Extensions of discrete classical orthogonal polynomials beyond the orthogonality

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ABSTRACT

It is well-known that the family of Hahn polynomials $\{h_n^{\alpha,\beta}(x;N)\}_{n\geq 0}$ is orthogonal with respect to a certain weight function up to degree N. In this paper we prove, by using the three-term recurrence relation which this family satisfies, that the Hahn polynomials can be characterized by a Δ-Sobolev orthogonality for every n and present a factorization for Hahn polynomials for a degree higher than N.

We also present analogous results for dual Hahn, Krawtchouk, and Racah polynomials and give the limit relations among them for all $n \in \mathbb{N}_0$. Furthermore, in order to get these results for the Krawtchouk polynomials we will obtain a more general property of orthogonality for Meixner polynomials.

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1. Introduction

In the last decade some of the classical orthogonal polynomials with non-classical parameters have been provided with certain non-standard orthogonality. For instance, K. H. Kwon and L. L. Littlejohn, in [9], established the orthogonality of the generalized Laguerre polynomials $\{L_n^{(-k)}\}_{n\geq 0}$, $k \geq 1$, with respect to the Sobolev inner product:

$$\langle f, g \rangle = (f(0), f'(0), \ldots, f^{(k-1)}(0))A \begin{pmatrix} g(0) \\ g'(0) \\ \vdots \\ g^{(k-1)}(0) \end{pmatrix} + \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx,$$

with A being a symmetric $k \times k$ real matrix. In [10], the same authors showed that the Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n\geq 0}$, are orthogonal with respect to the inner product

$$(f, g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^1 f'(x)g'(x)dx,$$

where d_1 and d_2 are real numbers.

Later, in [13], T.E. Pérez and M.A. Piñar gave a unified approach to the orthogonality of the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$, for any real value of the parameter α, by proving their orthogonality with respect to a non-diagonal Sobolev inner product, whereas in [14] they have shown how to use this orthogonality to obtain different properties of the generalized Laguerre polynomials.

M. Alfaro, M.L. Rezola, T.E. Pérez and M.A. Piñar have studied in [2] sequences of polynomials which are orthogonal with respect to a Sobolev bilinear form defined by

$$\mathcal{B}_S^{(N)} = (f(c), f'(c), \ldots, f^{(N-1)}(c)) A \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \left\langle \mathbf{u}, f^{(N)} g^{(N)} \right\rangle, \tag{1}$$

where $\mathbf{u}$ is a quasidefinite linear functional, $c \in \mathbb{R}$, N is a positive integer, and A is a symmetric $N \times N$ real matrix such that each of its principal submatrices is regular.

In particular, they deduced that Jacobi polynomials $\{P^{(-N,\beta)}\}_{n\geq 0}$, where $N + \beta$ is not a negative integer, are orthogonal with respect to the bilinear form given in (1), for $\mathbf{u}$ the Jacobi functional corresponding to the weight function $\rho^{(0,\beta+N)}(x) = (1+x)^{\beta+N}$ and $c = 1$.

The remaining cases for the Jacobi polynomials, i.e. both parameters, α and β, negative integers, were considered in [3] where they proved that such sequences satisfy a Sobolev orthogonality.

M. Álvarez de Morales, T.E. Pérez and M.A. Piñar in [7] have studied the sequence of the monic Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n\geq 0}$, where N is a positive integer. They have shown that this sequence is orthogonal with respect to a Sobolev inner product of the form

$$(f, g)_S^{2N} = (F(1)|F(-1)) A (G(1)|G(-1))^{\mathrm{T}} + \int_{-1}^{1} f^{(2N)}(x) g^{(2N)}(x)(1-x^2)^N \mathrm{d}x,$$

where

$$(F(1)|F(-1)) = (f(1), f'(1), \ldots, f^{(N-1)}(1), f(-1), f'(-1), \ldots, f^{(N-1)}(-1)),$$

$A = Q^{-1} D (Q^{-1})^{\mathrm{T}}$, Q is a non-singular matrix whose entries are the consecutive derivatives of the Gegenbauer polynomials evaluated at the points 1 and -1, and D is an arbitrary diagonal positive definite matrix.

M. Álvarez de Morales, T.E. Pérez, M.A. Piñar and A. Ronveaux in [8] have studied the sequence of the monic Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, for $0 < \mu < 1$ and $\gamma \in \mathbb{R}$. They have shown that this sequence is orthogonal with respect to the inner product

$$(f, g)_S^{(K,\gamma+K)} = \sum_{x=0}^{\infty} F(x) \Lambda^{(K)} G(x)^{\mathrm{T}} \rho^{(\gamma+K,\mu)}(x), \quad x \in [0, \infty),$$

where K is a non-negative integer, $F(x) = (f(x), \Delta f(x), \ldots, \Delta^K f(x))$, Δ and ∇ are, respectively, the finite forward and backward difference operators defined by

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1),$$

$\rho^{(\gamma+K,\mu)}$ denotes the weight function associated with the monic classical Meixner polynomials $\{M_n^{(\gamma+K,\mu)}\}_{n\geq 0}$, and $\Lambda^{(K)}$ is a positive definite $(K+1) \times (K+1)$ matrix, with $K \geq \max\{0, -\gamma+1\}$. Usually, when the inner product is defined by using the difference operator instead the differential operator, the orthogonality is said to be of Δ-Sobolev type.

These examples suggest that the classical orthogonal polynomials with non-classical parameters can be provided with a Sobolev or a Δ-Sobolev orthogonality. Furthermore, as was pointed out in [15], a Sobolev–Askey tableau should be established.

