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Simple generic structures

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Abstract

A study of smooth classes whose generic structures have simple theory is carried out in a spirit similar to Hrushovski (Ann. Pure Appl. Logic 62 (1993) 147; Simplicity and the Lascar group, preprint, 1997) and Baldwin–Shi (Ann. Pure Appl. Logic 79 (1) (1996) 1). We attach to a smooth class $\langle K_0, \prec\rangle$ of finite $\mathscr{L}$-structures a canonical inductive theory T_{Nat}, in an extension-by-definition of the language $\mathscr{L}$. Here T_{Nat} and the class of existentially closed models of $(T_{\mathrm{Nat}})_{\forall}=T_+, EX(T_+)$, play an important role in description of the theory of the $\langle K_0, \prec\rangle$-generic. We show that if M is the $\langle K_0, \prec\rangle$-generic then $M \in EX(T_+)$. Furthermore, if this class is an elementary class then $Th(M) = Th(EX(T_+))$. The investigations by Hrushovski (preprint, 1997) and Pillay (Forking in the category of existentially closed structures, preprint, 1999), provide a general theory for forking and simplicity for the nonelementary classes, and using these ideas, we show that if $\langle K_0, \prec\rangle$, where $\prec\, \in \{\leqslant, \leqslant^*\}$, has the joint embedding property and is closed under the Independence Theorem Diagram then $EX(T_+)$ is simple. Moreover, we study cases where $EX(T_+)$ is an elementary class. We introduce the notion of semigenericity and show that if a $\langle K_0, \prec\rangle$-semigeneric structure exists then $EX(T_+)$ is an elementary class and therefore the $\mathscr{L}$-theory of $\langle K_0, \prec\rangle$-generic is near model complete. By this result we are able to give a new proof for a theorem of Baldwin and Shelah (Trans. AMS 349 (4) (1997) 1359). We conclude this paper by giving an example of a generic structure whose (full) first-order theory is simple.

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1. Introduction

This article relates two subjects of model theory: simplicity and generic structures.

Let T be a first-order theory. Recall that T is *simple* if forking satisfies the local character axiom. This property was introduced by Shelah [17]. The class of simple theories extends the class of stable theories and includes many others such as the theory of the random graph, theory of algebraically closed fields with a generic automorphism and theory of pseudofinite fields. Shelah's notion of forking is the important tool in studying simplicity as well as stability theory. The forking theory provides a notion of independence for simple structures and in fact it was found that this notion of independence can be developed as satisfactorily as in the context of stable theories. In fact, for any simple theory, forking satisfies the symmetry, transitivity, existence and finite character axioms (see e.g. [9,10]).

Forking and simplicity, in fact, can be considered beyond first-order context. If T is a first-order universal theory then we can consider the category of existentially closed models of T, $EX(T)$, with embeddings as morphisms. This class of structures is not the class of models of some first-order theory $T' \supseteq T$. This will be true only if T' is a model companion of T. Hrushovski, in [6], considered the category of existentially closed models of a universal theory T which satisfies the separation of quantifiers (equivalently has the amalgamation property). He calls such a T Robinson theory. He showed that all basic and even advanced model-theoretic techniques and results hold in this category.

The important fact for such theories is that quantifier-free types replace types and the class of quantifier-free formulas are closed under negation. Pillay [14] developed the theory of forking and simplicity for the class of universal theories which do not have the amalgamation property. He showed that forking theory and simplicity have satisfactory analogues for the category of existentially closed models.

By generic structures we mean countable homogeneous-universal models constructed by the Fräissé–Hrushovski's method. In a series of surprising constructions, Hrushovski [5,7,8] showed how to vary the Fräissé construction to obtain stable structures. He introduced the key notion of assigning a dimension to a finite structure in terms of the difference between the number of points in the structure and the total number of tuples in the distinguished relations on the structure. Using this dimension function he defined the relation of a strong submodel on finite structures $A \subseteq B$. Now to get the constructions, he adapted Fräissé's method to amalgamate finite substructures relative to the notion of strong model. In an abstract setting, suppose $\mathscr{L}$ is a finite relational language and let K_0 be a class of finite $\mathscr{L}$-structures including $\emptyset$ structure and let $\leqslant$ be a binary relation on structures $A \subseteq B \in K_0$. $\langle K_0, \leqslant \rangle$ is called a smooth class if it satisfies the following axioms:

1. $\leqslant$ defines a partial order relation on K_0.
2. For each $A \in K_0, \emptyset \leqslant A$.
3. For $A, B \subseteq C \in K_0$ if $A \leqslant C$ then $A \cap B \leqslant B$.

A countable structure M is called $\langle K_0, \leqslant \rangle$-generic if K_0 is precisely the class of finite structures embeddable in M and M is homogeneous relative to $\leqslant$. So Fräissé's original construction deals with a special smooth class where the notions $\leqslant$ and $\subseteq$ coincide. To analyse definable sets of theory of generic structures we need to define $\mathscr{L}_+$, an extension-by-definition of the language $\mathscr{L}$, so that if $M \leqslant N$ then $M_+ \subseteq N_+$. Here M_+ is the natural expansion of M. For any smooth class $\langle K_0, \leqslant \rangle$ we assign an inductive theory T_{Nat} in $\mathscr{L}_+$ and then form the class of existentially closed models of $(T_{\mathrm{Nat}})_\forall = T_+, EX(T_+)$. We will see that this class is closely related to the theory of generic structures and in fact we prove that when this class is an elementary class, that is T_{Nat} has a model companion T^*, then T^* is the $\mathscr{L}_+$-theory of generic structures. One of the advantages of working with $EX(T_+)$ is that one does not need to suppose $\langle K_0, \leqslant \rangle$ has the amalgamation property. In this sense we can say M is $\langle K_0, \leqslant \rangle$-generic if $M_+ \in EX(T_+)$.

Now to connect these two subjects of model theory, one could think of finding some sufficient conditions for $\langle K_0, \leqslant \rangle$ under which the theory of generic structures (or more generally $EX(T_+)$) would be simple.

Here we try to develop this project. There are two motivations for this idea: Firstly to understand generic behaviour of simple theories and secondly to find new examples. As we mentioned earlier, Hrushovski used quite successfully generic structures in the stable context for finding new examples to refute many conjectures. So one could expect the same results for simple theories.

Here we prove that for any smooth class $\langle K_0, \prec \rangle$, with the *algebraic closure property* (AC) and defined via a dimension function, if it has the joint embedding property and is closed under the *independence theorem diagram*, then $EX(T_+)$ is formally simple. The AC condition roughly says that the closure of a set is a subset of its algebraic closure. This technical assumption plays an important role in describing the theory of generic structures as well as in proving the simplicity of this theory.

Kim and Pillay [10] conjectured that any ω-categorical supersimple structures have finite ranks and, are modular. Hrushovski [6] discovered a nonmodular supersimple ω-categorical of SU-rank 1 by adapting the ω-categorical pseudoplane's construction, hence refuting their conjecture. In this example he implicitly gave a sufficient condition (the independence theorem diagram), for the amalgamation class which guarantees the simplicity of the generic theory. Evans [4] gives an axiomatic framework for constructing simple ω-categorical examples. He includes some form of the independence theorem diagram as one of the axioms.

This paper is organized as follows. In Section 2 we review basic definitions and facts about forking and simplicity for the category of existentially closed models. In Section 3, we will prove the main theorem as described in the above paragraph. Namely we define the independence theorem diagram (ITD) and prove that for any smooth class $\langle K_0, \prec \rangle$ with AC, where $\prec \in \{\leqslant, \leqslant^*\}$, if this class has the joint embedding property and is closed under the ITD, then $EX(T_+)$ is simple (more precisely formally simple). If T_{Nat} has a model completion, T^* say, this result exactly means that T^* is a simple first-order theory. In Section 4, we define the notion of semigenericity for a smooth class $\langle K_0, \prec \rangle$ with AC, and prove that if the $\langle K_0, \prec \rangle$-semigeneric structure exists, then $EX(T_+)$ is an elementary class and, therefore, T_{Nat} has a model completion. Using this result we then show that the theory of generic structures is near model complete, i.e. every formula is equivalent to a Boolean combination of Σ_1-

formulas. Finally, the last section studies an example of a simple generic structure of SU-rank 1.

2. Simplicity in the category of existentially closed models

In this section we review basic definitions and facts of forking and simplicity in the category of existentially closed models. We first fix some notations. Throughout this paper T denotes a first-order theory (not necessarily complete). For any $A \subseteq M$, denote by $etp_M(a/A)$ ($qf_M(a/A)$ resp.), the existential type (the quantifier-free type resp.) of a over A in M, namely the set of existential formulas (quantifier-free formulas resp.) with parameters in A which are true of a in M. For a set A, by $eS(A)$ ($qfS(A)$ resp.) we mean the set of all complete existential types (quantifier-free types resp.) over set A; more generally a partial existentially (quantifier-free resp.) type over the set A is a consistent collection of existentially (quantifier-free resp.) formulas $\phi(\bar{x})$ with parameter in A. A model M of a universal theory T is called existentially closed if for any quantifier-free formula $\phi(\bar{x},\bar{y}), M \subseteq N \models T$ and tuple $\vec{a} \in M$ if $N \models (\exists \bar{x})\phi(\vec{x},\bar{a})$ then so $M \models (\exists \bar{x})\phi(\bar{x},\bar{a})$. Let $EX(T)$ the category of existentially closed models of T.

Definition 2.1. Let T be a universal theory. T has the amalgamation property (AP) if any model of $M \models T$ and embeddings $g: M \mapsto N$ and $h: M \mapsto K (N, K \models T)$, there are $P \models T$ and embeddings $g': N \mapsto P$ and $h': K \mapsto P$ such that $gg' = hh'$. T is called a *Robinson theory* if it has the AP.

Remark 2.2. If a universal theory T is a Robinson theory, then whenever $M, N \in EX(T)$, $\bar{a} \in M, \bar{b} \in N$ and $qftp_M(\bar{a}) = qftp_N(\bar{b})$, then $etp_M(\bar{a}) = etp_N(\bar{b})$ (see [14]).

Definition 2.3. Let T be a universal theory and $A \subseteq M \models T$. Suppose $\Sigma(\bar{x})$ is a set of existential formulas over A. $\Sigma(\bar{x})$ is called consistent with M if there is $N \supseteq M$ model of T such that $\Sigma(\bar{x})$ is realized in N.

Definition 2.4. Let T be a universal theory. M is a κ-existentially universal model for T, if M has cardinality κ and whenever A is a small, i.e. of cardinality $< \kappa$ subset of M and $\Sigma(x)$ is an existential set of formulas over A consistent with M, then $\Sigma(x)$ is realized in M.

Fact 2.5. *Let T be a universal theory. Then for any arbitrary large κ, any model of T embeds into a κ-existentially universal model of T.*

Definition 2.6. The $\mathscr{L}$-structure M is called an e-universal domain of cardinality κ if

(i) whenever $\Sigma(x)$ is a partial existential type over a small subset A of M of cardinality which is finitely satisfiable in M, then $\Sigma(x)$ is realized in M.
(ii) whenever A, B are subsets of M of cardinality $< \kappa$ and $f: A \to B$ is a bijection such that $etp_M(a) = etp_M(f(a))$ for all tuples a from A, there is an automorphism of M which extends f.

Fact 2.7. *Any κ-existentially universal model for a universal theory T is an e-universal domain M of cardinality κ. Furthermore, M is an existentially closed model for T.*

Now following Pillay [14] we want to define forking in the category of existentially closed structures of a universal theory. We will work inside a large e-universal domain M.

Definition 2.8. (i) Let $\Sigma(x,b)$ be an existential type over B where b enumerates B. Let A be any small set. We say that $\Sigma(x,b)$ divides over A if there is an infinite sequence $(b_i\colon\ i<\omega)$ which is e-indiscernible over A, with $b=b_0$, such that $\bigcup\{\Sigma(x,b_i)\colon\ i<\omega\}$ is not realized in M.

(ii) Let again $\Sigma(x)$ be an existential type. We say that $\Sigma(x)$ forks over A if there is a bounded set $\Phi(x)$ of existential formulas with parameters such that all of them divide over A and $M\models\Sigma(x)\rightarrow\bigvee\Phi(x)$.

Remark 2.9. Since we can only use compactness for existential types, the definition of forking in (ii) is different from the usual first-order case.

Definition 2.10. Call M simple if for any finite tuple $a\in M$ and B there is a subset A of B of cardinality at most $|\mathscr{L}|+\omega$, such that $etp(a/B)$ does not fork over A.

Definition 2.11. Let $p(x)$ be a maximal existential type over A. By a Morley sequence in p of length ω we mean a sequence $\langle a_i\colon\ i\in\omega\rangle$ of realizations of p which is an e-indiscernible over A such that $etp(a_i/A\cup\{a_j\colon\ j<i\})$ does not divide over A for each $i\in\omega$.

Using the same ideas as Kim [9], Pillay [14] shows that:

Lemma 2.12. *Let M be simple, $A\subseteq B\subset M$ two small set of parameters, and $\Sigma(x)$ an existential type. Then*

1. *If $\Sigma(x)$ does not fork over A then there is a realization a of Σ such that $etp(a/B)$ does not divide over A.*
2. *For any maximal existential $p(x)$ over A there exists a Morley sequence in p.*
3. *Σ divides over A if and only if Σ forks over A.*
4. *If $\Sigma(x)$ forks over A, then some formula in $\Sigma(x)$ forks over A.*

Definition 2.13. Let $A\subseteq B$ be two sets and a be a finite tuple. Then we say a is forking independent from B over A, if $etp(a/B)$ does not fork over A. In this situation we write $a \underset{A}{\smile\!\!\!\!\!|}\ B$.

Definition 2.14. Let M be an e-universal model and let Γ be a collection of triples (a,A,E), where a is a tuple from M and $A\subseteq B$ are small subsets of M. We will say that Γ is a notion of independence if the following hold:

1. (Invariance) Γ is invariant under automorphisms of M.
2. (Local character) For any a, B there is $A \subseteq B$ with $|A| \leqslant |T|$ and $(a,A,B) \in \Gamma$.
3. (Finite character) $(a,A,B) \in \Gamma$ if and only if for every finite tuple b from B we have $(a,A,A \cup \{b\}) \in \Gamma$.
4. (Symmetry) If $(a,A,A \cup \{b\}) \in \Gamma$, then $(b,A,A \cup \{a\}) \in \Gamma$.
5. (Transitivity) For $A \subseteq B \subseteq C$ we have $(a,B,C) \in \Gamma$, and $(a,A,B) \in \Gamma$ if and only if $(a,A,C) \in \Gamma$.
6. (Extension) For any a, any existentially closed substructure of N of M and any $B \supseteq N$ there is a' such that $etp(a'/N) = etp(a/N)$ and $(a',N,B) \in \Gamma$.

In this situation we say a is Γ-independent from A (or $etp(a/B)$ does not Γ-fork over A), denoted $a \underset{A}{\overset{\Gamma}{\smile\!\!\!\!|}} B$, whenever $(a,A,B) \in \Gamma$.

By definition there is a trivial notion of independence, namely the set of all (a,A,B) with $A \subseteq B$.

