mathscr{F}$, the only way this could fail is if there is some $X \in (\mathscr{A}_2 \downarrow \cap \mathscr{P}_3) \setminus \mathscr{F}$ where $|X| = 2$. Assume such an X

exists. Then there exists $Y \in \mathscr{A}_2$ such that $X \subset Y$ and $|Y|=3$. $|Y|=3$ and $Y \in \mathscr{A}_2$ imply that $Y=1_1 2_j 3_k$ for some j,k such that $2_j 3_k \in \mathscr{F}$. So $X \in \{1_1 2_j, 1_1 3_k, 2_j 3_k\}$. We can eliminate the case $X=2_j 3_k$ since $2_j 3_k \in \mathscr{F}$ (also $2_j 3_k \in \mathscr{P}_2$). We can eliminate the case $X=1_1 3_k$ since $1_1 2_1 \in \mathscr{F}$, so $1_1 2_1 3_k \in \mathscr{A}_1$ and $1_1 3_k \in \mathscr{A}_1{\downarrow} = \mathscr{P}_1$. So $X=1_1 2_j$. $1_1 2_j = X \in \mathscr{P}_3$ implies $1_1 2_j \notin \mathscr{P}_2$, so $2_j \in \mathscr{A}_1{\downarrow}$. But $Y=1_1 2_j 3_k \in \mathscr{A}_2 \subset \mathscr{P}_2$ implies $2_j \notin \mathscr{A}_1{\downarrow}$. With this contradiction, we conclude that $\mathscr{A}_2{\updownarrow} \cap \mathscr{P}_3 \subset \mathscr{F}$.

For each $i \in \{1,\dots,l\}$ such that $1_i{\uparrow} \cap \mathscr{P}_3 \neq \emptyset$, $1_i{\uparrow} \cap \mathscr{P}_3$ is skeletal as the following two cases show. If $1_i \notin \mathscr{P}_1$, then $1_i{\uparrow} \cap \mathscr{P}_3 = 1_i{\uparrow} \setminus (\mathscr{P}_1 \cup \mathscr{P}_2) = 1_i{\uparrow} \setminus \mathscr{P}_2 = 1_i{\uparrow} \setminus \bigcup_{2_j \in \mathscr{B}} 1_i 2_j{\uparrow} \cong K_{1,m-|\mathscr{B}|} \times K_{1,n}$. We know this product is skeletal since it follows almost immediately from Lemma 4 that $K_{1,p} \times K_{1,q}$ is skeletal for every $p,q \in \mathbb{N}$. (See also [3, Theorem 3.9].) Now suppose $1_i \in \mathscr{P}_1$. This means there is some $j \in \{1,\dots,m\}$ for which $1_i 2_j \in \mathscr{F}$, and hence $\{1_i 2_j 3_k : k=1,\dots,n\} \subseteq \mathscr{A}_1$, so $\{1_i 3_k : k=1,\dots,n\} \subseteq \mathscr{P}_1$. So $1_i{\uparrow} \cap \mathscr{P}_3 \subseteq \{1_i 2_j : j=1,\dots,m\} \cup \{1_i 2_j 3_k : j=1,\dots,m; k=1,\dots,n\}$. For $j=1,\dots,m$,

$$1_i 2_j \in \mathscr{F} \Leftrightarrow \{1_i 2_j 3_k : k=1,\dots,n\} \subset \mathscr{P}_1 \Leftrightarrow 1_i 2_j \in \mathscr{P}_1$$

and

$$\{1_i 2_j 3_k : k=1,\dots,n\} \subset \mathscr{P}_2 \Leftrightarrow 2_j \in \mathscr{B} \Leftrightarrow 1_i 2_j \in \mathscr{P}_2.$$

Hence, $1_i{\uparrow} \cap \mathscr{P}_3 \cong \sum_{\bar{r}} K_{1,n}$ where $r = m - |\mathscr{B}| - |\{1_i 2_j \in \mathscr{F}\}|$ and $\bar{r}$ denotes an r-element antichain. We know that $\sum_{\bar{r}} K_{1,n}$ is skeletal since it is clear that $K_{1,n}$ is skeletal, and it is easy to see that the cardinal sum of skeletal posets is skeletal. (This is an easy and special case of Theorem 2.3 in [3] which is concerned with lexicographic sums in general.)

Finally, we find an antichain $\mathscr{A}_3$ in $\mathscr{P}_3$ such that $\mathscr{A}_3$ is disjoint from $\mathscr{F}$ and $\mathscr{P}_3 \subset \mathscr{A}_3{\updownarrow}$. For each $i=1,\dots,l$, if $1_i{\uparrow} \cap \mathscr{P}_3 = \emptyset$, then put $\mathscr{A}_{3,i} = \emptyset$, otherwise pick $\mathscr{A}_{3,i}$ a maximal antichain of $1_i{\uparrow} \cap \mathscr{P}_3$ disjoint from $\mathscr{F}$, which we know is possible by the following. Since $1_i{\uparrow} \cap \mathscr{P}_3$ is skeletal, if $1_i{\uparrow} \cap \mathscr{P}_3 \cap \mathscr{F}$ is a fibre of $1_i{\uparrow} \cap \mathscr{P}_3 \neq \emptyset$, then it contains a maximal chain of $1_i{\uparrow} \cap \mathscr{P}_3$ whose union with $\{\emptyset, 1_i\}$ is a maximal chain of $\mathscr{P}$ contained in $\mathscr{F}$, a contradiction. Thus, $1_i{\uparrow} \cap \mathscr{P}_3 \cap \mathscr{F}$ is not a fibre of $1_i{\uparrow} \cap \mathscr{P}_3$ and we can pick $\mathscr{A}_{3,i}$ a maximal antichain of $1_i{\uparrow} \cap \mathscr{P}_3$ disjoint from $\mathscr{A}_3 = \bigcup_{i=1}^{l} \mathscr{A}_{3,i}$. $\mathscr{A}_3$ is an antichain since each $\mathscr{A}_{3,i}$ is an antichain, and each element of any $\mathscr{A}_{3,i}$ includes 1_i, making it impossible for elements of distinct $\mathscr{A}_{3,i}$'s to be comparable. So $\mathscr{A}_3$ is an antichain disjoint from $\mathscr{F}$ and $\mathscr{P}_3 \subseteq \mathscr{A}_3{\updownarrow}$. Recall $\mathscr{A}_2{\updownarrow} \cap \mathscr{P}_3 \subset \mathscr{F}$, and $\mathscr{A}_1{\updownarrow} = \mathscr{P}_1$, so $\mathscr{A}_3 \subset \mathscr{P}_3 \setminus (\mathscr{A}_1{\updownarrow} \cup \mathscr{A}_2{\updownarrow})$. Thus, $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ is a maximal antichain of $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$ and is disjoint from $\mathscr{F}$, which we assumed was a fibre.

This contradiction completes the proof. □

Acknowledgements

I am grateful to Prof. Bill Sands for advising me on how to make this paper presentable.

References

[1] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, Cambridge, MA, 1990.
[2] D. Duffus, B. Sands, P. Winkler, Maximal chains and antichains in Boolean lattices, SIAM J. Discrete Math. 3 (2) (May 1990) 197–205.
[3] P. Gibson, R. Maltby, Posets in which every cutset meets every fibre, preprint, 1993.
[4] R. Maltby, Cutsets and fibres in partially ordered sets, Master's thesis, University of Calgary, 1993.
[5] R. Maltby, S. Williamson, A note on maximal antichains in ordered sets, Order 9 (1992) 55–67.