In this paper we study discrete classical orthogonal polynomials which exist only up to a certain degree. This happens for the Krawtchouk, Hahn, dual Hahn and Racah polynomials which satisfy a discrete orthogonality with a finite number of masses. These families exhibit this finite character in several ways since there is a negative integer as a denominator parameter in the hypergeometric representation. Also in the three-term recurrence relation

$$x p_n = p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \quad n = 0, 1, 2, \ldots$$

there exists M such that $\gamma_M = 0$. But there are other ways to characterize the discrete classical orthogonal polynomials which, apparently, do not say anything about whether the sequence of polynomials is finite or infinite (see, for example, [1]).

We show that these polynomials can be considered for degrees higher than M, in fact for all degrees, and the main properties still hold except the orthogonality. However, using difference properties of the polynomials, we find a Δ-Sobolev orthogonality of the form

$$\langle f, g \rangle_S := \langle \mathbf{u}_0, fg \rangle + \left\langle \mathbf{u}_1, (\Delta^M f)(\Delta^M g) \right\rangle,$$

with respect to which the polynomials are characterized, where $\mathbf{u}_0$ and $\mathbf{u}_1$ are certain classical functionals.

Also we obtain a factorization for these polynomials of the form

$$p_n = p_M q_{n-M}, \quad n = M, M+1, M+2, \ldots,$$

where p_M vanishes in the M masses of the orthogonality measure associated with the linear functional $\mathbf{u}_0$, and q_{n-M} is a classical orthogonal polynomial that is at the same level in the Askey tableau as p_n.

The structure of the paper is as follows. The case of Hahn polynomials is studied in Section 2 in detail. In Sections 3–5, we get analogous results for the Racah, dual Hahn and Krawtchouk polynomials, which satisfy a discrete orthogonality with a finite number of masses, applying analogous reasoning that we have considered for Hahn polynomials. Finally in Section 6 we show that the known limit relations involving the above families hold for any $n \in \mathbb{N}_0$. The Appendix is devoted to proving more general orthogonal relations for Meixner polynomials which are used in Section 5.

2. Hahn polynomials

The monic classical Hahn polynomials $h_n^{\alpha,\beta}(x; N), n = 0, 1, \ldots, N, N \in \mathbb{N}_0$, can be defined by their explicit representation in terms of the hypergeometric function (see e.g. [12, p. 33]):

$$h_n^{\alpha,\beta}(x; N) = \frac{(-N, \alpha+1)_n}{(\alpha+\beta+n+1)_n} \, _3F_2 \left(\begin{matrix} -n, \alpha+\beta+n+1, -x \\ -N, \alpha+1 \end{matrix} \middle| 1 \right), \quad n = 0, 1, \ldots, N, \tag{2}$$

where $(a)_n$ denotes the Pochhammer symbol

$$(a)_0 := 1, \qquad (a)_n := a(a+1)\cdots(a+n-1), \quad n = 1, 2, 3, \ldots$$

and $(a_1, a_2, \ldots, a_j)_n := (a_1)_n (a_2)_n \cdots (a_j)_n$. These polynomials satisfy the following property of orthogonality:

$$\sum_{x=0}^{N} h_n^{\alpha,\beta}(x; N) h_m^{\alpha,\beta}(x; N) \rho^{\alpha,\beta}(x; N) = 0, \quad 0 \le m < n \le N, \tag{3}$$

where

$$\rho^{\alpha,\beta}(x; N) = \frac{\Gamma(\beta+N+1-x)\Gamma(\alpha+1+x)}{\Gamma(1+x)\Gamma(N+1-x)}.$$

When $\alpha, \beta > -1$ or $\alpha, \beta < -N$ this is a positive definite orthogonality. However, it is possible to consider general complex parameters α and β and (3) remains by using analytic continuation.

Furthermore, Atakishiyev and Suslov [6] considered Hahn polynomials for general complex parameters α, β and N and nowadays they are known as continuous Hahn polynomials [5]. The monic ones are

$$p_n(x; a, b, c, d) = D_n \, _3F_2 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \middle| 1 \right),$$

where

$$D_n = \frac{i^n (a+c, a+d)_n}{(n+a+b+c+d-1)_n},$$

and since the parameter a causes only a translation, Hahn and continuous Hahn polynomials are related in the following way

$$p_n(x; a, b, c, d) = i^n h_n^{a+d-1, b+c-1}(-a-ix; -a-c),$$
$$h_n^{\alpha,\beta}(x; N) = (-i)^n p_n(ix; 0, \beta+N+1, -N, \alpha+1). \tag{4}$$

Continuous Hahn polynomials satisfy a non-Hermitian orthogonality

$$\int_C p_n(x; a, b, c, d) x^m w(x; a, b, c, d) dx = 0, \quad 0 \le m < n, \tag{5}$$

where

$$w(x; a, b, c, d) = \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix), \tag{6}$$

and C is a contour on $\mathbb{C}$ from $-\infty$ to ∞ which separates the increasing poles

$$(a+k)i, (b+k)i, \quad k = 0, 1, 2, \ldots,$$

from the decreasing ones

$$(-c-k)i, (-d-k)i, \quad k = 0, 1, 2, \ldots,$$

which can be done when these two sets of poles are disjoint, i.e.

$$-a - c, -a - d, \ -b - c, -b - d \notin \mathbb{N}_0.$$

The continuous Hahn polynomials also satisfy, among others, a second order linear difference equation, a Rodrigues formula, the TTRR which will be useful later

$$xp_n(x) = p_{n+1}(x) + (B_n + a)ip_n(x) + C_n p_{n-1}(x),$$

with

$$B_n = \frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)} - \frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}$$

$$
\begin{aligned}
C_n &= \frac{n(n+b+c-1)(n+b+d-1)(n+a+b+c+d-2)}{(2n+a+b+c+d-1)(2n+a+b+c+d-2)} \\
&\quad \times \frac{(n+a+c-1)(n+a+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)},
\end{aligned}
\tag{7}
$$

and they have several generating functions.