Remark 2.15. The extension property in Definition 2.14 requires only the existence of Γ-independent extensions over an existentially closed model. This is weaker than that defined by Kim–Pillay [10], which requires the existence of Γ-independent extensions over all sets. However, we will see that for a Robinson theory both these definitions are equivalent.

Definition 2.16. (i) We call M *formally simple if* M has a nontrivial notion of independence which satisfies the Γ-independence theorem, namely 7. Let N be an existentially closed substructure of M. Let $A \supseteq N, B \supseteq N$, and suppose that for $a_0 \subset_\omega A$ and $b_0 \subset_\omega B$ we have $(a_0,N,N \cup \{b_0\}) \in \Gamma$. Let a_1 and a_2 be two tuples with $etp(a_1/N) = etp(a_2/N)$, $(a_1,N,A) \in \Gamma$ and $(a_2,N,B) \in \Gamma$. Then there is a such that $etp(a/A) = etp(a_1/A)$, $etp(a/B) = etp(a_2/B)$ and $(a,N,A \cup B) \in \Gamma$.

(ii) We say $EX(T)$ is (formally resp.) simple if any e-universal model of T, is (formally resp.) simple.

Lemma 2.17. *Let T be a Robinson theory. Then the two notions of simplicity are equivalent.*

Proof. Since T is a Robinson theory, existential types can be replaced by quantifier-free types. Call T weakly simple if dividing has the local character.

(1) Suppose T is formally simple. Then we show that T is weakly simple. Formal simplicity of T implies that there is a nontrivial notion of independence Γ which satisfies the independence theorem.

Claim. *For every existentially closed model $N, A \subseteq N$ and a finite tuple a, if $qftp(a/N)$ does not Γ-fork over A, then $qftp(a/N)$ does not divide over A.*

The proof of the claim is just an adaptation of the proof which appeared in [10, Claim II of Theorem 4.2]. For the complete proof of the claim see [16, Lemma 2.2.23].

Now we want to use local rank, $D(-,\Delta,k)$ where Δ is a finite set of quantifier-free formulas and $k\in\omega$ for the definition of this rank and its properties we refer to the work of Kim–Pillay [10]. In fact we can define D-rank for every Boolean closed set of formulas. Now since T is a Robinson theory, the basic properties of D-rank hold for T. Specifically,

1. For every $p\in qfS(B)$ if $D(p,\Delta,k)=\infty$ then p divides over any set $A\subset B$ with $|A|\leqslant|T|$.
2. For every Δ,k and partial quantifier-free type p over A, there is a complete quantifier-free type over A,q, such that $D(p,\Delta,k)=D(q,\Delta,k)$.
3. For every Δ,k and $p\in qftp(B)$ there is a finite subset $q(x)$ of $p(x)$ such that $D(p,\Delta,k)=D(q,\Delta,k)$.
4. Let $p(x)\in qftp(B)$ and $A\subseteq B$. If for every Δ and $k, D(p|A,\Delta,k)=D(p,\Delta,k)$ then p does not divide over A.

Now let $p(x)\in qftp(B)$. We show that for all Δ and $k, D(p,\Delta,k)<\infty$. Suppose $M\supseteq B$ an existentially closed model. We assume p to be a partial type over M. Then by 2, there is a type $q(x)\in qftp(M)$ such that $D(q,\Delta,k)=D(p,\Delta,k)$. Now $D(p,\Delta,k)=\infty$, implies that $D(q,\Delta,k)=\infty$. Hence by 1., $q(x)$ divides over any set A with $|A|\leqslant|T|$ which contradicts the claim. Therefore, for every Δ,k and $p\in qftp(B), D(p,\Delta,k)<\infty$. Hence in virtue of 3., we can choose $q_{(\Delta,k)}$ such that $D(p,\Delta,k)=D(q_{(\Delta,k)},\Delta,k)$. Hence if $r(x)=\bigcup\{q_{(\Delta,k)}\colon\ \Delta,k\}$ then for every Δ and k

$$D(p,\Delta,k)=D(r,\Delta,k)<\infty.$$

Thus if $A=dom(r)$ then by 4, p does not divide over A.

Now by the same proof used for the first-order case one can show that if T is weakly simple then T is simple (see [9, Proposition 2.17]).

2. If T is simple then T is formally simple.

With the same proof as first-order case we can show that forking independence defines a nontrivial notion of independence and, moreover, satisfies the independence theorem. □

Remark 2.18. Suppose T has a complete model companion, T^*. Then T is simple if and only if T^* a first-order simple theory.

3. Smooth classes with AC and simplicity

In this section we study simplicity of the class of generic structures. Here, we first fix some notations and review basic definitions.

Let $\mathscr{L}$ be a finite relational language. If A, B are subsets of N, we write AB for the $\mathscr{L}$-substructure of N with universe $A \cup B$. If B and C are $\mathscr{L}$-structure with $B \cap C = A$ we write $B \otimes_A C$ for the structure with universe $B \cup C$ and no relations other than those on B and C. If A, B, C are substructures of N such that the structure imposed by N on $B \cup C$ is isomorphic to $B \otimes_A C$, we say B and C are canonically in free amalgamation over A in N. Let K_0 be a collection of finite $\mathscr{L}$-structures and K be any class of $\mathscr{L}$-structures. We always assume that the empty structure is in K_0. Here, in general, we assume K_0 is closed under substructure and isomorphism. For a class K_0 by $\bar{K}_0$ we mean the class of all structures whose finite substructures are in K_0. We write $X \subseteq_\omega Y$ to indicate X is a finite subset of Y. For any $A \in K_0$, by $Diag_A(\bar{x})$, where $|\bar{x}| = |A|$, we mean the conjunction of all basic formulas in the diagram of A. From now on we, in general, assume that K_0 is a class of finite $\mathscr{L}$-structures.

Definition 3.1. Let δ be an arbitrary function assigning a real number to each isomorphism type of $\mathscr{L}$-structures, with $\delta(\emptyset) = 0$. Then for any $A, B, C \in K_0$ with $C = A \cup B$, define $\delta(A/B) = \delta(C) - \delta(B) = \delta(AB) - \delta(B)$. So this yields for any $A, B, C \in K_0$

$$\delta(AB/C) = \delta(A/BC) + \delta(B/C).$$

Definition 3.2. Let δ be a function assigning a real number to each isomorphism type of elements of K_0. We say that δ is a *predimension* on K_0, if it satisfies the following conditions:

1. $\delta : K_0 \mapsto \mathbb{R}^{\geqslant 0}$.
2. $\delta(\emptyset) = 0$.
3. For any $A, B \in K_0$ and $M \in \bar{K}_0$ with $A, B \subseteq_\omega M$

$$\delta(A \cup B) \leqslant \delta(A) + \delta(B) - \delta(A \cap B).$$

4. There is a real number $\varepsilon > 0$ such that for any $B \subseteq C \in K_0$, two subsets of $M \in \bar{K}_0$, if $A \subseteq_\omega M$ is disjoint from C and $\delta(A/B) - \delta(A/C) < \varepsilon$, then $\delta(A/B) = \delta(A/C)$.

Remark 3.3. Condition 3 in Definition 3.2, can be restated as follows: For all $A, B \in K_0$, two subsets of $M \in \bar{K}_0$,

$$\delta(A/B) \leqslant \delta(A/A \cap B).$$

Definition 3.4. Suppose that δ is a real-valued function on the class of all finite $\mathscr{L}$-structures. Then

1. $C^{\geqslant 0} = \{A|$ for all $B \subseteq A, \delta(B) \geqslant 0\}$.
2. $C^{>0} = \{A|$ for all $\emptyset \neq B \subseteq A, \delta(B) > 0\}$.

Definition 3.5. Let $\mathscr{L} = \{R_1, R_2, \ldots, R_p\}$ be a finite language. For each R_i we fix a real number α_i with $0 \leqslant \alpha_i \leqslant 1$, which we call weight of R_i. For any A, a finite $\mathscr{L}$-structure,

e_i^A denotes the number of tuples of realizing R_i in A. Now define

$$\delta(A) = |A| - \sum_{i=1}^{p} \alpha_i e_i^A.$$

Remark 3.6. The existence of weight zero relations is one of the main differences between a simple generic structure and a stable one. In the stable case, the canonical free amalgam is the only free amalgam. We would like to call the amalgam free regardless of whether weight zero relations hold between elements of B and C.

Lemma 3.7 (Baldwin and Shi [3]). *Let K_0 be any subclass of $C^{\geqslant 0}$. Then the function δ, defined in Definition 3.5, is a predimension on K_0.*

Definition 3.8. 1. $A \leqslant B$ if and only if for all C, with $A \subseteq C \subseteq B$, it is the case that $\delta(C/A) \geqslant 0$.

2. $A \leqslant^* B$ if and only if for all C, with $A \subset C \subseteq B$, it is the case that $\delta(C/A) > 0$.

The main properties of these definitions can be stated as follows.

Lemma 3.9. *Let $K_0 \subseteq C^{\geqslant 0}$ and $A, B, C \in K_0$. Suppose $\prec \in \{\leqslant, \leqslant^*\}$. Then*

1. *$A \prec B$ then $A \subseteq B$.*
2. *$A \prec A$.*
3. *If $A \prec B \prec C$ then $A \prec C$.*
4. *If $A \subseteq B \subseteq C$ and $A \prec C$ then $A \prec B$.*
5. *If $A \prec C$ and $B \subseteq C$ then $A \cap B \prec B$.*
   Furthermore, for $\leqslant$ we have
6. *$\emptyset \leqslant A$.*

Proof. (1) See [3, Theorem 3.12]. (2) can be proved similarly. □

Definition 3.10. Let $\prec$ be binary relation on K_0. We say that $\langle K_0, \prec \rangle$ is a *smooth class* if it satisfies conditions 1–6 of Lemma 3.9.

Lemma 3.11. 1. *$\langle K_0, \leqslant \rangle$ is a smooth class.*

2. *Let $C^{>0}$ be as defined in Definition 3.4. Suppose $K_0^{>0} = K_0 \cap C^{>0}$. Then $\langle K_0^{>0}, \leqslant^* \rangle$ is a smooth class.*

For any smooth class $\langle K_0, \prec \rangle$ one can define notion of $\prec$-closure.

Definition 3.12. Let $\langle K_0, \prec \rangle$ be a smooth class and $A \subseteq B \in K_0$. We write $A \prec_i B$, if there is no B' of proper subset of B with $A \subseteq B' \prec B$. If $A \prec_i B$, we say B is an intrinsic extension of A.

We denote by $\leqslant_i$ ($\leqslant_i^*$ resp.) the corresponding intrinsic closure for the relation $\leqslant$ ($\leqslant^*$ resp.). Furthermore, we can easily see the following lemma.

Lemma 3.13. *Let* $A, B \in K_0$.

1. $A \leqslant_i B$ *if* $A \subseteq B$ *and, either* $A = B$ *or* $\delta(B/C) < 0$ *for every* C, *with* $A \subseteq C \subset B$.
2. $A \leqslant_i^*$ *if* $A \subseteq B$ *and for every* C *with* $A \subseteq C \subseteq B$ *we have* $\delta(B/C) \leqslant 0$.

Notation 3.14. Let $A \in K_0$ and $N \subset M \in \bar{K}_0$ with $A \subseteq_\omega M$. Then

1. $cl_M(A) = \bigcup\{B \subseteq_\omega M \mid A \subseteq B \text{ and } A \leqslant_i B\}$.
2. $cl^*_M(A) = \bigcup\{B \subseteq_\omega M \mid A \subseteq B \text{ and } A \leqslant_i^* B\}$.
3. $cl_M(N) = \bigcup_{A \subseteq_\omega N} cl_M(A)$.
4. $cl^*_M(N) = \bigcup_{A \subseteq_\omega N} cl^*_M(A)$.

We call $cl_M(A)(cl^*_M(A)$ resp.), the $\leqslant$-closure (the $\leqslant^*$-closure resp.) of A in M.

Using property 6 of Lemma 3.9, we can see that for any $A \subset B \in K_0$, the $\leqslant$-closure (the $\leqslant^*$-closure) of A in B is the smallest $\leqslant$-closed ($\leqslant^*$-closed resp.) set in B containing A and, moreover, one can extend the definition of $\leqslant$ and $\leqslant^*$ to structures in $\bar{K}_0$, namely, for $M, N \in \bar{K}_0$, we say $M \leqslant N$ $(M \leqslant^* N)$ if $M \subseteq N$ and $cl_N(M) = M$ $(cl^*_N(M) = M$ resp.).

Remark 3.15. From Definition 3.13, it follows that $cl(A) \subseteq cl^*(A)$.

Definition 3.16. Let M be an $\mathscr{L}$-structure, $A \subseteq M$ and, $A \subseteq B \in K_0$.

1. By a copy of B over A in M we mean the image of an embedding of B over A into M,
2. $\chi_M(B/A)$ is the number of distinct copies of B over A in M,
3. $\chi^*_M(B/A)$ is the supremum of the cardinals of maximal families of disjoint (over A) copies of B over A in M, if it is finite. Otherwise, take $\chi^*_M(B/A) = \infty$.

The next lemma gives one of main properties of $\leqslant$.

Lemma 3.17 (Baldwin and Shi [3]). *Suppose* $K_0 \subseteq C^{\geqslant 0}$. *Then there is a function* $\mu : \omega \times \omega \mapsto \omega$ *such that for any* $A \leqslant_i B$ *and* $M \in \bar{K}_0$ *with* $A \subseteq_\omega M$, *we have* $\chi_M(B/A) \leqslant \mu(|A|, |B|)$.

This property plays an important role in the description of theory of generic structures. We say a smooth class $\langle K_0, \prec\rangle$ has the *algebraic closure property* (AC) if it satisfies Lemma 3.17. That means for any $A \subseteq M$, we have $\prec$-$cl_M(A) \subseteq acl_M(A)$. There are smooth classes which do not have the AC (see [15]).

In the following we define an extension-by-definition of the language $\mathscr{L}$. For any smooth class $\langle K_0, \prec\rangle$, the relation $\prec$ generally is a stronger notion than the substructure relation (in $\mathscr{L}$). One of the features of this extension would be to convert the $\prec$ relation to the notion of substructure in the extended language. First note that for any class K_0

there is universal theory T_0 such that $M \models T_0$ if and only if $M \in \bar{K}_0$. In fact

$$T_0 = \{\neg(\exists \bar{x}) Diag_A(\bar{x}) \mid A \notin K_0\}.$$

Notation 3.18. For any smooth class $\langle K_0, \prec\rangle$ and $A \in K_0$ put

1. $\mathscr{D}_A = \{\langle A, E\rangle \mid A \prec_i E \in K_0\}$,
2. $\mathscr{D} = \bigcup_{A\in K_0} \mathscr{D}_A$.

