THE IMAGE OF $H^*(BG, \mathbb{Z})$ IN $H^*(BT, \mathbb{Z})$ FOR $G$ A COMPACT LIE GROUP WITH MAXIMAL TORUS $T$

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§1. INTRODUCTION

Let $G$ be a compact Lie group with maximal torus $T$. Let $\rho : BT \rightarrow BG$ be the usual map between their classifying spaces. Let $H^*$ be singular cohomology with integral coefficients. Let $\text{INV} = H^*(BT)^w$ be the invariant elements of $H^*(BT)$ under the action of the Weyl group $W = N/T$ ($N$ being the normalizer of $T$ in $G$). A simple criterion (Theorem 2.8) is given for deciding when $\text{imp}^* = \text{INV}$. An example (Spin(12)) is given to show that the answer is sometimes negative. This answers a question of Borel which goes back to 1954[1]. It is of course well-known that the analogous statement is always true for rational coefficients. Complete proofs are included in this note.

§2. PROOF OF MAIN THEOREM

A transfer $\tau : H^*(BT) \rightarrow H^*(BG)$ has been defined in the above situation [2, 3]. There are two formulas relating $\tau$ with $\rho^* : H^*(BG) \rightarrow H^*(BT)$. Let $n$ equal the order of the Weyl group.

Property 2.1. $\tau \circ \rho^* = \text{multiplication by } n = \chi(G/T)$.

Property 2.2. (Brumfiel-Madsen) $\rho^* \circ \tau = \Sigma C_\phi$ where the sum is over the elements of the Weyl group. $C_\phi$ is a conjugation isomorphism[4, II.3; 5, 3.4].

2.1 is a basic property of the transfer. 2.2 was first proved in [5, 3.5] and also follows easily as a special case of the general double coset formula[4, VI.5].

Corollary 2.3 (to 2.1) (Borel). The kernel of $\rho^*$ is precisely the torsion subgroup of $H^*(BG)$, denoted $\text{Tors}$. Hence if $\tilde{\rho}^* : H^*(BG)/\text{Tors} \rightarrow H^*(BT)$ is the induced map then $\tilde{\rho}^* : H^*(BG)/\text{Tors} \rightarrow \text{imp}^*$ is an isomorphism.

Corollary (2.4) (to 2.2).

$$n \text{ INV} \subset \text{imp}^*$$

Pf: Let $x \in \text{INV}$. Then $C_\phi(x) = x$. Hence $\rho^*\tau(x) = nx$.

Definition (2.5). We say a ring $R$ which is a free abelian group is $p$-indivisible for a prime $p$ if the following is true for all $r \in R$. If $p \mid r$ then $p^q \mid r^q$ for all $q \geq 1$, i.e. powers of $p$-indivisible elements are $p$-indivisible. We say $R$ is indivisible if it is $p$-indivisible for all $p$.

Lemma (2.6). INV is an indivisible ring.

Pf: $\text{INV} \subset H^*(BT)$ which is a polynomial algebra over $\mathbb{Z}$ on a finite number of

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generators. $H^*(BT)$ is thus a unique factorization domain. If $x \in \text{INV}$ and $p|x^q$ for some prime $p$, $q \geq 1$, then $p|x$ in $H^*(BT)$. Since $W$ acts trivially on $\mathbb{Z}$, $p|x$ in $\text{INV}$.

**Lemma 2.7.** Suppose $R' \subset R$ are two indivisible rings and suppose that there exists $n \geq 1$ so that $nR \subset R'$. Then $R = R'$.

**Proof:** Let $r \in R$ be indivisible, i.e. a generator of a direct summand $\mathbb{Z}$. Then $nr \in R'$. Let $m =$ smallest positive multiple of $r$ contained in $R'$. Hence $mr$ is not divisible in $R'$ by any prime number. We wish to show $m = 1$. Let $q \geq 1$. Then $nr^q \in R'$. Hence $\gcd(m^q, n)r^q \in R'$. But since $R'$ is indivisible $\gcd(m^q, n) = m^q$, i.e. $m^q|n$ for all $q \geq 1$. This implies $m = 1$.