Let us focus our attention on (2), it can be rewritten as

$$h_n^{\alpha,\beta}(x; N) = \frac{(\alpha+1)_n}{(\alpha+\beta+n+1)_n} \sum_{k=0}^{n} \frac{(-n, \alpha+\beta+n+1, -x)_k (-N+k)_{n-k}}{(\alpha+1, 1)_k}, \tag{8}$$

which is valid for every n and, in this way, it can be used to define Hahn polynomials for any $n \in \mathbb{N}_0$.

These new polynomials satisfy the following result:

Theorem 1. *Let N be a non-negative integer, then the Hahn polynomials $h_n^{\alpha,\beta}(x; N)$ for $n \geq 0$ satisfy the following properties:*

(i)

$$h_{-1}^{\alpha,\beta}(x; N) = 0, \qquad h_0^{\alpha,\beta}(x; N) = 1,$$

$$xh_n^{\alpha,\beta}(x; N) = h_{n+1}^{\alpha,\beta}(x; N) + \beta_n h_n^{\alpha,\beta}(x; N) + \gamma_n h_{n-1}^{\alpha,\beta}(x; N), \quad n = 0, 1, 2, \ldots \tag{9}$$

where

$$\beta_n = \frac{(\alpha+1)N(\alpha+\beta) + n(2N-\alpha+\beta)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)},$$

$$\gamma_n = \frac{n(N+1-n)(\alpha+\beta+n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}. \tag{10}$$

(ii) *For any integer k, $0 \leq k \leq n$,*

(ii.1) $\Delta^k h_n^{\alpha,\beta}(x; N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x; N-k),$ \hfill (11)

(ii.2) $\nabla^k h_n^{\alpha,\beta}(x; N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x-k; N-k).$ \hfill (12)

(iii) *Second order linear difference equation:*

$$x(\beta+N+1-x)\nabla\Delta h_n^{\alpha,\beta}(x; N) + ((\alpha+1)N \tag{13}$$

$$-(\alpha+\beta+2)x)\Delta h_n^{\alpha,\beta}(x; N) + \lambda_n h_n^{\alpha,\beta}(x; N) = 0, \tag{14}$$

with $\lambda_n = n(n+\alpha+\beta+1)$.

(vi) *Rodrigues formula:*

$$h_n^{\alpha,\beta}(x; N)\rho^{\alpha,\beta}(x; N) = \frac{(-1)^n}{(\alpha+\beta+n+1)_n} \nabla^n(\rho^{\alpha+n,\beta+n}(x; N-n)), \tag{15}$$

with

$$\rho^{\alpha,\beta}(x; N) = \frac{\Gamma(\beta+N+1-x)\Gamma(\alpha+1+x)}{\Gamma(1+x)\Gamma(N+1-x)}.$$

(v) *Generating function:*

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+n+1)_n}{(\beta+1, \alpha+1, 1)_n} h_n^{\alpha,\beta}(x; N)t^n = {}_1F_1\left(\begin{matrix} -x \\ \alpha+1 \end{matrix} \middle| -t\right) {}_1F_1\left(\begin{matrix} -N+x \\ \beta+1 \end{matrix} \middle| t\right), \tag{16}$$

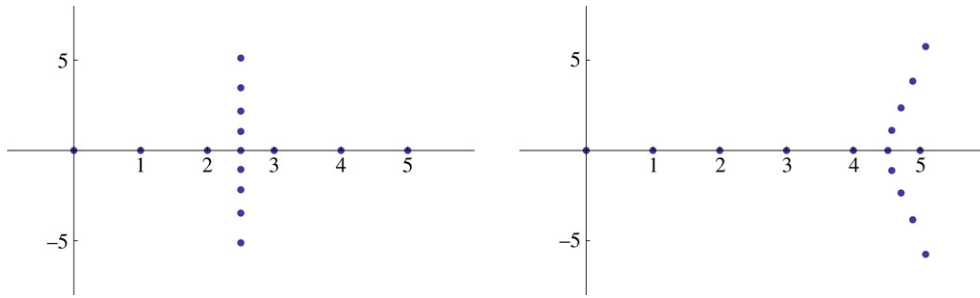*valid for $x \in \mathbb{C}$ and $-\alpha-1, -\beta-1 \notin \mathbb{N}$.*

Fig. 1. Zeros of $h_{15}^{1,1}(x; 5)$ (left) and $h_{15}^{1,15}(x; 5)$ (right).

Remark 2. Apparently, the conditions $-\alpha, -\beta, -\alpha - \beta \notin \mathbb{N}$ are necessary in (i)–(iv), but this problem disappears by using a suitable normalization, for instance, if there is a polynomial dependence on the parameters.

The proof is straightforward using the well-known properties of continuous Hahn polynomials (see [12]) and the limit relation

$$h_n^{\alpha,\beta}(x; N) = \lim_{\varepsilon \to 0} (-\mathrm{i})^n \, p_n(\mathrm{i}x; 0, \beta + N + \varepsilon + 1, -N - \varepsilon, \alpha + 1),$$

easily obtained from (4). Note that for small ε the weight function associated with the continuous Hahn polynomials, $w(x; 0, \alpha + N + \varepsilon + 1, -N - \varepsilon, \beta + 1)$, satisfies the poles separation condition.

Now we center our attention on the behavior of the zeros. Fig. 1 shows the standard configuration of the zeros of Hahn polynomials for degree greater than $N + 1$. In fact, they have $N + 1$ zeros on $0, 1, \ldots, N$, and the other $n - N - 1$ zeros are located on unknown curves of the complex plane. In the special case $\alpha = \beta \in \mathbb{R}^+$ this curve is the line $\{ti + N/2 : t \in \mathbb{R}\}$.