Definition 3.19. Suppose $A \in K_0$ and $E \in \mathscr{D}_A$. Let $\phi_E(\bar{x},\bar{y}) = Diag_A(\bar{x}) \wedge Diag_E(\bar{x},\bar{y})$ be the $\mathscr{L}$-formula describing the $\mathscr{L}$-quantifier-free diagram of E. We wish to extend the language $\mathscr{L}$ so that the formula $(\exists \bar{y})\phi_E(\bar{x},\bar{y})$ would be treated as an atomic formula. So for any ϕ_E we add a new $|A|$-array relation symbol $R_E(\bar{x})$ to the language $\mathscr{L}$. Put

$$\mathscr{L}_+ = \mathscr{L} \cup \{R_E \mid E \in \mathscr{D}\}.$$

For two $\mathscr{L}_+$-structures M and N, we write $M \subseteq_+ N$ to indicate that M is an $\mathscr{L}_+$-substructure of N.

Definition 3.20. (i) Let T_{Nat} denote a theory generated by the union of T_0 and the collection of universal closures of formulas:

$$(\exists \bar{y}) Diag_E(\bar{x},\bar{y}) \leftrightarrow R_E(\bar{x})$$

for any $E \in \mathscr{D}$. We say the $\mathscr{L}_+$-structure M is natural if $M \models T_{\text{Nat}}$.

(ii) Let T_+ denote the universal part of T_{Nat}, i.e. $T_+ = (T_{\text{Nat}})_\forall$.

(iii) For any $\mathscr{L}$-structure M, let M_+ denote the expansion of M to an $\mathscr{L}_+$-structure, with $M_+ \models T_{\text{Nat}}$. So for any $E \in \mathscr{D}$, we have

$$M_+ \models \forall \bar{x}(R_E(\bar{x}) \leftrightarrow (\exists \bar{y})(Diag_A(\bar{x}) \wedge Diag_E(\bar{x},\bar{y}))).$$

Remark 3.21. Every $\mathscr{L}_+$-existential formula is T_{Nat}-equivalent to a disjunction of formulas of the form

$$(\exists \bar{y})\left(Diag_A(\bar{x}) \wedge Diag_B(\bar{x},\bar{y}) \wedge \bigwedge_i \neg R_{E_i}(\bar{x},\bar{y})\right),$$

where $A \subseteq B \in K_0$ and $B \prec_i E_i$.

The following lemmas show the main properties of this extension. The proofs are quite straightforward and left to the reader. Complete proofs can be found in [16, Lemmas 2.1.23 and 2.1.24].

Lemma 3.22. *Let $M, N \models T_{\text{Nat}}$ be two $\mathscr{L}_+$-structures. Let $A \prec_i B \in K_0$ and $A \subset_\omega M$. If $M \subset_+ N$ then $\chi_M(B/A) = \chi_N(B/A)$.*

Lemma 3.23. *Let $M \subseteq N \models T_0$ and $A, B \in K_0$.*

1. *If $M \prec N$, then $M_+ \subset_+ N_+$.*

2. $A \prec B$, *if and only if* $A_+ \subset_+ B_+$
3. *If* $\langle K_0, \prec\rangle$, *has the* AC, *then* $M_+ \subseteq_+ N_+$ *if and only if* $M \prec N$.

Lemma 3.24. *Let* $\mathscr{L}_+^*$ *be the expansion of* $\mathscr{L}$ *as in Definition* 3.19 *when* $\prec = \leqslant^*$ *and let* T_0 *be the universal theory which is determined by* $C^{>0}$. *Then for any* $M, N \models T_0$, *if* $M_+ \subseteq_+ N_+$ *then* $M \leqslant N$.

Proof. Suppose $\mathscr{L}_+$ is the extension of $\mathscr{L}$ (according Definition 3.19) with respect to $\leqslant$. By Remark 3.15, $\mathscr{L}_+ \subset \mathscr{L}_+^*$. Now by Lemma 3.17 $\langle K_0, \leqslant\rangle$ has the AC. So we can use property 3 in Lemma 3.23 and the proof is then clear. □

The next lemma relates the amalgamation property of a smooth class $\langle K_0, \prec\rangle$ to the amalgamation property of the corresponding T_{Nat}. We first need the following definition.

Definition 3.25. For any $M \models T_0$, any $m \in \omega$ and any $A \subseteq M$,

$$cl_M^m(A) = \bigcup \{B \mid A \prec_i B \subseteq M \text{ and } |B - A| < m\}.$$

If a smooth class $\langle K_0, \prec\rangle$ has the AC, we can easily see that for any m and $A \subset_\omega M \models T_0, cl_M^m(A)$ is finite and its cardinality is uniformly bounded by $\mu(|A|, m)$. □

Lemma 3.26. *Suppose a smooth class* $\langle K_0, \prec\rangle$ *has the AP and the AC. Then* T_+ *is a Robinson theory.*

Proof. We must show that T_+ has the amalgamation. Let $A, B, C \models T_+$ with $A \subset_+ B$, $A \subset_+ C$. Without loss of generality we may assume that B and C are natural. Moreover, by compactness, we may suppose A is finite. So in B and C each $R_E \in \mathscr{L}_+$ has a natural interpretation. We claim

$$cl_B(A) \cong_A cl_C(A).$$

To prove the claim we first show that $cl_B^m(A) \cong_A cl_C^m(A)$ for any $m \in \omega$. Let $E_0 \cong cl_B^m(A)$. We have

$$B \models R_{E_0}(\bar{a}) \wedge \bigwedge_{E \in D_{A,E_0}^m} \neg R_E(\bar{a}).$$

Intuitively this formula says that "$B \models cl^m(A) \cong E_0$". Since $A \subset_+ B$ and $A \subset_+ C$,

$$A \models R_{E_0}(\bar{a}_0) \wedge \bigwedge_{E \in D_{A,E_0}^m} \neg R_E(\bar{a}),$$

and

$$C \models R_{E_0}(\bar{a}) \wedge \bigwedge_{E \in D_{A,E_0}^m} \neg R_E(\bar{a}). \qquad (*)$$

Now since C is natural, $(*)$ means $cl_C^m(A)\cong E_0$. Now for A,B,C we define $\langle A_B^*, n\in\omega\rangle$ and $\langle A_C^n, n\in\omega\rangle$ by induction on n as follows:

Take $A_B^0=A_C^0=A_0$. For each $n>0$ let $A_B^{n+1}=cl_B^{n+1}(A_B^n)$ and $A_C^{n+1}=cl_C^{n+1}(A_C^n)$. By what we proved above and induction on n, there are $f_n: A_B^n\mapsto A_C^n$, with $f_0=Id$ and $f_n\subset f_{n+1}$. But as $cl_B(A_0)=\bigcup_n A_B^n$ and $cl_C(A_0)=\bigcup_n A_C^n$, $f=\bigcup_n f_n: cl_B(A_0)\mapsto cl_C(A_0)$ is an isomorphism. Moreover, if we interpret $\mathscr{L}_+$ naturally in $cl_B(A)$ and $cl_C(A)$, then $cl_B(A)\subset_+ B$ and $cl_A(A)\subset_+ C$.

So to prove the claim it is enough to show A is an amalgamation base, when A is natural. Thus we must prove that $Diag_+(A,B)\cup Diag_+(A,C)$ is consistent. But since $\langle K_0,\prec\rangle$ is closed under taking substructure and has the amalgamation property, this can be easily verified by compactness. □

Definition 3.27. On account of condition 1 in Definition 3.2 we modify δ to $d:\{(N,A)\mid N\in\bar{K_0}$ and $A\subset_\omega N\}\mapsto\mathbb{R}^+$ by defining for each $N\in\bar{K_0}$,

$$d(N,A)=\inf\{\delta(B)\mid A\leqslant_i B\subseteq_\omega N\}.$$

We usually write $d(N,A)$ as $d_N(A)$ and omit the subscript N if it is clear from the context.

Definition 3.28. For finite A,B contained in N, we define the relative dimension of A over B by $d_N(A/B)=d_N(AB)-d_N(B)$.

Definition 3.29. For $M\in K$ and $A,B\subseteq_\omega M$ and $C\subseteq M$, we say that A is d-independent from B over C (in M) (or A and B are independent over C) if

$$d(A/C)=d(A/BC).$$

For arbitrary $A,B\subseteq M$ we say that A and B are d-independent over C (in M) if for all $\bar{A}\subseteq_\omega A$ and $\bar{B}\subseteq_\omega B$ we have $\bar{A}$ is d-independent from $\bar{B}$ over C (in M).

Notation 3.30. If A is d-independent from B over C, then we write $A \underset{C}{\smile\!\!\!\!|^{d}} B$.

For the standard facts about the d-independence see [3,18,16]. We just quote the following facts that are needed later.

Lemma 3.31. *Let $A\subseteq A'\subset_\omega cl(A)$. Then $d(A)=d(A')$.*

Lemma 3.32. *Let $A\leqslant M$ and $\bar{a}$, a finite tuple. Then for any $\varepsilon>0$ there are $A'\subset_\omega A$ and $B'\subset_\omega cl(\bar{a}A)$ such that $A'=cl(A)\cap B'$ and*

$$|\delta(B'/A')-d(\bar{a}/A)|<\varepsilon_0 \tag{$*$}$$

Lemma 3.33. *Suppose $A, B \leqslant M$ and $C = A \cap B$. Then the following conditions are equivalent.*

1. $A \mathop{\smile}\limits^{|^{d}}_{C} B$.
2. $AB \leqslant M$ *and*, A *and* B *are in free amalgamation over* C *within* M.

3.1. The independence theorem diagram and simplicity

Let $\mathscr{L} = \{R_1, \ldots, R_n\}$ be a finite relational language. Suppose the function δ be as in Definition 3.5. So for each R_i there is a real number α_i with $0 \leqslant \alpha_i \leqslant 1$ such that for each finite $\mathscr{L}$-structure A

$$\delta(A) = |A| - \sum_{i=1}^{n} \alpha_i e_i{}^{A}.$$

Let K_0 be a subclass of $C^{>0}$ closed under taking substructures and isomorphisms, containing the empty set and let $\prec$ be either $\leqslant$ or $\leqslant^*$, as defined in Section 1. Let T_0 be the universal theory determined by K_0, and let $\mathscr{L}_+$ be the $\mathscr{L}$-extension and $T_{\mathrm{Nat}} \supset T_0$ the theory as defined in Definition 3.19.

Assumption 3.34. Throughout this section we assume that $\langle K_0, \prec \rangle$ has the algebraic closure property (AC) (Definition 3.17).

The main subject of this section is to study simplicity of $EX(T_+)$, the category of existentially closed models of T_+. Note that as T_{Nat} is a universal theory, any $M \in EX(T_+)$ models T_{Nat}.

Definition 3.35. (i) We say that $I = \langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12} \rangle$ is a $\langle K_0, \prec \rangle$-independence theorem diagram (ITD) if I satisfies the following conditions:

1. All co-ordinates of I are in K_0,
2. $A_i \prec A_{ij}$ for $i \neq j \in \{0, 1, 2\}$,
3. A_i and A_j are in free amalgamation over B in A_{ij},
4. $A_{ij} \cap A_{ik} = A_i$ for $i, j, k \in \{0, 1, 2\}$ with $j \neq k$,
5. $A_i \cup A_j \leqslant A_{ij}$ for $i \neq j \in \{0, 1, 2\}$.

(ii) Let $I = \langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12} \rangle$ be an ITD. Suppose $E_I = \bigcup A_{ij}$ and the relations are just the union of the relations on each A_{ij}. Then $\langle K_0, \prec \rangle$ is said to be closed under the independence theorem diagram (ITD), if whenever $I = \langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12} \rangle$ is a $\langle K_0, \prec \rangle$-ITD, then E_I is in K_0.

Remark 3.36. If $\mathscr{L}$ does not contain any weight zero relation, then $E_I = (A_{02} \otimes_{A_0} A_{01}) \otimes_{A_1 \cup A_2} A_{12}$.

The next lemma shows why we call the above diagram an independence theorem diagram.

Lemma 3.37. (i) *Let* $I=\langle B,A_0,A_1,A_2,A_{01},A_{02},A_{12}\rangle$ *be a* $\langle K_0,\prec\rangle$*-ITD and let* E_I *be as in Definition* 3.35. *Then*

1. $A_0\cup A_1\cup A_2\leqslant E_I$,
2. $d_E(A_0/A_1\cup A_2)=d_E(A_0/B)$.

(ii) *If* $\prec\;=\;\leqslant^*$ *then for any* $\{i,j,k\}\in\{0,1,2\}$ *we have* $A_i\cup A_j\leqslant^* A_{ik}\otimes_{A_k}A_{kj}$.

Proof. (i) Since weight zero relations do not have any effect on what we should prove, we may assume that $\mathscr{L}$ does not contain any weight zero relation and therefore by Remark 3.36 we have $E_I=(A_{02}\otimes_{A_0}A_{01})\otimes_{A_1\cup A_2}A_{12}$.

Now to show 1, we should verify that for any U with $A_0\cup A_1\cup A_2\subseteq U\subseteq E$ we have $\delta(U/A_0\cup A_1\cup A_2)\geqslant 0$.

For any $i\neq j$, let $X_{ij}=(U\cap A_{ij})-(A_i\cup A_j)$. Then $U-(A_0\cup A_1\cup A_2)=\bigcup_{i\neq j}X_{ij}$. Therefore, we have

$$\begin{aligned}\delta(U/A_0\cup A_1\cup A_2)&=\delta\left(\bigcup_{i\neq j}X_{ij}/A_0\cup A_1\cup A_2\right)\\&=\delta(X_{01}/A_0\cup A_1\cup A_2)+\delta(X_{02}/A_0\cup A_1\cup A_2\cup X_{01})\\&\quad+\delta(X_{12}/A_0\cup A_1\cup A_2\cup X_{01}\cup X_{02}).\end{aligned}$$

Now for $j\neq k$

- X_{ij} and $X_{ik}\cup A_k$ are freely amalgamated over $A_i\cup A_j$,
- X_{ij} and $(X_{ik}\cup X_{kj})\cup A_k$ are freely amalgamated over $A_i\cup A_j$.

Hence we have

- $\delta(X_{02}/A_0\cup A_1\cup A_2\cup X_{01})=\delta(X_{02}/A_0\cup A_2)$,
- $\delta(X_{12}/A_0\cup A_1\cup A_2\cup X_{01}\cup X_{02})=\delta(X_{12}/A_1\cup A_2)$,
- $\delta(U/A_0\cup A_1\cup A_2)=\sum_{i\neq j}\delta(X_{ij}/A_i\cup A_j)$.

Moreover since $A_i\cup A_j\leqslant A_{ij}$, we have $\delta(X_{ij}/A_i\cup A_j)\geqslant 0$ and hence $\delta(U/A_0\cup A_1\cup A_2)\geqslant 0$.