It is well known that $\text{imp}^* \subset \text{INV}$. We now state the main result.

**Theorem 2.8 (Criterion for surjectivity).** $\bar{p}^* : H^*(BG)/\text{Tors-} \rightarrow \text{INV}$ is always injective. It is an isomorphism iff $H^*(BG)/\text{Tors}$ is indivisible.

**Proof:** apply 2.3, 2.4, 2.6, 2.7.

**Remarks 2.9.** (1) $H^*(BG)/\text{Tors}$ is $p$-indivisible iff $H^*(BG)/\text{Tors} \otimes \mathbb{Z}/p\mathbb{Z}$ has no nilpotent elements. The latter is precisely the $E_1$ term of the Bockstein spectral sequence for $H^*(BG, \mathbb{Z}/p\mathbb{Z})$. (2) Borel [6, 29.2] has shown that if $G$ is connected $H^*(BG)/\text{Tors} \otimes \mathbb{Z}/p\mathbb{Z}$ is isomorphic to $\text{INV} \otimes \mathbb{Z}/p\mathbb{Z}$ for all primes $p$ for which $H^*(BG)$ has no $p$-torsion. (Bott has shown that $G/T$ has no torsion [7]). Hence it is enough in this case to calculate the Bockstein spectral sequence for those primes $p$ for which $H^*(BG)$ has $p$-torsion. (3) A similar argument to that of 2.8 can be given to show that $H^*(BG)/\text{Tors} \otimes \mathbb{Z}/p\mathbb{Z}$ has no nilpotent elements iff $\bar{p}^* \otimes \mathbb{Z}/p\mathbb{Z}$ is an isomorphism onto $\text{INV} \otimes \mathbb{Z}/p\mathbb{Z}$. (4) Wilkerson in a different context has independently discovered a principle similar to Lemma 2.7. His lemma can be applied without using the double coset formula. (5) $\bar{p}^*$ is an isomorphism iff $H^*(BG)/\text{Tors} \otimes \mathbb{Z}/p\mathbb{Z}$ is an integral domain for all primes $p$.

§3. **Examples**

Many examples exist where $\bar{p}^*$ is an isomorphism, the most famous being the classical unitary groups $U(n)$. $\bar{p}^*$ is also an isomorphism for the special orthogonal groups $SO(n)$.

**Theorem 3.2** $\bar{p}^*$ for $\text{Spin} (12)$ is not surjective in dimension 32.

**Proof:** Quillen has calculated $H^*(B\text{Spin}(12), \mathbb{Z}/2\mathbb{Z})$ [8, 6.5]. The explicit result is written down in [9, 4.1], namely $H^*(B\text{Spin}(12), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \{w_4, w_6, w_7, w_9, w_{10}, w_{11}, w_{12}\}/(R_1, R_2) \otimes \mathbb{Z}/2\mathbb{Z}[e_6]$ where