The following result is straightforward taking into account (8) and that if $n \geq N + 1$, then $(-N + k)_{n-k} = 0$, for $k = 0, \ldots, N$.

Proposition 3. *Let N be a non-negative integer. For every $n \geq N + 1$,*

$$h_n^{\alpha,\beta}(x; N) = (x - N)_{N+1}(-\mathrm{i})^{n-N-1} p_{n-N-1}(\mathrm{i}x; N + 1, \beta + N + 1, 1, \alpha + 1)$$

$$= (x - N)_{N+1}(-\mathrm{i})^{n-N-1} p_{n-N-1}\left(\left(x - \frac{N}{2}\right)\mathrm{i}; 1 + \frac{N}{2}, \beta + 1 + \frac{N}{2}, 1 + \frac{N}{2}, \alpha + 1 + \frac{N}{2}\right).$$

Remark 4. Note that, as was expected from (3), $h_{N+1}^{\alpha,\beta}(x; N) = (x - N)_{N+1}$ which vanishes on $0, 1, \ldots, N$ and is independent of α and β – this fact is non-trivial from the other ways to characterize these polynomials (see e.g. [4]). On the other hand, in the case $\alpha = \beta$ the continuous Hahn polynomial in the right-hand side is a linear transformation of a real polynomial.

Now we establish the main result.

Theorem 5. *Let N be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that*

$$- \alpha, -\beta \notin \{1, 2, \ldots, N, N + 2, \ldots\}, \tag{17}$$

and

$$- \alpha - \beta \notin \{1, 2, \ldots, 2N + 1, 2N + 3, \ldots\}. \tag{18}$$

Then the family of monic Hahn polynomials $h_n^{\alpha,\beta}(x; N)$ for $n \geq 0$ is a MOPS with respect to the following Δ-Sobolev inner product:

$$(f, g)_S = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha,\beta}(x; N) + \int_C (\Delta^{N+1}f(z))(\Delta^{N+1}g(z))\omega^{\alpha,\beta}(z; N)\mathrm{d}z,$$

where

$$\rho^{\alpha,\beta}(x; N) = \frac{\Gamma(\beta + N + 1 - x)\Gamma(\alpha + x + 1)}{\Gamma(N + 1 - x)\Gamma(x + 1)},$$

$$\omega^{\alpha,\beta}(z; N) = \Gamma(-z)\Gamma(\beta + N + 1 - z)\Gamma(1 + z)\Gamma(\alpha + N + 2 + z),$$

and C is a complex contour from $-\infty\mathrm{i}$ to $\infty\mathrm{i}$ which separates the poles of the functions $\Gamma(-z)\Gamma(\beta + N + 1 - z)$ and $\Gamma(1 + z)\Gamma(\alpha + N + 2 + z)$. Furthermore, this Δ-Sobolev inner product characterizes the polynomials $h_n^{\alpha,\beta}(x; N)$ for all $n \in \mathbb{N}_0$.

Remark 6. Note that the conditions (17) and (18) are equivalent to the existence of a unique n such that $\gamma_n = 0$, and therefore $n = N + 1$. Furthermore, (17) is equivalent to $h_n^{\alpha,\beta}(x; N)$ is uniquely determined by (3) for $n \leq N$ together with the poles separation condition of $w^{\alpha,\beta}(z; N)$ given in the above theorem.

Proof. If $0 \leq m < n \leq N + 1$, since $\Delta^{N+1} x^m = 0$,

$$(h_n^{\alpha,\beta}(x; N), x^m)_S = \sum_{x=0}^{N} h_n^{\alpha,\beta}(x; N) x^m \rho^{\alpha,\beta}(x; N) = 0.$$

If $n \geq N + 1$ and $m \leq N$, by Proposition 3,

$$(h_n^{\alpha,\beta}(x; N), x^m)_S = \sum_{x=0}^{N} h_n^{\alpha,\beta}(x; N) x^m \rho^{\alpha,\beta}(x; N) = 0,$$

and if $N + 1 \leq m < n$, then

$$
\begin{aligned}
(h_n^{\alpha,\beta}(x; N), x^m)_S &= \int_C (\Delta^{N+1} h_n^{\alpha,\beta}(z; N))(\Delta^{N+1} z^m) \omega^{\alpha,\beta}(z; N) \mathrm{d}z \\
&= \int_C p_{n-N-1}(zi; 0, \beta + N + 1, 1, \alpha + N + 2) q_{m-N-1}(z) \omega^{\alpha,\beta}(z; N) \mathrm{d}z = 0,
\end{aligned}
$$

where q_{m-N-1} is a polynomial of degree $m - N - 1$.

Now we show that the inner product characterizes the polynomials. If $n \leq N$ then, due to (17), we get

$$(h_n^{\alpha,\beta}(x; N), x^n)_S = \sum_{x=0}^{N} h_n^{\alpha,\beta}(x; N) x^n \rho^{\alpha,\beta}(x; N) \neq 0,$$

and if $n \geq N + 1$ we obtain

$$
\begin{aligned}
(h_n^{\alpha,\beta}(x; N), x^n)_S &= \int_C (\Delta^{N+1} h_n^{\alpha,\beta}(z; N))(\Delta^{N+1} z^n) \omega^{\alpha,\beta}(z; N) \mathrm{d}z \\
&= \int_C p_{n-N-1}(zi; 0, \beta + N + 1, 1, \alpha + N + 2) q_{n-N-1}(z) \omega^{\alpha,\beta}(z; N) \mathrm{d}z \neq 0,
\end{aligned}
$$

since q_{n-N-1} is a polynomial of degree $n - N - 1$ and the coefficient C_k, given in (7), is different from zero for $a = 0$, $b = \beta + N + 1, c = 1$, and $d = \alpha + N + 2$, for $k = 1, 2, \ldots$. □