To show 2, one can invoke 1, and since $A_1\cup A_2\leqslant E$, we have

$$\begin{aligned}d_E(A_0/A_1\cup A_2)&=d_E(A_0\cup A_1\cup A_2)-d_E(A_1\cup A_2)\\&=\delta(A_0\cup A_1\cup A_2)-\delta(A_1\cup A_2)=\delta(A_0/B).\end{aligned}$$

For (ii), we shall prove that $A_0 \cup A_2 \leqslant^* A_{01} \otimes_{A_1} A_{12}$. Let $L_1 = A_{01} \otimes_{A_1} A_{12}$ and H be a nonempty subset of $L_1 - (A_0 \cup A_2)$. We show $\delta(H/A_0 \cup A_2) > 0$. Suppose $Y_{01} = H \cap A_{01}$, $Y_{12} = H \cap A_{12}$ and $Y_1 = H \cap A_1$. Then $H = Y_{01} \otimes_{Y_1} Y_{12}$. By transitivity of predimension we have

$$\delta(H/A_0 \cup A_2) = \delta(Y_{01}/A_0 \cup A_2) + \delta(Y_{12}/A_0 \cup A_2 \cup Y_{01}).$$

Now since A_{01} and A_{12} are in free amalgamation over A_1, we get $\delta(Y_{01}/A_0 \cup A_2) = \delta(Y_{01}/A_0) \geqslant 0$ and

$$\delta(Y_{12}/A_0 \cup A_2 \cup X_{01}) = \delta(Y_{12} - Y_{01}/A_0 \cup A_2 \cup Y_{01}) = \delta(Y_{12} - Y_{01}/A_2) \geqslant 0.$$

But as either Y_{01} or $Y_{12} - Y_{01}$ is nonempty, at least one of the summands in $\delta(H/A_0 \cup A_2)$ must be strictly positive. Thus $\delta(H/A_0 A_2) > 0$. □

From now on we work with a fixed e-universal model $\mathfrak{C}$ of $T_+(T_{\mathrm{Nat}})$. Now we state the main result of this section.

Theorem 3.38. *Suppose $\langle K_0, \prec \rangle$ has the JEP and is closed under the ITD. Then $EX(T_+)$ is formally simple.*

The main idea of the proof of Theorem 3.38 is to show that the d-independence is a notion of independence which satisfies the independence theorem.

Proposition 3.39. *Let $\rho \in \mathbb{R}^+$ and E a small subset of $\mathfrak{C}$. Suppose $n \in \omega$. Then $\{\bar{x} \in \mathfrak{C}^n \mid d(\bar{x}/E) \geqslant \rho\}$ is type definable.*

Proof. By the definition of a d-function for any $\bar{a}$ with $|\bar{a}| = n$ we have

$$d(\bar{a}/E) \geqslant \rho \quad \text{iff} \quad \text{for any } E_0 \subset_\omega E, d(\bar{a}/E_0) \geqslant \rho.$$

Therefore, it is enough to prove the proposition for any $E_0 \subset_\omega \mathfrak{C}$. Put $A_{E_0} = \{F \supseteq E_0 \mid$ there is a tuple $\bar{x}, |\bar{x}| = n$, $(E_0 \cup \bar{x}) \leqslant_i F$ and $\delta(F) < d(E_0) + \rho\}$. Let $\bar{e}_0$ be an enumeration of E_0. We set

$$\Gamma_{(E_{0,\rho})}(\bar{x}) \equiv \bigwedge_{F \in A_{E_0}} \neg \exists \bar{z}\, Diag_F(\bar{e}_0, \bar{x}, \bar{z}) \equiv \bigwedge_{F \in A_{E_0}} \neg R_F(\bar{e}_0, \bar{x}).$$

We want to show that for any tuple $\bar{a}$:

$$\bar{a} \models \Gamma_{(E_{0,\rho})}(\bar{x}) \quad \text{if and only if} \quad d(\bar{a}/E_0) \geqslant \rho.$$

To demonstrate this, we first suppose $d(\bar{a}/E_0) < \rho$. Then $d(\bar{a}E_0) - d(E_0) < \rho$ and $d(\bar{a}E_0) < \rho + d(E_0)$. Moreover, since $d(\bar{a}E_0) = \inf\{\delta(F) \mid \bar{a}E_0 \leqslant_i F\}$, there is $F \supseteq \bar{a}E_0$ such that $\delta(F) < \rho + d(E_0)$. Hence $F \in A_{E_0}, \mathfrak{C} \models \exists \bar{z}\, Diag_F(\bar{e}_0, \bar{a}, \bar{z})$ and $\bar{a} \not\models \Gamma_{(E_{0,\rho})}(\bar{x})$.

Now for a tuple $\bar{a}$ if $d(\bar{a}/E_0) \geqslant \rho$ then $d(\bar{a}E_0) - d(E_0) \geqslant \rho$ and $d(\bar{a}E_0) \geqslant \rho + d(E_0)$. So $\inf\{\delta(F) | \bar{a}E_0 \leqslant_i F\} \geqslant \rho + d(E_0)$, and for all $F \supseteq \bar{a}E_0$ with $\bar{a}E_0 \leqslant_i F$ we have $\delta(F) \geqslant \rho + d(E_0)$. Hence for given $F \in A_{E_0}$ we have $\mathfrak{C} \models \neg \exists \bar{z}\, Diag_F(\bar{e}_0, \bar{a}, \bar{z})$ and $\bar{a} \models \Gamma_{(E_{0,\rho})}(\bar{x})$. □

Now by the above proposition and the definition of the d-function, for any small subset E the following relations hold:

$$\begin{aligned} d(\bar{x}/E) \geqslant \rho \quad & \text{iff } \forall n \in \omega \ d(\bar{x}/E) \geqslant \rho - 1/n, \\ & \text{iff } \forall n \in \omega \ \forall E_0 \subset_\omega E \ d(\bar{x}/E_0) \geqslant \rho - 1/n, \\ & \text{iff } \bigcup_{n\in\omega} \bigcup_{E_0 \subset_\omega E} \Gamma_{E_{0,\rho}-1/n}(\bar{x}). \end{aligned}$$

We denote the above type by $\Gamma_{E,\rho}(\bar{x})$; note that is a quantifier-free type in $\mathscr{L}_+$.

Definition 3.40. Let A be any small set. For any $p \in eS(A)$ define $d(p)$ to be $d(\bar{a}/A)$ for (some) any $\bar{a} \models p$.

Lemma 3.41. *$d(p)$ is well-defined.*

Proof. It is enough to show that for any $\bar{a} \equiv_A \bar{b}$ we have $d(\bar{a}/A) = d(\bar{b}/A)$. But since $\bar{a} \equiv_A \bar{b}$, we have $cl(\bar{a}A) \cong cl(\bar{b}A)$. Therefore, for any $A_0\bar{a} \subseteq B$, with $\delta(B) \leqslant \delta(A_0\bar{a})$ there is B' such that $A_0\bar{b} \subseteq B'$ and $\delta(B') = \delta(B)$. In fact $B' \cong_{A_0} B$. Now on account of Lemma 3.32, we have $d(\bar{a}/A) = d(\bar{b}/A)$. □

Lemma 3.42 (Independence theorem). *Let X be a $\prec$-closed subset of C. Let $Y_1, Y_2 \supseteq X$ be such that $Y_i \prec \mathfrak{C}$ for $i = 1,2$ and Y_1 is d-independent from Y_2 over X. Suppose $\bar{a}_1$ and $\bar{a}_2$ with $etp(\bar{a}_1/X) = etp(\bar{a}_2/X)$. Moreover, $\bar{a}_i$ is d-independent from Y_i over X $(i = 1,2)$. Then there is a tuple $\bar{a}$ such that $etp(\bar{a}/Y_i) = etp(\bar{a}_i/Y_i)$, $(i = 1,2)$ and $\bar{a}$ is d-independent from $Y_1 \cup Y_2$ over X.*

Proof. Let $\rho = d(p)$ and $E = Y_1 \cup Y_2$. Suppose $p_i(\bar{x}) = etp(\bar{a}_i/Y_i)$, $i = 1,2$. We also consider types $\Gamma_{(E,\rho)}$ as defined in Propositions 3.39.

Claim. *$\Omega(E,\bar{x}) \equiv etp(E/\emptyset) \cup p_1(\bar{x}) \cup p_2(\bar{x}) \cup \Gamma_{(E,\rho)}(\bar{x})$ is consistent.*

Proof. By compactness it is enough to show that $\Omega(E,\bar{x})$ is finitely consistent. Let $\bar{e}_0 \subset_\omega E, \phi(\bar{e}_0) \in etp(E/\emptyset), \theta_i(\bar{x},\bar{y}_i) \in p_i(\bar{x})$ with $i = 1,2$ and, $\Gamma_0(\bar{x},\bar{e}_0) \subset_\omega \Gamma_{(E,\rho)}(\bar{x})$. We show that $\{\phi(\bar{e}_0), \theta_1(\bar{x},\bar{y}_1), \theta_2(\bar{x},\bar{y}_2)\} \cup \Gamma_0(\bar{x},\bar{e}_0)$ is consistent. Here $\bar{e}_0$ is an enumeration of $E_0 \subset_\omega E$ and $\bar{y}_i = \bar{e}_0 \cap Y_i$ $(i = 1,2)$. By the definition of $\Gamma_{(E,\rho)}(\bar{x})$ it is clear there are $X' \subset_\omega X, E' \subset_\omega E$ and, $n \in \omega$ such that

$$\Gamma_{(E,\rho)}(\bar{x}) \subset \Gamma_{(E',\rho-1/n)}(\bar{x}).$$

So we show that for any $E_0 \subset_\omega E$ and $n \in \omega$ there is a large set $X' \subset_\omega X$ such that

$$\{\phi(\bar{e}_0), \theta_1(\bar{x},\bar{y}_1), \theta_2(\bar{x},\bar{y}_2)\} \cup \Gamma_{(E_0,\rho-1/n)}(\bar{x}) \text{ is consistent.}$$

This would mean that to find $\bar{a}$, $\bar{e}_0$ and X' such that

- $\mathfrak{C} \models \phi(\bar{e}_0) \wedge \theta_1(\bar{a},\bar{y_1}) \wedge \theta_2(\bar{a},\bar{y_2})$,
- $d(\bar{a}/E_0) \geqslant \rho - 1/n$,

Let $\bar{x}_0 = (\bar{y_1} \cap X) \cup (\bar{y_2} \cap X)$. By renaming $\bar{y}_1$ and $\bar{y}_2$, we may suppose that $\bar{e_0} = \bar{x_0} \cup \bar{y_1} \cup \bar{y_2}$, where $\bar{y_i} \subset_\omega Y_i - X$, for $i = 1, 2$. As $\bar{a}_i$ are d-independent over X, we have $d(\bar{a}_i/Y_i) = d(\bar{a}_i/X)$. Since $\mathfrak{C}$ is natural, by Remark 3.21, we may assume that $\phi(\bar{e}_0)$ and $\theta_i(\bar{x}, \bar{y}_i)$, $i = 1, 2$, have the following forms:

$$\phi(\bar{e}_0) \rightleftharpoons \exists \bar{z} \left(\phi'(\bar{z}, \bar{e}_0) \wedge \bigwedge_i \neg R_{E_i}(\bar{z}, \bar{e}_0) \right),$$

$$\theta_1(\bar{x}, \bar{x_0}, \bar{y_1}) \rightleftharpoons \exists \bar{z} \left(\theta_1'(\bar{z}, \bar{x}, \bar{x_0}, \bar{y_1}) \wedge \bigwedge_i \neg R_{F_i}(\bar{z}, \bar{x}, \bar{x_0}, \bar{y_1}) \right),$$

$$\theta_2(\bar{x}, \bar{x_0}, \bar{y_2}) \rightleftharpoons \exists \bar{z} \left(\theta_2'(\bar{z}, \bar{x}, \bar{x_0}, \bar{y_2}) \wedge \bigwedge_i \neg R_{G_i}(\bar{z}, \bar{x}, \bar{x_0}, \bar{y_2}) \right).$$

Hence there are $\bar{z}_{ij}$, $i \neq j \in \{0, 1, 2\}$ such that

$$\mathfrak{C} \models \phi'(\bar{z}_{12}, \bar{e}_0) \wedge \bigwedge_i \neg R_{E_i}(\bar{z}_{12}, \bar{e}_0),$$

$$\mathfrak{C} \models \theta_1'(\bar{z}_{01}, \bar{a}_1, \bar{x_0}, \bar{y_1}) \wedge \bigwedge_i \neg R_{F_i}(\bar{z}_{01}, \bar{x}, \bar{x_0}, \bar{y_1}),$$

$$\mathfrak{C} \models \theta_2'(\bar{z}_{02}, \bar{a}_2, \bar{x_0}, \bar{y_2}) \wedge \bigwedge_i \neg R_{G_i}(\bar{z}_{02}, \bar{x}, \bar{x_0}, \bar{y_2}).$$

Let $C_{12} = \bar{z}_{12} \cup \bar{e}_0$, $C_{01} = \bar{z}_{01} \cup \bar{x_0} \cup \bar{y_1}$ and $C_{02} = \bar{z}_{02} \cup \bar{x}_0 \cup \bar{y}_2$. So $C_{ij} \in K_0$. Suppose $D_{0i} = C_{0i} \cap cl_\prec(a_i X)$ and let $\bar{d}_{0i}$ be enumeration of D_{0i}. Now since $etp(a_1/X) = etp(a_2/X)$, there is a function $g \in Aut_X(\mathfrak{C})$ such that $g(a_1) = a_2$. It follows that $etp(a_1 \bar{d}_{01} g^{-1}(\bar{d}_{02}/X) = etp(a_2 g(\bar{d}_{01})\bar{d}_{02}/X)$.

Thus $\bar{d}_{01} g^{-1}(\bar{d}_{02}) \cong_X g(\bar{d}_{01})\bar{d}_{02}$.

Now by Lemma 3.32, there are $X' \subseteq_\omega X$, with $\bigcup_{i \neq j} (C_{ij} \cap X) \subseteq X'$ and $K_0 \subseteq_\omega cl_\prec(a_1 X)$ with $d_{01} g^{-1}(d_{02}) \subset K_0$. Let $K_0' = g(K_0)$. Then $K_0 \cong_{X'} K_0'$ and hence

$$|\delta(K_0'/X') - \rho| < 1/n.$$

Now we introduce a suitable ITD. We first identify K_0 and K_0' via function g. Let $B = X'$, $A_0 = K_0$, $A_1 = X' \cup (C_{01} \cap Y_1) \cup (C_{12} \cap Y_1)$, $A_2 = X' \cup (C_{02} \cap Y_2) \cup (C_{12} \cap Y_2)$. Furthermore, $A_{01} \cong C_{01} \cup K_0 \cup A_1$, $A_{02} \cong C_{02} \cup K_0' \cup A_2$, and $A_{12} \cong C_{12} \cup A_1 \cup A_2$ in such a way that $A_{ij} \cap A_{ik} = A_i$ for $i, j, k \in \{0, 1, 2\}$ and $j \neq k$.

We claim that $I = \langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12} \rangle$ is an ITD. So we must verify that for $i \neq j \in \{0, 1, 2\}$ the following conditions are met.