- $R_1 = w_6w_4 + w_{11}w_5$
- $R_3 = w_1 + w_1^2w_4 + w_7w_8^2 + w_{12}w_9^3$

The factor that concerns us is the l.h.s. of the tensor product. This is precisely the image of $\pi^*: H^*(B\text{SO}(12), \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(B\text{Spin}(12), \mathbb{Z}/2\mathbb{Z})$. $w_i$ is the image of the $i$th Stiefel–Whitney class. We use the naturality of the Bockstein spectral sequence. Let $\beta = \text{Bockstein for } B\text{SO}(12)$, $\beta = \text{Bockstein for } B\text{Spin}(12)$. Then $\pi^*\beta = \bar{\beta} \circ \pi^*$. We shall use the same notation for elements in $H^*(B\text{SO}(12), \mathbb{Z}/2\mathbb{Z})$ as for their images in $H^*(B\text{Spin}(12), \mathbb{Z}/2\mathbb{Z})$. The Bockstein spectral sequence for $B\text{SO}(12)$ collapses after applying one Bockstein. Hence every element $x$ in an odd dimension is either hit by some even dimensional element or hits one. If it is hit by one then its image under $\pi^*$ is
hit under $\beta$ by the corresponding element. However if $\beta(x) \neq 0$ it is conceivable that $\beta(x) = 0$. We are concerned with dimensions 31 and 33. We can of course assume that $x$ does not involve $w_2, w_3, w_5, w_7$. If $\beta(x) = 0$ then $\beta(x) = aR_3$ for some $a$ (since there is no element in dimension 1 $R_3$ does not come into play). Note that $\beta(R_3) = 0$. Hence $\beta(aR_3) = \beta(a)R_3$. So $\beta(a) = 0$. Thus $\beta(aw_{10}w_5) = aR_17$ and $x = aw_{10}w_5 + k$ where $\beta(k) = 0$. Note that since $a$ and $k$ are odd dimensional there exist $l, b$ such that $\beta(l) = k, \beta(b) = a$. Then $\beta(bw_{10}w_5 + l) = x$. Hence after the first Bockstein there are no elements in dimensions 31 and 33.

Note that if $x = w_{10}w_1 + w_1w_3w_4 + w_{10}w_6w_7^3 + w_{12}w_3w_5$, then $\beta(x) = R_{33}$. Hence $\beta(x) = 0$. It is also easily seen that $x \notin im\beta$. Suppose it were. Then $\beta(y) = x + a$ for some $y$ and some $a \in \mathfrak{g}$, the ideal in $H^*(BSO(12), \mathbb{Z}/2\mathbb{Z})$ generated by $R_{17}, w_2, w_3, w_5, w_9$. Since $\beta(\mathfrak{g}) \subset \mathfrak{g}$ and $\beta \circ \beta = 0$, this implies that $\beta(x + a) = R_{33} + \beta(a) = 0$, i.e. $R_{33} \in \mathfrak{g}$ which is a contradiction. Hence $x$ survives to $E_3$.

However $x^2 = \beta(w_{10}w_1^3 + w_1w_3w_4^2 + w_{10}w_6^2w_7^3 + w_{12}w_3w_5^2w_6^3)$. Hence $x^2 = 0$ in $E_3$. Thus $H^*(BSpin(12)) \otimes \mathbb{Z}/2\mathbb{Z}$ contains a nilpotent element and $\tilde{p}^*$ is not surjective.

Remarks 3.3. (1) An entirely similar argument shows that $\tilde{p}^*$ is not surjective for Spin(11) in dimension 32. (2) The counterexamples do not depend upon the transfer. Rather they violate an obvious necessary condition. (3) It is not hard to check that $\tilde{p}^*$ is surjective for Spin(n) $n \leq 10$. $H^*(BSpin(10)) \otimes \mathbb{Z}/2\mathbb{Z}$ is not a polynomial algebra, however, and $H^*(BSpin(n))$ contains 2-torsion for $n \geq 7$. (4) $\tilde{p}^*$ is surjective for $G_2$. This follows easily from [10, 17.3] i.e. $H^*(G_2, \mathbb{Z}/2\mathbb{Z})$ has a simple system of universally transgressive generators $(e_3, e_5, e_6)$, where $\beta e_3 = e_6$. Thus $H^*(BG_2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_4, x_6, x_7]$ where $\beta x_6 = x_7$ and it follows easily that $H^*(BG_2) \otimes \mathbb{Z}/2\mathbb{Z}$ is a polynomial algebra.

Since it is well known that $\tilde{p}^*$ is surjective for $SU(n)$, we have that $\tilde{p}^*$ is surjective for all simply connected compact Lie groups for which $N$ splits as a semidirect product of $W$ and $T[11]$. It would be nice to know whether this is true for nonsimply connected groups and to have a proof that does not depend on the classification of Lie groups. Also I do not know of any examples where $\tilde{p}^* \otimes \mathbb{Z}/p\mathbb{Z}$ is not onto $INV \otimes \mathbb{Z}/p\mathbb{Z}$ for $p$ odd.

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REFERENCES