3. Racah polynomials

We can apply an analogous process for the Racah polynomials

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = r_n \, {}_4F_3 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix} \middle| 1 \right), \tag{19}$$

in which

$$r_n = \frac{(\alpha + 1, \beta + \delta + 1, \gamma + 1)_n}{(n + \alpha + \beta + 1)_n},$$

and $\lambda(x) = x(x + \gamma + \delta + 1)$, by using the Wilson polynomials [12, p. 28]

$$W_n(x^2; a, b, c, d) = w_n \, {}_4F_3 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} \middle| 1 \right),$$

in which

$$w_n = \frac{(-1)^n (a + b, a + c, a + d)_n}{(n + a + b + c + d - 1)_n}.$$

In fact,

$$
\begin{aligned}
&R_n(\lambda(x); \alpha, \beta, \gamma, \delta) \\
&\quad = (-1)^n W_n \left(\left(ix + i \frac{\gamma + \delta + 1}{2} \right)^2; \frac{\gamma + \delta + 1}{2}, \alpha - \frac{\gamma + \delta - 1}{2}, \beta + \frac{-\gamma + \delta + 1}{2}, \frac{\gamma - \delta + 1}{2} \right),
\end{aligned}
$$

where $\alpha + 1 = -N$ or $\beta + \delta + 1 = -N$ or $\gamma + 1 = -N$, with N a non-negative integer, and

$$W_n(x^2; a, b, c, d) = (-1)^n R_n\left(\lambda(-a - ix); a + b - 1, c + d - 1, a + d - 1, a - d\right),$$

where $\lambda(t) = t(t + 2a)$.

On the other hand taking, for instance, $\alpha + 1 = -N$ we get the following factorization for $n > N$:

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = R_{N+1}(\lambda(x); -N - 1, \beta, \gamma, \delta)(-1)^{n-N-1}$$
$$\times W_{n-N-1}\left(\left(i\left(x + \frac{\gamma + \delta + 1}{2}\right)\right)^2; N + \frac{\gamma + \delta + 3}{2}, \frac{-\gamma - \delta + 1}{2}, \beta + \frac{-\gamma - \delta + 1}{2}, \frac{\gamma - \delta + 1}{2}\right).$$

In this case, the Racah polynomials satisfy the following Δ-Sobolev orthogonality:

$$\langle p, q \rangle_S = \langle p, q \rangle_d + \left\langle \left(\frac{\Delta}{\Delta\lambda}\right)^{N+1} p, \left(\frac{\Delta}{\Delta\lambda}\right)^{N+1} q \right\rangle_c,$$

with

$$\langle p, q \rangle_d = \sum_{x=0}^{N} p(x)q(x) \frac{(\alpha + 1, \beta + \delta + 1, \gamma + 1, \gamma + \delta + 1, (\gamma + \delta + 3)/2)_x}{(-\alpha + \gamma + \delta + 1, -\beta + \gamma + 1, (\gamma + \delta + 1)/2, \delta + 1, 1)_x},$$

$$\langle p, q \rangle_c = \int_C p(z^2)q(z^2)\nu\left(zi + i + i\frac{\gamma + \delta + N}{2}\right)\nu\left(-\left(zi + i + i\frac{\gamma + \delta + N}{2}\right)\right)dz,$$

where

$$\nu(z) \equiv \nu(z; a, b, c, d) = \frac{\Gamma(a + iz)\Gamma(b + iz)\Gamma(c + iz)\Gamma(d + iz)}{\Gamma(2z)},$$

being

$$a = 1 + \frac{\gamma + \delta + N}{2}, \qquad b = \frac{-\gamma - \delta - N}{2}, \qquad c = \beta + 1 + \frac{-\gamma + \delta + N}{2}, \qquad d = 1 + \frac{\gamma - \delta + N}{2},$$

and C is the imaginary axis deformed so as to separate the increasing sequences of poles

$$k, -1 - \gamma - \delta - N + k, \beta - \gamma + k, -\delta + k, \quad k = 0, 1, 2, \ldots$$

from the decreasing sequences

$$-\gamma - \delta - N - 2 - k, -1 - k, -\beta - \delta - N - 2 - k, -\gamma - N - 2 - k, \quad k = 0, 1, 2, \ldots.$$

Of course, we need to assume that these two sets of poles are disjoint, i.e.,

$$2a, a + b, a + c, \ldots, c + d, 2d \notin \{0, -1, -2, \ldots\}.$$

On the other hand, in this case, i.e. $\alpha + 1 = -N$, we get the following generating functions (see [12, p. 29]) which are valid for all $x \in \mathbb{C}$:

$$\sum_{n=0}^{\infty} \frac{(\alpha + 1, \gamma + 1)_n}{(\alpha - \delta + 1)_n n!} R_n(\lambda(x); \alpha, \beta, \gamma, \delta)t^n = {}_2F_1\left(\begin{array}{c} -x, -x + \beta - \gamma \\ \beta + \delta + 1 \end{array}\middle| t\right) {}_2F_1\left(\begin{array}{c} x + \alpha + 1, x + \gamma + 1 \\ \alpha - \delta + 1 \end{array}\middle| t\right),$$

$$\sum_{n=0}^{\infty} \frac{(\alpha + 1, \beta + \delta + 1)_n}{(\alpha - \beta - \gamma + 1)_n n!} R_n(\lambda(x); \alpha, \beta, \gamma, \delta)t^n = {}_2F_1\left(\begin{array}{c} -x, -x - \delta \\ \gamma + 1 \end{array}\middle| t\right) {}_2F_1\left(\begin{array}{c} x + \alpha + 1, x + \beta + \delta + 1 \\ \alpha + \beta - \gamma + 1 \end{array}\middle| t\right).$$

See e.g. [17] or [11] to get more information about algebraic properties and applications for Wilson polynomials.