1. All I-components are in K_0. This holds because all of these structures are subsets of $\mathfrak{C}$ and $\langle K_0, \prec \rangle$ is closed under taking substructure.
2. $A_i \prec A_{ij}$. For example we prove $A_0 \prec A_{01}$. Since $cl_\prec(a_1 X) \prec \mathfrak{C}$, by Lemma 3.9, we have $cl_\prec(a_1 X) \cap A_{01} \prec A_{01}$. But $cl_\prec(a_1 X) \cap A_{01} = cl_\prec(a_1 X) \cap (C_{01} \cup K_0 \cup A_1) =$

$(cl_{\prec}(a_1X)\cap C_{01})\cup(cl_{\prec}(a_1X)\cap K_0)\cup(cl_{\prec}(a_1X)\cap A_1)$. Now we claim that $cl_{\prec}(a_1X)\cap A_1)=X'$ holds. This is because

$$cl_{\prec}(a_1X)\cap A_1\subset cl_{\prec}(\bar{a}_1X)\cap Y_1\subset(X\cap A_1)=X'.$$

So $cl_{\prec}(a_1X)\cap A_{01}=D_{01}\cup K_0\cup X'$. Moreover, $D_{01}\subset K_0$. Thus $D_{01}\cup K_0\cup X'=A_0$ and $A_0\prec A_{01}$. Other relations can be proved similarly.

3. $A_{ij}\cap A_{jk}=A_i$ for $j\neq k$. Just holds by definitions of A_{ij}'s.
4. A_i and A_j are freely amalgamated over B in A_{ij}. For instance, one can show that A_0 and A_2 are freely amalgamated over B in A_{02}. Given that $(A_0-X)\subset(cl_{\prec}(\bar{a_2}X)-X)$ and $A_2-X\subset Y_2-X$ and, since these two sets are freely amalgamated over X in $\mathfrak{C}$, A_0 and A_2 are in A_{02}.
5. $A_i\cup A_j\leqslant A_{ij}$. One can prove that $A_0\cup A_1\leqslant A_{01}$. Since Y_1 and a_1 are d-independent over X, by Lemma 3.33, we have $cl_{\prec}(a_1X)\cup Y_1\leqslant\mathfrak{C}$. So, again, by Lemma 3.9 $(cl_{\prec}(a_1X)\cup Y_1)\cap A_{01}\leqslant A_{01}$. But $(cl_{\prec}(a_1X)\cup Y_1)\cap A_{01}=(A_{01}\cap cl_{\prec}(a_1X))\cup(Y_1\cap A_{01})=A_0\cup A_1$. Hence $A_0\cup A_1\leqslant A_{01}$.

Now since $\langle K_0,\prec\rangle$ is closed under ITD, we get $E_I\in K_0$. Let $E_I=L$. The JEP entails that there is a function $f:L\mapsto\mathfrak{C}$ with $f(L)\prec\mathfrak{C}$. Suppose $\bar{L}=f(L)$ and $\bar{a}=f(\bar{a}_1)$. Now on account of Lemma 3.37, in both L and $\bar{L}$ the following equations hold:

$$d_L(\bar{a}_1/B)=d_{\bar{L}}(\bar{a}/\bar{B})=\delta(A_0/A_1\cup A_2)=d_{\bar{L}}(\bar{a}/\bar{A}_1\cup\bar{A}_2)=\delta(A_0/B).$$

But $|\delta(A_0/B)-\rho|<1/n$. Hence $d_{\bar{L}}(\bar{a}/\bar{A}_1\cup\bar{A}_2)\geqslant\rho-1/n$. So as $f(E_0)\subseteq\bar{A}_1\cup\bar{A}_2$, we have $d(\bar{a}/f(E_0))\geqslant d(\bar{a}/A_1\cup A_2)\geqslant\rho-1/n$. Moreover, since $\bar{L}\prec\mathfrak{C}$, we get $d_{\bar{L}}(\bar{a}/\bar{A}_1\cup\bar{A}_2)=d_{\mathfrak{C}}(\bar{a}/\bar{A}_1\cup\bar{A}_2)$. Hence $\bar{a}$ is d-independent from $\bar{A}_{12}$ over $f(B)$. Now we show that

$$\mathfrak{C}\models\phi(f(\bar{e}_0))\wedge\theta_1(\bar{a},f(\bar{y}_1))\wedge\theta_2(\bar{a},f(\bar{y}_2)).$$

For example, to verify that $\mathfrak{C}\models\phi(f(\bar{e}_0))$, we need to prove that $\mathfrak{C}\models\exists\bar{z}(\phi'(\bar{z},f(\bar{e}_0))\wedge\bigwedge_i\neg R_{E_i}(\bar{z},f(\bar{e}_0)))$. But $\mathfrak{C}\models\phi'(f(\bar{z}_{12}),f(\bar{e}_0))$. So it is enough to show that $\mathfrak{C}\models\bigwedge_i\neg R_{E_i}(f(\bar{z}_{12}),f(\bar{e}_0))$. If not, then for some i we have $\mathfrak{C}\models R_{E_i}(f(\bar{z}_{12}),f(\bar{e}_0))$. Now as $f(A_{12})\prec\bar{L}\prec\mathfrak{C}$, there should be $E_i\subset f(A_{12})$ realizing the diagram of E_i over $f(\bar{z}_{12})\cup f(\bar{e}_0)$. Hence by considering f^{-1} we should have

$$\mathfrak{C}\models R_{E_i}(\bar{z}_{12},\bar{e}_0),$$

a contradiction. Thus $\{\phi(\bar{e}_0),\theta_1(\bar{x},\bar{y}_1),\theta_2(\bar{x},\bar{y}_2)\}\cup\Gamma_{(E_0,\rho-1/n)}(\bar{x})$ is consistent and the proof of claim is complete. □

Now by claim there should be $E'\equiv_1 E$ and $\bar{a}'$ with $\bar{a}'\models p_1(\bar{x})\cup p_2(\bar{x})$ such that $d(\bar{a}'/\bar{X})=d(\bar{a}'/\bar{E})$. But $\bar{E}\equiv_1 E$ implies that there is an automorphism $h\in Aut(\mathfrak{C})$ with $h(\bar{E})=E$. So if we take $\bar{a}=h(\bar{a}')$ then $\bar{a}\models p_1(\bar{x})\cup p_2(\bar{x})\cup\Gamma_{(E,\rho)}(\bar{x})$ and the proof of the lemma is complete. □

Definition 3.43. Let $A\subset\mathfrak{C}$ and $\bar{b}$ be a finite tuple. We say $\bar{b}$ is algebraic over A, if there is an existential $\mathscr{L}_+$-formula, with parameters in A, say $\phi(\bar{x})$ such that $\mathfrak{C}\models\phi(\bar{b})$

and this formula has only finitely many solutions in $\mathfrak{C}$. By the algebraic closure $acl(A)$ of A in $\mathfrak{C}$ is the union of all the algebraic tuples over A.

Lemma 3.44. *Let M be an existentially closed substructure of $\mathfrak{C}$. Then T_0 has free amalgamation over M, i.e. for each B and E with $M \prec B$ and $M \prec E$, there is an $\mathscr{L}$-structure $F \models T_0$ such that F is a free amalgam of B and E over M.*

Proof. Since M is an existentially closed model, it is an amalgamation base. Hence there are functions $f : B \mapsto \mathfrak{C}$ and $g : E \mapsto \mathfrak{C}$ with $f(B) \prec \mathfrak{C}$ and $g(E) \prec \mathfrak{C}$ such that $f|M = g|M = Id|M$. So we may suppose $M \prec B \prec \mathfrak{C}$ and $M \prec E \prec \mathfrak{C}$. Let $\Gamma_E(X/B)$ be the type such that for $E' \subset \mathfrak{C}$, E' realizes $\Gamma_E(X/B)$ if and only if $E' \cong_M E$ and E' and B are freely amalgamated over M. So this type includes the $\mathscr{L}$-quantifier-free diagram of E over M and, in addition, the negation that all possible relations that state that X and B are not freely amalgamated over M. Clearly $\Gamma_E(X/B)$ can be stated by an $\mathscr{L}$-quantifier-free type. We show that this type is finitely consistent. Let $M_0 \subset_\omega M, \bar{e}_0 \subset_\omega E - M$, $\bar{b}_0 \subset_\omega B - M$ and $\Gamma_0(\bar{x}) \subset_\omega \Gamma_E(X/B)$. We suppose $\Gamma_0(\bar{x})$ includes the quantifier-free type of $\bar{e}_0$ over M_0 and also states $\bar{x}$ and $\bar{b}_0$ are freely amalgamated over M_0. Since M is existentially closed, $acl(M) = M$. Hence $\bar{e}_0$ is not algebraic over M, i.e. there are infinitely many pairwise disjoint $\bar{e}_i \subset M$ (for $i > 0$) all of which realize the diagram of $\bar{e}_0$ over M_0. But $\bar{b}_0 \subset_\omega B - M$ and $M \prec B$. Thus for any $i > 0$,

$$\delta(\bar{b}_0/M_0 e_1, \ldots, e_i) \geqslant 0. \qquad (*)$$

Now we claim that there is an $n > 0$ such that $\bar{e}_n$ and $\bar{b}_0$ are in free amalgamation over M_0. If otherwise, then for each n we have $\delta(\bar{b}_0/M_0) > \delta(\bar{b}_0/M_0 e_n)$. Now by Definition 3.2 there is an $\varepsilon > 0$ such that for each $n \in \omega$ we have $\delta(\bar{b}_0/M_0 e_n) - \delta(\bar{b}_0/M_0) < -\varepsilon$. Thus

$$\delta(\bar{b}_0/M_0 e_0 \ldots e_n) \leqslant \delta(\bar{b}_0/M_0) - n\varepsilon.$$

Hence if we take n large enough then we have

$$\delta(\bar{b}_0/M_0 e_0 \ldots e_n) < 0$$

which contradicts $(*)$. So by interpreting the corresponding variables of $\bar{e}_0$ in $\Gamma_0 \subset_\omega \Gamma_E(X/B)$ by $\bar{e}_n$ we see that $\Gamma_E(X/B)$ is finitely consistent. Therefore there is $E' \cong_A E$ such that E' and B are in free amalgamation. Suppose $F = B \cup E'$. Then $F \models T_0$, and F is a free amalgam of E' and B over M. □

Remark 3.45. In the absence of weight zero relations $F \cong B \otimes_M E \models T_0$.

Remark 3.46. From definition of d-independence it is clear that it satisfies the invariance property of a notion of independence.

Lemma 3.47. *Let M be an existentially closed substructure of $\mathfrak{C}(M \prec \mathfrak{C})$ and $p(\bar{x}) \in eS(M)$. Let B with $M \subseteq_+ B$. Then there is a type $q(\bar{x}) \in eS(B)$ extending $p(\bar{x})$ and $q(\bar{x})$ is d-independent over M.*

Proof. Without loss of generality we suppose that B is also existentially closed. By Proposition 3.39 there is an $\mathscr{L}_+$-existential type $\Gamma_{(M,B)}(\bar{x})$ over B, such that for any $\bar{a}$ we have

$$\bar{\Gamma}_{(M,B)}(\bar{a}) \quad \text{if and only if} \quad \bar{a} \text{ is } d\text{-independent from } B \text{ over } M.$$

Put $Q(\bar{x}) = p(\bar{x}) \cup \bar{\Gamma}_{(M,B)}(\bar{x})$. This type can be shown to be finitely consistent. Let $\phi(\bar{x}) \in p(\bar{x})$. We may suppose $\phi(\bar{x}) = (\exists \bar{z})\theta(\bar{x}, \bar{z})$, where $\theta(\bar{x}, \bar{z})$ is a quantifier-free $\mathscr{L}_+$-formula. Let $\bar{a} \models p(\bar{x})$ and $\bar{d}$ be such that $\models \theta(\bar{a}, \bar{d})$ and $E = cl_{\prec}(\bar{a}\bar{d}M)$. Note that as $M \prec \mathfrak{C}$ and $M \subseteq E$, we have $M \prec E$. Since $M \prec B$ and $M \prec E$, by Lemma 3.44, there is $H \models T_0$ which a free amalgam of B and E over M. Then for H_+ and B_+, the natural expansions of H and B, we have, $B_+ \subset_+ H_+$. But since B is an existentially closed substructure, it is an amalgamation base. Hence there is a function $f : H_+ \mapsto \mathfrak{C}$ with $f(H_+) \prec \mathfrak{C}$ and $f|M = Id$. Then we have

1. $\mathfrak{C} \models \theta(f(\bar{a}), f(\bar{d}))$,
2. $cl_{\prec}(f(\bar{a})f(\bar{d})M)$ is in free amalgamation with B over M,
3. $cl_{\prec}(f(\bar{a})f(\bar{d})M) \cup B \prec \mathfrak{C}$.

So by Lemma 3.33 and since $\prec$ is stronger than $\leqslant (\prec \in \{\leqslant, \leqslant^*\})$, we can infer that

1. $\mathfrak{C} \models \phi(f(\bar{a}))$,
2. $d(f(\bar{a})f(\bar{d})/B) = d(f(\bar{a})f(\bar{d})/M)$,
3. $d(f(\bar{a})/M) = d(f(\bar{a})/B)$.

Thus by compactness there is an $\bar{a}'$ such that $\bar{a}' \models p(\bar{x})$ and $\bar{a}'$ is d-independent from B over M. $\square$

Proof of Theorem 3.38. Properties 1–5 generally hold. Furthermore, by Lemmas 3.42 and 3.47, d-independence satisfies axiom 6–7 of a notion of independence. Thus $EX(T_+)$ is formally simple. $\square$

Remark 3.48. If $\langle K_0, \prec \rangle$ has the AP and JEP then by Lemma 3.26 T_+ is a Robinson theory and therefore $EX(T_+)$ is simple. If moreover $\langle K_0, \prec \rangle$ has the free amalgamation property, then we can show the existence of d-independent types over any small subset $A \prec \mathfrak{C}$.

Corollary 3.49. *Let $\langle K_0, \prec \rangle$ has the free amalgamation property and be closed under the ITD. Let $A \prec B \prec \mathfrak{C}$ be small sets and $\bar{a}$, a finite tuple. Then $EX(T_+)$ is (formally) simple and*

$$\bar{a} \mathop{\smile}^{\vert}_{A} B \quad \textit{if and only if} \quad d(\bar{a}/A) = d(\bar{a}/B).$$

In the following we can prove if T_+ is also simple then forking independence always implies d-independence over any algebraically closed substructure. The main idea of this lemma is borrowed from the work of Baldwin–Shelah [2].