4. Dual Hahn polynomials

We can apply an analogous process for the dual Hahn polynomials

$$R_n(\lambda(x); \gamma, \delta, N) = (\gamma + 1, -N)_n \, {}_3F_2\left(\begin{array}{c} -n, -x, x + \gamma + \delta + 1 \\ \gamma + 1, -N \end{array}\middle| 1\right), \tag{20}$$

with $\lambda(x) = x(x + \gamma + \delta + 1)$, by using the continuous dual Hahn polynomials [12, p. 31]

$$S_n(x^2; a, b, c) = (-1)^n(a + b, a + c)_n \, {}_3F_2\left(\begin{array}{c} -n, a + ix, a - ix \\ a + b, a + c \end{array}\middle| 1\right).$$

In fact,

$$R_n(\lambda(x); \gamma, \delta, N) = (-1)^n S_n \left(\left(ix + i\frac{\gamma + \delta + 1}{2} \right)^2 ; \frac{\gamma + \delta + 1}{2}, \frac{\gamma - \delta + 1}{2}, -N - \frac{\gamma + \delta + 1}{2} \right),$$

and

$$S_n(x^2; a, b, c) = (-1)^n R_n \left(\lambda(-a - ix); a + b - 1, a - b, -a - c \right),$$

where $\lambda(t) = t(t + 2a)$.

We get the following factorization for $n > N$:

$$R_n(\lambda(x); \gamma, \delta, N) = R_{N+1}(\lambda(x); \gamma, \delta, N) (-1)^{n-N-1}$$

$$\times S_{n-N-1} \left(\left(xi + i\frac{\gamma + \delta + 1}{2} \right)^2 ; N + \frac{\gamma + \delta + 2}{2}, \frac{-\gamma - \delta + 1}{2}, \frac{\gamma - \delta + 1}{2} \right).$$

Dual Hahn polynomials polynomials satisfy the following Δ-Sobolev orthogonality:

$$\langle p, q \rangle_S = \langle p, q \rangle_d + \left\langle \left(\frac{\Delta}{\Delta\lambda} \right)^{N+1} p, \left(\frac{\Delta}{\Delta\lambda} \right)^{N+1} q \right\rangle_c,$$

with

$$\langle p, q \rangle_d = \sum_{x=0}^{N} p(x)q(x) \frac{(2x + \gamma + \delta + 1)(\gamma + 1, -N)_x}{(-1)^x (x + \gamma + \delta + 1)_{N+1}(\delta + 1, 1)_x},$$

$$\langle p, q \rangle_c = \int_C p(z^2)q(z^2)w(z; \gamma, \delta, N)dz,$$

where

$$w(z; \alpha, \beta, N) = v \left(zi + i + i\frac{\gamma + \delta + N}{2}; \gamma, \delta, N \right) v \left(-zi - i - i\frac{\gamma + \delta + N}{2}; \gamma, \delta, N \right),$$

being

$$v(z; a, b, c) = \frac{\Gamma(a + iz)\Gamma(b + iz)\Gamma(c + iz)}{\Gamma(2zi)},$$

and

$$a = 1 + \frac{\gamma + \delta + N}{2}, \qquad b = 1 + \frac{\gamma - \delta + N}{2}, \qquad c = -\frac{\gamma + \delta + N}{2},$$

and C is the imaginary axis deformed so as to separate the increasing sequences of poles

$$k, \quad -\gamma - \delta - N - 1 + k, \quad -\delta + k, \qquad k = 0, 1, 2, \ldots,$$

from the decreasing sequences

$$-\gamma - \delta - N - 2 - k, \quad -1 - k, \quad -\gamma - N - 2 - k, \qquad k = 0, 1, 2, \ldots.$$

Of course, we need to assume that these two sets of poles are disjoint, i.e.,

$$2a, a + b, a + c, \ldots, 2c \notin \{0, -1, -2, \ldots\}.$$

On the other hand we get the following generating functions (see [12, p. 36]) which are valid for all $x \in \mathbb{C}$:

$$(1 - t)^{N-x} {}_2F_1 \left(\begin{matrix} -x, -x - \delta \\ \gamma + 1 \end{matrix} \middle| t \right) = \sum_{n=0}^{\infty} \frac{(-N)_n}{n!} R_n(\lambda(x); \gamma, \delta, N) t^n.$$

$$(1 - t)^x {}_2F_1 \left(\begin{matrix} x - N, x + \gamma + 1 \\ -\delta - N \end{matrix} \middle| t \right) = \sum_{n=0}^{\infty} \frac{(\gamma + 1, -N)_n}{(-\delta - N)_n n!} R_n(\lambda(x); \gamma, \delta, N) t^n.$$

5. Krawtchouk polynomials

Similarly, properties for the Krawtchouk polynomials

$$K_n(x; p, N) = (-N)_n p^n \, _2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p} \right),$$

(21)

with $p \in \mathbb{C}$, $p \neq 0, 1$, can be obtained via Meixner polynomials

$$M_n(x; \beta, c) = \frac{c^n (\beta)_n}{(c-1)^n} \, _2F_1 \left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - \frac{1}{c} \right), \quad \beta > 0, 0 < c < 1.$$

In fact,

$$K_n(x; p, N) = M_n \left(x; -N, \frac{p}{p-1} \right),$$

$$M_n(x; \beta, c) = K_n \left(x; -\beta, \frac{c}{c-1} \right),$$

setting $\beta = -N$ and $c = p/(p-1)$.

We have the following factorization for the Krawtchouk polynomials for $n > N$:

$$K_n(x; p, N) = K_{N+1}(x; p, N) M_{n-N-1}(x - N - 1; N + 2, p/(p-1)).$$

Furthermore, these polynomials satisfy the following Δ-Sobolev orthogonality:

$$\langle r, q \rangle_S = \langle r, q \rangle_d + \left\langle \Delta^{N+1} r, \Delta^{N+1} q \right\rangle_c,$$

with

$$\langle r, q \rangle_d = \sum_{x=0}^{N} r(x) q(x) \frac{p^x (1-p)^{N-x}}{\Gamma(x+1) \Gamma(N-x+1)},$$

(22)

$$\langle r, q \rangle_c = \int_C r(z) q(z) \Gamma(-z) \Gamma(1+z) \left(\frac{p}{1-p} \right)^z \, dz,$$

(23)

where C is the imaginary axis deformed so as to separate the increasing from the decreasing sequence of poles of the weight function, in fact in this case we can consider the curve $C = \{-1/2 + ti : t \in \mathbb{R}\}$.

Remark 7. Note that the property of orthogonality for the Krawtchouk polynomials (22) is valid for all $p \in \mathbb{C}$, with $p \neq 0, 1$ by using an analytic continuation for the standard weight function associated with the Krawtchouk polynomials.

On the other hand, we get the following generating function (see [12, p. 47]) which is valid for all $x \in \mathbb{C}$:

$$\left(1 - \frac{1-p}{p} t \right)^x (1+t)^{N-x} = \sum_{n=0}^{\infty} \binom{N}{n} K_n(x; p, N) t^n.$$

6. Limit relations between hypergeometric orthogonal polynomials

In this section, we study the limit relations involving the orthogonal polynomials, considered in this paper, associated with some families of polynomials of the Askey scheme of hypergeometric orthogonal polynomials [12].

Let us now consider such limits for any $n \in \mathbb{N}_0$:

(1) Racah → Hahn. If we take $\gamma + 1 = -N$ and $\delta \to \infty$ in the definition (19) of the Racah polynomials, we obtain the Hahn polynomials defined by (2). Hence

$$\lim_{\delta \to \infty} R_n(\lambda(x); \alpha, \beta, -N - 1, \delta) = h_n^{\alpha, \beta}(x; N).$$

The Hahn polynomials can also be obtained from the Racah polynomials by taking $\delta = -\beta - N - 1$ in the definition (19) and letting $\gamma \to \infty$:

$$\lim_{\gamma \to \infty} R_n(\lambda(x); \alpha, \beta, \gamma, -\beta - N - 1) = h_n^{\alpha, \beta}(x; N).$$

Another way to do this is to take $\alpha + 1 = -N$ and $\beta \to \beta + \gamma + N + 1$ in the definition (19) of the Racah polynomials and then take the limit $\delta \to \infty$. In that case we obtain the Hahn polynomials given by (2) in the following way:

$$\lim_{\delta \to \infty} R_n(\lambda(x); -N - 1, \beta + \gamma + N + 1, \gamma, \delta) = h_n^{\gamma, \beta}(x; N).$$

(2) Racah → Dual Hahn. If we take $\alpha + 1 = -N$ and let $\beta \to \infty$ in the definition (19) of the Racah polynomials, then we obtain the dual Hahn polynomials defined by (20). Hence

$$\lim_{\beta \to \infty} R_n(\lambda(x); -N-1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).$$

If we take $\beta = -\delta - N - 1$ and $\alpha \to \infty$ in (19), then we also obtain the dual Hahn polynomials:

$$\lim_{\alpha \to \infty} R_n(\lambda(x); \alpha, -\delta - N - 1, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).$$

Finally, if we take $\gamma + 1 = -N$ and $\delta \to \alpha + \delta + N + 1$ in the definition (19) of the Racah polynomials and take the limit $\beta \to \infty$ we find the dual Hahn polynomials given by (20) in the following way:

$$\lim_{\beta \to \infty} R_n(\lambda(x); \alpha, \beta, -N-1, \alpha + \delta + N + 1) = R_n(\lambda(x); \alpha, \delta, N).$$

(3) Hahn → Krawtchouk. If we take $\alpha = (1-p)t$ and $\beta = pt$ in the definition (2) of the Hahn polynomials and let $t \to \infty$ we obtain the Krawtchouk polynomials defined by (21):

$$\lim_{t \to \infty} h_n^{(1-p)t, pt}(x; N) = K_n(x; p, N).$$

(4) Dual Hahn → Krawtchouk. In the same way we find the Krawtchouk polynomials from the dual Hahn polynomials by setting $\gamma = pt$, $\delta = (1-p)t$ in (20) and letting $t \to \infty$:

$$\lim_{t \to \infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N).$$

Remark 8. The proof of each one of these limits is straightforward once one reduces the hypergeometric representation of each family as we did for the Hahn polynomials (see (8)) which is valid for all $n \in \mathbb{N}_0$.

Acknowledgements

Appendix. Orthogonality relations for Meixner polynomials with general parameter

In this appendix we will show that Meixner polynomials, $\{M_n(x; \beta, c)\}_{n \geq 0}$, with $c < 0$ and $\beta \in \mathbb{C}$, can be provided with a property of orthogonality which can be obtained through a process limit from the continuous Hahn polynomials.