Lemma 3.50. *Suppose T_+ is a simple Robinson theory. Let M be an algebraically closed substructure of $\mathfrak{C}$. Let $B \supseteq M$ be a $\prec$-closed subset of $\mathfrak{C}$. Let $p(\bar{x}) \in eS(B)$. If $p(\bar{x})$ does not fork over M then it is also d-independent over M.*

Proof. Let $\bar{a} \models p(\bar{x})$. Since $acl(M) = M$, we may assume $p(\bar{x})$ is not algebraic over M. So it is enough to show that $d(\bar{a}/M) = d(\bar{a}/B)$. By Lemma 3.33, it is enough to show that

1. B and $cl(\bar{a}M)$ are freely amalgamated over M.
2. $B \cup cl(\bar{a}M) \leqslant \mathfrak{C}$.

Let $M_0 \subset_\omega M$ and $B_0 \subset_\omega B - M$, and suppose $K_0 \subset_\omega cl(\bar{a}M) - M$ and $C_0 \subset_\omega cl(\bar{a}B)$ such that $\delta(C_0/B_0M_0K_0) < 0$. Let $\bar{m_0}, \bar{b_0}, \bar{k_0}, \bar{c_0}$ be enumerations for M_0, B_0, K_0, C_0. We claim that K_0M_0 and $B_0M_0C_0$ are freely amalgamated over M_0. Suppose they are not. Then by the properties of δ (Definition 3.2), there is an $\varepsilon > 0$ such that $\delta(K_0/M_0) - \delta(K_0/M_0B_0C_0) > \varepsilon$. Since $\bar{a} \underset{M}{\downarrow} B$ and $K_0 \subset cl(\bar{a}M) \subseteq acl(\bar{a}M)$, we have $K_0 \underset{M}{\downarrow} B$. Therefore we may assume $K_0 = \bar{a}$. Let $\langle \bar{a}_i\bar{b}_i | i \in \omega \rangle$ be a Morley sequence of $etp(\bar{a}\bar{b}/M)$. Since T_+ is a Robinson theory, by Lemmas 2.17 and 2.12, such a sequence exists. We show $p_i(\bar{x}) = etp(\bar{a}_i/M\bar{b}_i)$ for $i \in \omega$ are n-contradictory for suitable $n \in \omega$. Let $f_n \in Aut(\mathfrak{C})$ such that $\bar{a}_0 = \bar{a}$ and $\bar{b}_0 = \bar{b}$, $f_n(\bar{a}\bar{b}) = \bar{a}_n\bar{b}_n$ and $\bar{c}_n = f_n(\bar{c}_0)$. We assume, on the contrary, that for each $n \in \omega$ there is an $\bar{a}^*$ which is a common solution for, say $p_1, \ldots, p_{n+1}$. We fix n such that $n\varepsilon > \delta(\bar{a}/M_0)$. But we have

$$\delta(\bar{a}^*/\bar{b}_1, \ldots, \bar{b}_n, \bar{c}_1, \ldots, \bar{c}_n) \leqslant \delta(\bar{a}/M_0) - n.\varepsilon < 0.$$

Therefore $\bar{a}^* \subset cl(M_0\bar{b}_1 \ldots \bar{b}_n\bar{c}_1, \ldots \bar{c}_n) \subset cl(M_0\bar{b}_1 \ldots \bar{b}_n\bar{a}_1 \ldots \bar{a}_n) \subseteq acl(M_0\bar{b}_1 \ldots \bar{b}_n\bar{a}_1 \ldots \bar{a}_n)$. Thus $\bar{a}_{n+1} \subset acl(M_0\bar{b}_1 \ldots \bar{b}_n\bar{a}_1 \ldots \bar{a}_n)$. But $\bar{a}_{n+1} \underset{M}{\downarrow} \bar{b}_1 \ldots \bar{b}_n\bar{a}_1 \ldots \bar{a}_n$ implies $\bar{a}_{n+1} \subset acl(M)$, a contradiction. Thus we conclude that B and $cl(\bar{a}M)$ are freely amalgamated over M and that $\delta(C/M_0\bar{b}) = \delta(C/M_0\bar{b}K_0) < 0$. Hence $C \subset cl(M_0\bar{b}) \subset B$ and $B \cup cl(\bar{a}M) \leqslant \mathfrak{C}$.

Now we suppose T_+ is a simple Robinson theory. Thus by Lemma 2.12 for any arbitrary set $A \subset \mathfrak{C}$ and $p \in eS(A)$, Morley sequences for p exist. This fact enables us to give the relationship between forking and d-independence. □

Lemma 3.51. *Suppose $\langle K_0, \prec \rangle$ is closed under the ITD and T_+ is a Robinson theory. Let $A \subseteq B \subset \mathfrak{C}$ with $acl(A) = A$ and $\bar{a}$, a finite tuple. Then*

(i) $\bar{a} \underset{A}{\downarrow} B$ *if and only if* $d(\bar{a}/A) = d(\bar{a}/B)$.
(ii) *the forking independence theorem holds for A.*

Proof. (i) As T_+ is a Robinson theory and d-independence is a notion of independence and satisfies the independence theorem over any set $A = acl(A)$ then by the claim in Lemma 2.17, d-independence always implies the forking independence over such sets. But since $acl(A)=A$, in virtue of Lemma 3.50, forking independence implies d-independence.

(ii) By (i), since d-independence is the same as forking independence, Lemma 3.42 implies the (forking) Independence Theorem holds for A.

Corollary 3.52. *Suppose* $\langle K_0, \prec\rangle$ *has the amalgamation property and is closed under the ITD. Then* Lstp = Stp *in* $\mathfrak{C}$.

Proof. Since $\langle K_0, \prec\rangle$ has the amalgamation property, by Lemma 3.26, T_+ is a Robinson theory. Hence on account of Lemma 3.51, the forking independence theorem holds for every algebraically closed set. So by the same method invoked in first-order case one can show that Lstp = Stp. □

Remark 3.53. Let K_0 be the class of triangle-free graphs. For each $A \in K_0$ we define $\delta(A)=|A|$. Then with respect to this predimension $\leqslant = \subseteq$. Since all members of K_0 are triangle-free, it is easy to see that $\langle K_0, \subseteq\rangle$ is not closed under the ITD. Moreover Hrushovski [6] proves that T, the generic theory is not simple. This example shows that in order to get a simple generic theory we need (some form of) independence theorem diagram.

The next example shows that the independence theorem diagram is not necessary for getting a simple generic theory.

Example 3.54. Let $\mathscr{L}=\{R_0, R_1\}$ where R_0 is a ternary relation and R_1 is a binary relation. Let $\delta(A)=|A|-|R_0(A)|$. Hence R_1 has a zero weight. Suppose K'_μ is a class of finite $\mathscr{L}$-structures with $K'_\mu=\{A | A|R_0 \in K_\mu\}$. Since $\langle K_\mu, \leqslant\rangle$ has the amalgamation, it implies that $\langle K'_\mu, \leqslant\rangle$ has also the free amalgamation. Let $T_1=\langle K_\mu, \leqslant\rangle$-generic theory and let T^* be the $\langle K'_\mu, \leqslant\rangle$-generic theory. In fact T_1 is Hrushovki's strongly minimal example. Then it is easy to check that the independence in T_1 defines also a notion of independence for T^* and it satisfies the independence theorem. So T^* is a simple theory. However, $\langle K'_\mu, \leqslant\rangle$ is not closed under the ITD. In fact we show a stronger result. Let $B=\{a,b\}$ be a discrete structure, $A_0=\{a,b,x_0\}$ with just one relation $R_0(a,b,x_0)$, $A_1=\{a,b,x_1\}$ with the only relation $R_0(a,b,x_1)$ and $A_2=\{a,b,y_1,y_2,\dots,y_{p-1},t\}$ such that for $i \leqslant p-1$ we have $R_0(a,b,y_i)$. Here $p=\mu(B,A_0)=\mu(B,A_1)$. Moreover, define $A_{01}=A_0 \otimes_B A_1$, $A_{02}=A_0 \otimes_B A_2$ and A_{12} such that $A_{12}|R_0=(A_1 \otimes_B A_2)|R_0$ and $R_1(t,x_1)$. It is easy to check that all A_{ij}'s are in K'_μ and $I=\langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12}\rangle$ is a $\langle K'_\mu, \leqslant\rangle$ independence theorem diagram. We claim that there is no $E \in K'_\mu$ such that we can embed all of A_{ij}'s in this structure. Suppose otherwise. Then we may assume these embeddings are identity functions. If in E we have $x_0 \neq x_1$ then in E there are $\mu(B,A_0)+1$ copies of A_0 over B which contradicts the definition of K_μ. On the other hand, in E we have $R_1(x_1,t)$ and $\neg R_1(x_0,t)$. So it is not possible to have $x_0=x_1$, a contradiction. Hence we cannot amalgamate A_{ij}'s and $\langle K'_\mu, \leqslant\rangle$ is not closed under the ITD.

4. Semigenericity and some model completeness results

In this section we consider cases where the class of existentially closed models of T_+, $EX(T_+)$, is an elementary class, i.e. T_+ has a model companion. We introduce a notion of semigeneric structures, which is in fact a first-order variant of genericity. This notion first appeared in different formulations in [3, Definition 4.3, 1, Definition 1.26]. What we present here is in fact a generalized version of the second formulation. We will also give a new proof for the quantifier elimination result proved by Baldwin–Shelah [1, Theorem 1.29]. Throughout this section we work with a smooth class $\langle K_0, \prec\rangle$ of finite $\mathscr{L}$-structures with the algebraic closure property and which satisfy Lemma 3.9; we do not require $\langle K_0, \prec\rangle$ to be defined via a predimension. The terminology and notations used here are borrowed from [1].

Definition 4.1. Let $A\subseteq B\in K_0$. Let $M\models T_0$, $f: B\mapsto \bar{B}\subset_\omega M$, and $m\in\omega$. We say that $\bar{B}$ does not have a new m-closure relative to $\bar{A}=f(A)$ if $cl^m_M(\bar{B})=\bar{B}\cup cl^m_M(\bar{A})$ and $\bar{B}\cap cl^m(\bar{a})=\bar{A}$, and $cl^m_M(\bar{B})-\bar{B}=E$ implies $cl^m_M(\bar{A})=\bar{A}\cup E$. In this situation we write $cl^m_M(\bar{B})=\bar{B}\bar{\cup} cl^m_M(\bar{A})$. So for any X with $|X|<m$, if $X\cap\bar{B}=\emptyset$ and $\bar{B}\prec_i \bar{B}\cup X$, then $\bar{A}\prec_i \bar{A}X$. That is if $X\subset cl^m_M(\bar{B})-\bar{B}$ then $X\subseteq cl^m_M(\bar{A})$.

Definition 4.2. We say the model M is semigeneric if 1. and 2. are fulfilled:

1. $M\models T_0$.
2. If $A\prec B\in K_0$ and $g: A\mapsto M$, then for each $m\in\omega$ there exists an embedding $\hat{g}$ of B into M which extends g such that

$$cl^m_M(\hat{g}(B)) = \hat{g}(B)\bar{\cup} cl^m_M(g(A)).$$

The following lemma gives an easy example of a semigeneric model. Recall that a smooth class $\langle K_0, \prec\rangle$ has the *full Amalgamation property* if for any $A, B, C\in K_0$ with $A\prec B$ and $A\subseteq C$ we have $B\otimes_A C\in K_0$ and $C\prec B\otimes_A C$.

Lemma 4.3. *Let K_0 be a subclass of $C^{>0}$ and suppose $\langle K_0, \leqslant^*\rangle$ has the full amalgamation property. Then the $\langle K_0, \leqslant^*\rangle$-generic model is semigeneric.*

Proof. Let M_0 be the $\langle K_0, \leqslant^*\rangle$-generic model. Suppose $A\leqslant^* B\in K_0$. Let $f: A\mapsto M_0$ and $C=cl^*_{M_0}(f(A))\subset_\omega M_0$. Since $A\subseteq C$ and $A\leqslant^* B$, we have $C\otimes_A B\in K_0$ and $C\leqslant^* C\otimes_A B$. Thus there is G: $C\otimes_A B\mapsto M_0$, $G|C=Id$ and $G(C\otimes_A B)\leqslant^* M_0$. So $G(C\otimes_A B)= C\otimes_A \tilde{B}$, where $\tilde{B}=G(B)$. Now assume that $D\subset_\omega M_0$ with $\tilde{B}\leqslant^*_i D$ and $|D-B|<m$. Hence $D\subseteq C\otimes_A \tilde{B}$ and $(D-\tilde{B})\subseteq C$. But $\delta(D-\tilde{B}/\tilde{B})=\delta(D-\tilde{B}/A)$. Thus $A\leqslant^*_i A\cup(D-\tilde{B})$ and $cl^m_M(\tilde{B})=\tilde{B}\bar{\cup} cl^m_M(g(A))$.

This lemma in fact shows that whenever $\langle K_0, \leqslant^*\rangle$ has the full amalgamation property, there is an embedding of B over A which works uniformly for each m. This suggests finding a weaker condition on $\langle K_0, \prec\rangle$ which guarantees the existence of a semigeneric model. □

Definition 4.4. $\langle K_0, \prec\rangle$ has the semifull amalgamation property, if for any $A, B, C \in K_0$ with $A \prec B$ and $A \subseteq C$, and for each $m \in \omega$, there exists an amalgam of B and C over A, say $D_m \in K_0$, with $B \subseteq D_m$, $C \prec D_m$ and $cl^m_{D_m}(B) = cl^m_G(A) \bar{\cup} B$.

Lemma 4.5. *$\langle K_0, \prec\rangle$ has the semifull amalgamation property if and only if there is a $\langle K_0, \prec\rangle$-semigeneric model.*

Proof. For left to right, we can use the usual Fräissá–Hrushovski amalgamation method to construct a semigeneric model and the proof is left to reader. The other direction is also trivial by definition of semigenericity. □

From now on we assume that $\langle K_0, \prec\rangle$ has the semifull amalgamation property.

Now we show that the class of semigeneric models can be axiomatised in $\mathscr{L}$ by an infinite set of first-order formulas. Let us first establish some notation.

Notation 4.6. 1. We write $A \prec^m_i D$ if for each $d \in D - A$ there is an E with $Ad \subseteq E$, $A \prec_i E$, and $|E - A| < m$.

2. Let $A \in K_0$, i.e. $D = cl^m_D(A)$.

$$\mathscr{D}^m_A = \{D \in K_0 \mid A \prec^m_i D\}.$$

3. For $C \in \mathscr{D}^m_A$ let $\mathscr{D}^m_{A,C}$ be the set of $D \in \mathscr{D}^m_A$ into which C can be embedded properly.

4. For $C \in \mathscr{D}^m_A$, $\theta^m_{A,C}(\bar{x}, \bar{y})$ is the formula

$$Diag_A(\bar{x}) \wedge Diag_C(\bar{y}) \wedge \bigwedge_{D \in \mathscr{D}^m_{A,C}} \neg\exists \bar{w}_D\, Diag_D(\bar{x}, \bar{y}, \bar{w}_D).$$

Then for $\bar{a}\bar{c}$ an enumeration of $C, A \subseteq C \subseteq N$, $N \models \theta^m_{A,C}(\bar{a}, \bar{c})$ if and only if $C = cl^m_N(A)$.

5. For $A \prec B$ and $C \in \mathscr{D}^m_A$, let $D^m_{A,B,C}$ be the set of $E \in \mathscr{D}^m_B$ with $C \subseteq E$ and $E = B \bar{\cup} C$.

6. Let $\phi^m_{A,B,C}$ be the sentence

$$(\forall \bar{x})(\forall \bar{y})(\exists \bar{z}) \left[Diag_A(\bar{x}) \wedge \theta^m_{A,C}(\bar{x}, \bar{y}) \rightarrow \left(Diag_B(\bar{x}, \bar{z}) \wedge \bigvee_{E \in D^m_{A,B,C}} \theta^m_{B,E}(\bar{x}, \bar{y}, \bar{z}) \right) \right].$$

7. Let T_{sem} be the set of all $\phi^m_{A,B,C}$ sentences for $m \in \omega$ and $A, B, C \in K_0$.

Remark 4.7. For any $A \subseteq B \in K_0$, if $A \prec^m_i B$ then $A \prec_i B$.