From the TTRR of the continuous Hahn polynomials it is straightforward to obtain the following relation

$$p_{n+1}\left(ix; 0, -\frac{t}{c}, t, \beta\right) = (x - B_n)p_n\left(ix; 0, -\frac{t}{c}, t, \beta\right) - C_n p_{n-1}\left(ix; 0, -\frac{t}{c}, t, \beta\right), \tag{A.1}$$

where $t \in \mathbb{R}^+$, and

$$B_n = \frac{n\left(n - \frac{t}{c} + t - 1\right)\left(n - \frac{t}{c} + \beta - 1\right)}{\left(2n - \frac{t}{c} + t + \beta - 2\right)\left(2n - \frac{t}{c} + t + \beta - 1\right)} - \frac{\left(n - \frac{t}{c} + t + \beta - 1\right)(n + t)(n + \beta)}{\left(2n - \frac{t}{c} + t + \beta - 1\right)\left(2n - \frac{t}{c} + t + \beta\right)},$$

$$C_n = \frac{n\left(n - \frac{t}{c} + t - 1\right)\left(n - \frac{t}{c} + \beta - 1\right)\left(n - \frac{t}{c} + t + \beta - 2\right)(n + t - 1)(n + \beta - 1)}{\left(2n - \frac{t}{c} + t + \beta - 1\right)\left(2n - \frac{t}{c} + t + \beta - 2\right)^2\left(2n - \frac{t}{c} + t + \beta - 3\right)}.$$

Since

$$\lim_{|t| \to \infty} B_n = \frac{n + c(n + \beta)}{1 - c},$$

$$\lim_{|t| \to \infty} -C_n = \frac{nc(n + \beta - 1)}{(c - 1)^2},$$

coincide with the coefficients of the TTRR for the monic Meixner polynomials, with initial conditions $p_{-1} = 0$ and $p_0 = 1$, one deduces

$$\lim_{|t| \to \infty} (-i)^n p_n(ix; 0, -t/c, t, \beta) = M_n(x; \beta, c), \quad n = 0, 1, 2, \ldots, \tag{A.2}$$

by using induction in (A.1).

Proposition 9. *For any $\beta, c \in \mathbb{C}, c \notin [0, \infty)$ and $-\beta \notin \mathbb{N}$, the following property of orthogonality for the Meixner polynomials fulfills:*

$$\int_C M_n(z; c, \beta) z^m \Gamma(-z) \Gamma(\beta + z)(-c)^z \, \mathrm{d}z = 0, \quad 0 \le m < n, n = 0, 1, 2, \ldots \tag{A.3}$$

where C is a complex contour from $-\infty \mathrm{i}$ to $\infty \mathrm{i}$ separating the increasing poles $\{0, 1, 2, \ldots\}$ from the decreasing poles $\{-\beta, -\beta - 1, -\beta - 2, \ldots\}$.

Proof. We prove the result for $c < 0$, thus the general case is obtained by analytic continuation.

Let us assume that β is such that the contour $C = \{-1/2 + y\mathrm{i} : y \in \mathbb{R}\}$ separates the poles of $\Gamma(\beta + z)$ from the poles of $\Gamma(-z)$, i.e., $\Re\beta > 1/2$, and let us take the normalized weight for the continuous Hahn polynomials (see (6))

$$W_t(z) = \frac{w(\mathrm{i}z; 0, -t/c, t, \beta)}{\Gamma(-t/c)\Gamma(t)} = \Gamma(-z) \frac{\Gamma(-t/c - z)}{\Gamma(-t/c)} \frac{\Gamma(t + z)}{\Gamma(t)} \Gamma(\beta + z).$$

Notice that

$$\lim_{t \to \infty} W_t(z) = \Gamma(-z)\Gamma(\beta + z)(-c)^z =: W(z),$$

pointwise in C by using the Stirling formula

$$\Gamma(z) = \sqrt{2\pi} z^{z - \frac{1}{2}} \mathrm{e}^{-z}(1 + o(1)), \quad z \to \infty, |\arg(z)| < \pi.$$

It is known that

$$|\Gamma(x + \mathrm{i}y)| \le |\Gamma(x)|, \quad \forall x, y \in \mathbb{R},$$

hence

$$|W_t(z)| \le |\Gamma(-z) \Gamma(\beta + z)|. \tag{A.4}$$

Using once again induction on (A.1) and due to the exponential behavior of the right-hand side of (A.4) at the endpoints of C, one obtains that

$$p_n(\mathrm{i}z; 0, -t/c, t, \beta) W_t(z)$$

is dominated by an integrable function on C. Thus, from the dominated convergence theorem

$$\lim_{t \to \infty} \int_C (-\mathrm{i})^{n+1} p_n(\mathrm{i}z; 0, -t/c, t, \beta) z^m W_t(z) \mathrm{d}z = \int_C M_n(z; \beta, c) z^m W(z) \mathrm{d}z.$$

On the other side since C also separates the poles of $\Gamma(-t/c - z)$ from the poles of $\Gamma(t + z)$, we get

$$\int_C (-\mathrm{i})^{n+1} p_{n+1}(\mathrm{i}z; 0, -t/c, t, \beta) z^m W_t(z) = 0.$$

Thus (A.3) holds for $\Re\beta > 1/2$.

The general case is straightforward by using that if C is a contour separating the poles and $C_1 = \{\lambda + y\mathrm{i} : y \in \mathbb{R}\}$, where $\lambda \in (0, 1), \lambda \ne \Re\beta$, which does not separate the poles, then the integral through C and C_1 differs on a finite number of residues. $\square$

The case $c > 0$ cannot be considered by an integral of the form (A.3) since it diverges. However, when $|c| < 1$, (A.3) is rewritten on the form (see [16, Section 5.6] for details)

$$\sum_{x=0}^{\infty} M_n(x; c, \beta) x^m \frac{\Gamma(\beta + x) c^x}{x!} = 0,$$

which is also valid for $c \in (0, 1)$ and coincides with the very well-known orthogonal relations for Meixner polynomials.

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