Lemma 4.8. *The structure $N \models T_0$ is semigeneric if and only if $N \models T_{\text{sem}}$.*

Proof. If N is semigeneric, then we can show that $N \models \phi^m_{A,B,C}$. Suppose $A \subset_\omega M$ and $cl^m_N(A) \cong C$. Then there is an embedding $f: B \mapsto N$ such that $cl^m_N(f(B)) = f(B) \bar{\cup} cl^m_N(A)$. Let a be an enumeration for A, b for $f(B) - A$ and d for $cl^m_N(f(B)) - f(B)$. Then

$$N \models Diag_A(a) \wedge \theta^m_{A,C}(a, b) \rightarrow \left(Diag_B(a, b) \wedge \bigvee_{E \in D^m_{A,B,C}} \theta^m_{B,E}(a, b, d) \right).$$

Hence $N \models \phi^m_{A,B,C}$. The other direction can also be proved similarly.

$T_{\rm sem}$ is a Π_3 $\mathscr{L}$-theory. Let $T^* = T_{\rm Nat} \cup T_{\rm sem}$. So for any $\mathscr{L}_+$-structure M,

$$M \models T^* \text{ if and only if } M \text{ is semigeneric and Natural.}$$

This means that we can replace any formula with the form $\exists \bar{y}(Diag_A(\bar{x}) \wedge Diag_B(\bar{x}, \bar{y}))$, where $A \prec_i B \in K_0$ by the corresponding formula $R_B(\bar{x})$. Hence T^* is a $\Pi_2 \mathscr{L}_+$-theory, i.e. it is inductive. □

In the following we establish the main theorem of this section. We show that T^*, the $\mathscr{L}_+$-theory of semigeneric models, is the theory of existentially closed models of T_+ in $\mathscr{L}_+$.

Theorem 4.9. *T^* is the model companion of $T_{\rm Nat}$.*

Proof. To prove the theorem, we first show that for each $M \models T_{\rm Nat}$, there is a model $N \models T^*$ such that $M \subset_+ N$. As $\langle K_0, \prec \rangle$ has the AC, by Lemma 3.23, this is equivalent to saying $M \prec N$. Let $Diag_\prec(M) = Diag_{\mathscr{L}}(M) \cup \{\neg(\exists \bar{y})\ Diag_E(\bar{a}, \bar{y}) \mid \bar{a} \subset M,\ E \in \mathscr{D}_A$, and there is no $\bar{b} \subset M$ with $M \models Diag_E(\bar{a}, \bar{b})\}$.

We claim that $T^* \cup Diag_\prec(M)$ is consistent. By compactness it is enough to show that for $N \models T^*, A \in K_0$ and $E_i \in \mathscr{D}_A$ with $i = 1, \ldots, n$ and $A \neq E_i$ we have

$$N \models (\exists \bar{y}) \left(Diag_A(\bar{y}) \wedge \bigwedge_{i=1}^{n} \neg \exists \bar{z}_i\, Diag_{E_i}(\bar{y}, \bar{z}_i) \right).$$

We take $m \in \omega$ such that $m > \mathrm{Max}\{|E_i - A| \mid 1 \leqslant i \leqslant n\}$. Since $\emptyset \prec A$ and N is semigeneric, there is a function $f : A \mapsto N$ such that

$$cl_N^m(f(A)) = f(A) \bar{\cup} cl^m(\emptyset) = f(A) \bar{\cup} \emptyset = f(A).$$

Hence for each $i \leqslant n$ we have $N \models \neg \exists \bar{z}_i Diag_{E_i}(f(\bar{a}), \bar{z}_i)$. Thus

$$N \models (\exists \bar{y}) \left(Diag_A(\bar{y}) \wedge \bigwedge_{i=1}^{n} \neg \exists \bar{z}_i\, Diag_{E_i}(\bar{y}, \bar{z}_i) \right).$$

Now in order to prove that T^* is a model complete theory, by Robinson test it is enough to show that for each $M \subset_+ N \models T^*$ we have $M \preccurlyeq_1 N$, that is M is a Σ_1-substructure of N. Therefore we should prove for any existential $\mathscr{L}_+$-formula $\theta(x)$ and $\bar{a} \subset M$,

$$M \models \theta(\bar{a}) \quad \text{if and only if} \quad N \models \theta(\bar{a}).$$

On the basis of Remark 3.21, every existential $\mathscr{L}_+$-formula is equivalent to a Boolean combination of formulas of the form

$$\exists \bar{y} \left(Diag_A(\bar{x}) \wedge Diag_B(\bar{x}, \bar{y}) \wedge \bigwedge_{i} \neg R_{E_i}(\bar{x}, \bar{y}) \right),$$

where $A \subseteq B \in K_0$ and $E_i \in D_B^m$ (for $i=1,\ldots,n$), and some large $m \in \omega$. So it is enough to show that for any formula $\phi(\bar{x},\bar{y})$ with that form and $\bar{a} \subset M$, if $\bar{a} \subset_\omega M$ and there is a sequence $\bar{b} \subset_\omega N$ with

$$N \models Diag_A(\bar{a}) \wedge Diag_B(\bar{a},\bar{b}) \wedge \bigwedge_i \neg R_{E_i}(\bar{a},\bar{b}),$$

then there is $\hat{b} \subset_\omega M$ with $M \models \phi(\bar{a},\hat{b})$. So in fact let $\bar{a}$ be an enumeration of A. Let $B_0 = cl_B(A) \subset cl_N(A)$. By Lemma 3.23, since $M \subset_+ N$, we have $M \prec N$. Hence $cl_N(A) = cl_M(A)$ and $B_0 \subset_\omega M$. So we may suppose that $A = B_0, A \prec B$ and $\bar{b} \cap M = \emptyset$.

Now by induction we define $I_0, I_1, \ldots, I_k, \ldots$ sets $A_0 \subseteq A_1 \cdots \subseteq A_k \subseteq \cdots \subset_\omega M$, and $B_0 \subseteq B_1 \cdots \subseteq B_k \subseteq \cdots \subset_\omega N$ as follows. Put $I_0 = \emptyset$, $A_0 = A$ and $B_0 = B$. Suppose we already defined the sequences up to k.

Let $I_{k+1} = \{i \leqslant n \mid i \notin \bigcup_{j=0}^k I_j$, and $E_i \in \mathscr{D}_{B_k}^m$ and $A_k(E_i - B_k) \in \mathscr{D}_{A_k}^m\}$. Let $i \in I_{k+1}$. Since $A_k(E_i - B_k) \in \mathscr{D}_{A_k}^m$ and $\langle K_0, \prec\rangle$ has the AC, there are only finitely many copies of $A_k(E_i - B_k)$ over A_k. So we put $A_{k+1} = A_k \cup \bigcup_{i \in I_{k+1}} \{E' \subseteq_\omega M \mid E' \cong_{A_k} (E_i - B_k)A_k\}$ and $B_{k+1} = \bar{b} \cup A_{k+1}$. Now $M \prec N$ and $B_k \subset N$. Thus by Lemma 3.9, we get $B_k \cap M \prec B_k$. Now since $\bar{b} \cap M = \emptyset$, we have $B_k \cap M = A_k$. Therefore for all $k, A_k \prec B_k$. We fix k_0 to be the first natural number for which $I_{k_0+1} = \emptyset$. Let $I' = \{1,2,\ldots,n\} - \bigcup_{j=0}^{k_0} I_j$, and, $A' = A_{k_0}$ and $B' = B_{k_0}$. Since M is semigeneric, there is a function $f : B' \mapsto \hat{B} \subset_\omega M$, such that $f|A' = Id$ and

$$cl_M^m(f(B')) = f(B') \bar{\cup} cl_M^m(A'). \qquad (*)$$

Let $\bar{b}'$ be an enumeration of B' and $\hat{b} = f(\bar{b}')$. We shall show that

$$M \models Diag_B(\bar{a},\hat{b}) \wedge \bigwedge_{i=1}^n \neg R_{E_i}(\bar{a},\hat{b}).$$

We first show that for $i \in I_0 \cup \cdots \cup I_{k_0}$ we have $M \models \neg R_{E_i}(\bar{a},\hat{b})$. To prove this, first by induction on $j < k_0$ and because A_{j+1} is definable set over A_j we can see that each A_j is the unique set which satisfies the formula $Diag_{A_j}(\bar{a},\bar{y}_j)$ over A. So each A_j is definable over A. Hence if for some $i \in I_{j+1}, M \models R_{E_i}(\bar{a},\hat{b})$, then by definition of I_{j+1} we have

$$M \models (\exists \bar{y}_j \bar{z}_i) Diag_{A_j}(\bar{a},\bar{y}_j) \wedge Diag_{B_j}(\bar{a},\hat{b},\bar{y}_j) \wedge Diag_{E_i}(\bar{a},\hat{b},\bar{y}_j,\bar{z}_i).$$

Let $\bar{c}_j$ and $\bar{e}_i$ be two tuples with

$$M \models Diag_{A_j}(\bar{a},\bar{c}_j) \wedge Diag_{B_j}(\bar{a},\hat{b},\bar{c}_j) \wedge Diag_{E_i}(\bar{a},\hat{b},\bar{y}_j,\bar{e}_i).$$

Thus $\bar{a} \cup \bar{c}_j = A_j$ and $A_j \cup \bar{e}_i \subset A_{j+1}$. But since $B' \cong_{A'} \hat{B}$, we have $\hat{b}A_{j+1} \cong_{A_{j+1}} \bar{b}A_{j+1}$. Hence we must also have

$$N \models Diag_{A_j}(\bar{a},\bar{c}_j) \wedge Diag_{B_j}(\bar{a},\bar{b},\bar{c}_j) \wedge Diag_{E_i}(\bar{a},\bar{b},\bar{y}_j,\bar{e}_i).$$

This means $N \models R_{E_i}(\bar{a},\bar{b})$, which is a contradiction.

Now for $i \in I'$, if there is $\bar{e}' \cap \hat{B} = \emptyset$ with $M \models Diag_{E_i}(\bar{a}, \hat{b}, \bar{e}')$ then by (∗) and $f(B') = \hat{B} \prec_i^m E_i$ we have $A' \cup (E_i - \hat{B}) = \bar{a} \cup \bar{e}' \subset cl_M^m(A')$. Now $i \notin \bigcup_{j \leqslant k_0} I_j$. Thus by definition of I_{k_0} we have $i \in I_{k_0+1} = \emptyset$, a contradiction. Hence $M \models \phi(\bar{a}, \hat{b})$ and the proof is complete. □

Corollary 4.10. *T^* is a complete theory.*

Proof. Since $\langle K_0, \prec \rangle$ has the JEP, $T_+ = (T_{\mathrm{Nat}})_\forall = T_\forall^*$ has the JEP and therefore, as T^* is the T_+-model companion, it is a complete theory.

Remark 4.11. Baldwin and Shelah [1] call an $\mathscr{L}$-structure $M \models T_0$ semigeneric if whenever $A \prec B \in K_0$ and $g: A \mapsto M$, then for each m there exists an embedding $\hat{g}$ of B into M which extends g such that

1. $cl_M^m(\hat{g}(B)) = \hat{g}(B) \cup cl_M^m(g(A))$.
2. $M | \hat{g}(B) \cup cl_M^m(g(A))$ is the free amalgam of $cl_M^m(g(A))$ and $\hat{g}(B)$ over $g(A)$. In particular, $cl_M^m(\hat{g}(B)) = \hat{g}(B) \bar{\cup} cl_M^m(g(A))$.

They prove that the class of semigeneric structure is first-order axiomatizable and, moreover, that this theory is near model complete, i.e. every formula is the Boolean combination of Σ_1 formulas.

We will in fact show that under the weaker notion of semigenericity, introduced here, the $\mathscr{L}$-theory of semigeneric models is near model complete. To see this, we first show that $T_+ = T_\forall^*$ has the amalgamation property. This, together with model completeness of T^*, implies that T^* has in fact elimination of quantifiers.

Corollary 4.12. *T^* has the quantifier elimination.*

Proof. Since T^* is model complete and, by Lemma 3.26 $T_+ = T_\forall^*$ has the amalgamation property, this is immediate. □

Theorem 4.13. *T_{sem}, the $\mathscr{L}$-theory of semigeneric models, is a near model complete theory.*

Proof. By Corollary 4.12, any $\mathscr{L}$-formula $\phi(\bar{x})$ is T^*-equivalent to a quantifier-free $\mathscr{L}_+$-formula, say $\theta_\phi(\bar{x})$. So there are $E_1, \ldots, E_n \in K_0$ such that

$$T_{\mathrm{sem}} \cup \{R_{E_i}(\bar{x}) \leftrightarrow (\exists \bar{y})\, Diag_{E_i}(\bar{y}, \bar{x}) \mid i = 1, \ldots, n\} \vdash \phi(\bar{x}) \leftrightarrow \theta_\phi(\bar{x}).$$

Thus after replacing each of R_{E_i}'s with the corresponding $(\exists \bar{y})\, Diag_{E_i}(\bar{y}, \bar{x})$'s in $\theta_\phi(\bar{x})$, we get an $\mathscr{L}$-formula, $\bar{\theta}_\phi(\bar{x})$, which is a Boolean combination of Σ_1 $\mathscr{L}$-formulas such that

$$T_{\mathrm{sem}} \vdash \phi(\bar{x}) \leftrightarrow \bar{\theta}_\phi(\bar{x}). \qquad \square$$

Corollary 4.14. *Let T be the $\langle C^{\geqslant 0}, \leqslant\rangle$-generic theory. Then T is a near model complete theory.*

Proof. By Lemma 3.17 $\langle C^{\geqslant 0}, \leqslant\rangle$ has the AC. Moreover, it has the full amalgamation property. Hence with the same proof as we gave for Lemma 4.3, we can see that the $\langle C^{\geqslant 0}, \leqslant\rangle$-generic structure is a semigeneric model. Hence, on account of Theorem 4.13, T is a near model complete theory. □

Using ideas from Section 3, we can show that

Theorem 4.15. *Let $\langle K_0, \prec\rangle$ be a class of finite $\mathscr{L}$-structure with the semifull amalgamation property which is closed under the ITD. Then the theory of $\langle K_0, \prec\rangle$ generic is simple.*

Proof. By Theorem 4.9 the $\mathscr{L}_+$-theory of the $T^* = \langle K_0, \prec\rangle$-generic structure is the model companion of T_+. Hence by Remark 2.18 T_+ is simple if and only if T^* is a simple first-order theory. □

5. Example of an unstable simple theory

As an application, we give an example of a simple unstable generic theory.

Assumption 5.1. We assume here that $\mathscr{L}$ consists of only a ternary relation $R(x, y, z)$ and we define a predimension on finite $\mathscr{L}$-structures by $\delta(A) = |A| - |R(A)|$. Let $C^{>0} = \{A\colon\ \emptyset \leqslant^* A\}$.

Following Hrushovski [5] we define

Definition 5.2. Let $A, B \subset M \in C^{>0}$ and $A \cap B = \emptyset$. We say that B is simply algebraic over A (in M) if $A \leqslant AB$ with $\delta(B/A) = 0$, and there is no proper nonempty subset B' of B such that $\delta(B'/A) = 0$. For $A \subseteq B$, we say that B is simply algebraic over A if $B - A$ is simply algebraic over A in B. If, moreover, A is minimal then we say B is minimally simply algebraic.

Remark 5.3. By lemma definition of $\leqslant^*$, one could see that for any $A \leqslant^* B$ and $A \leqslant B$, there are $B_0, \ldots, B_n$ with $A = B_0 \subseteq \cdots \subseteq B_n = B$ such that B_{i+1} is simply algebraic over B_i.

Now we fix a function $\mu\colon C^{>0} \times C^{>0} \mapsto \omega$ such that for any $A \subseteq B$, if B is simply algebraic over A, then $\mu(A, B) \geqslant \delta(A)$.

Definition 5.4. Let K_μ be the class of finite structures M of $\mathscr{L}$, such that

1. $\emptyset \leqslant^* M$.

2. Let $A, B_i (i=1,\dots,n)$ be pairwise disjoint subsets of $M (B_j \neq \emptyset)$. Suppose the atomic type of (A, B_i) is constant with i, and that B_i is simply algebraic over A in M. Then $n \leqslant \mu(A, AB_i)$.

Let T_μ be the K_μ-universal theory, and $(T_\mu)_+$ the corresponding T_+.

Fact 5.5 (Hrushovski [5]). *$\langle K_\mu, \leqslant\rangle$ has the amalgamation property. Moreover, the theory of generic structure is strongly minimal and near model complete.*

Lemma 5.6. *$\langle K_\mu, \leqslant^*\rangle$ has the AC.*

Proof. Let $A \leqslant_i^* B \in K_\mu$. Let $B' = cl_B(A)$. Then $A \leqslant_i B' \leqslant B$. Moreover, since $A \leqslant_i^* B$, $\delta(B/B') = 0$. By Lemma 3.17, there is $N \in \omega$ such that for all $M \in T_\mu$ we have $\chi_M^*(B'/A) \leqslant N$. So to prove that this class has the AC, we should prove that there is $N' \in \omega$ such that $\chi_M^*(B/B') \leqslant N'$. Furthermore, by Remark 5.3, it is enough to prove this when B is simply algebraic over B'. But by definition of K_μ, in this case, $\chi_M^*(B/B') \leqslant \mu(B', B)$.

Thus in order to prove the theorem by using the results from Section 3.1, we show this class has the AP (and hence the JEP) and is closed under the ITD. □

The next lemma shows that $\langle K_\mu, \leqslant^*\rangle$ has a nice amalgamation property. We need this lemma to prove first that $\langle K_\mu, \leqslant^*\rangle$ is closed under ITD, and then show the near model completeness of the generic theory.

Fact 5.7 (Hrushovski [5]). *Suppose $A, B_1, B_2 \in K_\mu$, $A = B_1 \cap B_2$, and $B_1 - A$ is simply algebraic over A in B_1. Let E be the free amalgam of B_1 and B_2 over A. Then $E \in K_\mu$, unless either*

1. *$B_1 - A$ is minimally simply algebraic over some $F \subseteq A$ and B_2 contains $\mu(F, F \cup (B_1 - A))$ disjoint sets, each realizing the atomic type of $B_1 - A$ over F;*
2. *or there exists a set $X \subseteq B_2$ such that $X \cap A \nleqslant X$ and B_1 contains an isomorphic copy of X.*

Following [18], we consider the following definition.

Definition 5.8. For $A \subseteq B$ and a natural number m we say A is m-closed in $B (A \leqslant^m B)$ if for any C with $A \subseteq C \subseteq B$ and $|C - A| < m$ we have $A \leqslant C$.

Lemma 5.9. *Let $A \leqslant^* C \in K_\mu$. Then for any $B \in K_\mu$ with $A \leqslant^{|C-A|} B$ we have $B \otimes_A C \in K_\mu$.*

Proof. We use induction on $|B - A| + |C - A|$. Let $B' = cl_B^*(A)$.

Case 1: $B' \neq A$.

In this case B' is simply algebraic over A. Now by Fact 5.7, since $A \leqslant^* C$ and $A \leqslant^{|C-A|} B'$, we have $D = B' \otimes_A C \in K_\mu$. Now $|B - B'| + |C - A| < |B - A| + |C - A|$. Thus by induction, and since $B' \leqslant B$ and $B' \leqslant^* D$, we have $B \otimes_{B'} D = B \otimes_A C \in K_\mu$.

Case 2: $B' = A$.

In this case $A \leqslant^* B$ and for all X with $A \subset X \subseteq B, \delta(X) > \delta(A) \geqslant \delta(A) + 1$. Now we pick up a point, d say, from $B - A$. Thus $\delta(x/A) = 1$. Hence there is no edge between d and A. A simple computation shows that $E = A \cup \{d\} \otimes_A C \in K_\mu$. Now for any X with $X \cap (A \cup \{d\}) = \emptyset$ with $|X| < |C - A|$, we have

$$\delta(X/A \cup \{d\}) = \delta(Xd/A) - \delta(d/A) = \delta(Xd/A) - 1 \geqslant 0.$$

Hence $A \cup \{d\} \leqslant^{|C-A|} B$. Thus, again, by induction we have $B \otimes_{A \cup \{d\}} E = B \otimes_A C \in K_\mu$. □

Corollary 5.10. *$\langle K_\mu, \leqslant^* \rangle$ has the free amalgamation.*

Proof. 1. Let $A \leqslant^* B$ and $A \leqslant^* C$. Then by Lemma 5.9 we have $B \otimes_A C \in K_\mu$. □

Lemma 5.11. *$\langle K_\mu, \leqslant^* \rangle$ is closed under the ITD.*

Proof. Let $I = \langle B, A_0, A_1, A_2, A_{01}, A_{02}, A_{12} \rangle$ be an $\langle K_\mu, \leqslant^* \rangle$ independence theorem diagram. We show that $E \cong (A_{01} \otimes_{A_1} A_{12}) \otimes_{A_0 A_2} A_{02}$ is in K_μ. Since $\langle C^{>0}, \leqslant \rangle$ has the full amalgamation property, we have $A_1 \leqslant^* A_{12}$ and $A_0 A_2 \leqslant A_{02}, (A_{01} \otimes_{A_1} A_{12})$ and, $E \cong (A_{01} \otimes_{A_1} A_{12}) \otimes_{A_0 A_2} A_{02}$ are in $C^{>0}$. On account of Lemma 3.37, $A_0 A_2 \leqslant^* A_{01} \otimes_{A_1} A_{12}$. Moreover, since I is an ITD, $A_0 A_2 \leqslant A_{02}$ implies. Thus $E \cong (A_{01} \otimes_{A_1} A_{12}) \otimes_{A_0 A_2} A_{02}$ is in K_μ by Lemma 5.9. □

Lemma 5.12. *For any $M \models T_\mu$ and $A \subset_\omega M$, the $\leqslant$-closure of A (in M), $cl(A)$, is finite.*

Proof. For any $A \subset_\omega M$, we let $B \supseteq A$ with the minimum predimension and cardinality. Then it is easy to check that $B = cl_\leqslant(A)$. □

Definition 5.13. We call a model of T_μ semigeneric, if for each $A \leqslant^* B \in K_\mu$, $g : A \mapsto M$ with $g(A) \leqslant^{|B-A|} M$ and m, there exists an embedding $\hat{g}$ of B into M which extends g such that

1. $cl_M^m(\hat{g}(B)) = \hat{g}(B) \cup cl_M^m(g(A))$,
2. $M | \hat{g}(B) \cup cl_M^m(g(A))$ is the free amalgam of $cl_M^m(g(A))$ and $\hat{g}(B)$ over $g(A)$.

Using Lemma 5.9 and the usual Fräissé–Hrushovski method, the following lemma is immediate.

Lemma 5.14. *There exists a $\langle K_\mu, \leqslant^* \rangle$-semigeneric model. In fact the $\langle K_\mu, \leqslant^* \rangle$-generic model is semigeneric.*

Let M_μ be the generic structure, T_μ^* the $\mathscr{L}_+$-first-order theory of M_μ and T_{sem} the $\mathscr{L}$-first-order theory of M_μ.

Lemma 5.15. *T^*_μ is a model complete theory. In fact, T^*_μ is the model companion of $(T_\mu)_+$ and it has quantifier elimination.*

Proof. First note that the semigenericity in this context is a first-order notion. So any model of T^*_μ is also semigeneric. Here we use the same terminology as in Theorem 4.9. For the model completeness proof we use the same idea as invoked in Theorem 4.9 and we only need to modify the definitions of A_j's (for $j \leqslant k$). Here we define

$$A_{j+1} = cl\left(\bigcup_{i\in I_j} \{E' \subset_\omega M | E' \cong_{A_j} (E_i - B_j)A_j\}\right).$$

By virtue of Lemma 5.12 all of A_j's are finite (for $j \leqslant k$) and $A_k \leqslant M$ (and hence $A \leqslant^{|B-A|} M$). The other part of the proof is identical to that of Theorem 4.9 and Corollary 4.12. □

Corollary 5.16. *T_{sem} is near model complete. Furthermore, T_{sem} is the theory of the $\langle K_\mu, \leqslant^*\rangle$-generic.*

Lemma 5.17. *For any $E \subseteq_\omega M \in \bar{K}_\mu$ we have $cl^*(E) = acl(E)$.*

Proof. Let $E \subseteq_\omega M \in \bar{K}_\mu$. Without loss of generality we may suppose M is ω-saturated in $\mathscr{L}_+$. Clearly since $\langle K_\mu, \leqslant^*\rangle$ has the AC, $cl^*(E) \subseteq acl(E)$. Now let $c \in acl(E)$. Suppose $c \notin cl^*(E)$. Now since $cl^*(cl(E)) = cl^*(E)$, then $cl(E) \leqslant^* cl(E) \cup \{c\} \in K_\mu$. But $cl(E) \leqslant M$. Let E_n be the free amalgamation of n-copies of $clE \cup \{c\}$ over $cl(E)$. Then by Corollary 5.10 we have $E_n \in K_\mu$ and $cl(E) \leqslant^* E_n$. Furthermore, since $cl(E) \leqslant M$, we have $E_n \otimes_{cl(E)} M \in \bar{K}_\mu$ and $M \leqslant^* E_n \otimes_{cl(E)} M \in \bar{K}_\mu$. Hence $M \subseteq_+ E_n \otimes_{cl(E)} M \in \bar{K}_\mu$. But by Lemma 5.15 M is existentially closed. Therefore, for each n we can embed E_n in M. But since M is ω-saturated, we can find infinitely many $d \in M$, all of which have the same type as c. Hence $c \notin acl(E)$, a contradiction. □

Theorem 5.18. *T_{sem} is an unstable supersimple theory of SU-rank 1. Furthermore, T_{sem} is not ω-categorical.*

Proof. Let $\mathfrak{C}$ be a universal model for T_{sem}. By Lemma 5.11 and since T_{sem} is near model complete T_{sem} is simple. Suppose $A \leqslant^* B \leqslant^* \mathfrak{C}$ and E are small sets. Let $\bar{a}$ be a finite tuple. Since $\langle K_\mu, \leqslant^*\rangle$ has the free amalgamation property, we have $cl(E) = acl(E)$ in $\mathfrak{C}$ and moreover,

$$\bar{a} \underset{A}{\smile\!\!\!\!\!|} B \text{ if and only if } d(\bar{a}/A) = d(\bar{a}/B). \tag{$*$}$$

But because the predimension is integer valued, there is $\bar{A} \subseteq_\omega A$ such that $d(\bar{a}/cl^*(\bar{A})) = d(\bar{a}/A)$. Hence $\bar{a} \underset{cl^*(\bar{A})}{\smile\!\!\!\!\!|} A$ and $\bar{a} \underset{\bar{A}}{\smile\!\!\!\!\!|} A$. Thus T_{sem} is supersimple. Now for any $a \in \mathfrak{C}$

$$d(a/E) = 0 \quad \text{if and only if} \quad a \in cl^*(E) = acl(E). \tag{$**$}$$

Therefore if we assume $a \underset{A}{\not\smile} E$ then $d(a/E) < d(a/A)$. But since $0 \leqslant d(a/A) \leqslant 1$, we can deduce $d(a/E) = 0$, which on account of (**) means $a \in cl^*(E) = acl(E)$. Hence T_{sem} has SU-rank 1.

Let $\phi(x, y) = (\exists z)R(x, y, z)$. To verify that T_{sem} is an unstable theory, we show $\phi(x, y)$ has the independence property. By Corollary 7.33 in [13], the independence property for $\phi(x, y)$ is equivalent to find an $\emptyset$-indiscernible $\langle b_i \mid i \in \mathbb{Z}\rangle$, and a^* such that

$$\mathfrak{C} \models \phi(a^*, b_i) \quad \text{if and only if} \quad i \geqslant 0.$$

Let $F = \{a, b, c\} \subset_\omega \mathfrak{C}$ with $R(a, b, c)$ and $F \leqslant^* \mathfrak{C}$. Thus $cl^*(\{a\}) = \{a\}, cl^*(\{b\}) = \{b\}$ and $d(a/\emptyset) = d(a/b)$. Hence by (*) we have $a \underset{\emptyset}{\smile} b$.

Let $\langle b_i \mid i \in \omega\rangle$ be an $\emptyset$-indiscernible sequence with $b_0 = b$, a discrete sub-structure of $\mathfrak{C}$. Since $a \underset{\emptyset}{\smile} b$, by definition of dividing, there is an $a^* \equiv a$ such that for each $i \in \omega, \mathfrak{C} \models (\exists z)R(a^*, b_i, z)$. We let $G = \{b_i \mid i \in \omega\}, H = \{a^*\} \cup G$ with the induced structure and $K = H \otimes_\emptyset G$. We may embed H strongly in $\mathfrak{C}$, i.e. $H \leqslant^* \mathfrak{C}(H \subset_+ \mathfrak{C})$. Then $K \models T_\mu$ and $H \leqslant^* K$. By the amalgamation property of $(T_\mu)_+$, there is a function $f : K \mapsto \mathfrak{C}$ such that $f|H = Id$. If for $i > 0 b_{-i} = f(\bar{b}_i)$, then it is easy to see that $\langle b_i \mid i \in \mathbb{Z}\rangle$ is an $\emptyset$-indiscernible and for $i < 0$ we have $\mathscr{C} \models \neg(\exists z)R(a^*, b_i, z)$. Thus $\phi(x, y)$ has the independence property.

To show that T_{sem} is not ω-categorical, we prove that there is a finite set A with an infinite algebraic closure. By induction on $n \in \omega$, we construct a sequence of structures $\langle A_n \mid n \in \omega\rangle$ in K_μ. Let $A_0 = \{x, y_0\}$, $x \neq y_0$, be the two elements discrete $\mathscr{L}$-structure. Trivially, $A_0 \in K_\mu$. Suppose $A_n = \{x, y_0, \ldots, y_n\}$ with $|A_n| = n + 1$ has been defined such that $y_{n-1} \in A_n$ is the unique element with $R(x, y_{n-1}, y_n)$. Now let $B_n = \{x, y_n, y_{n+1}\} \cong_{\{x\}} A_1$ such that $y_{n+1} \notin A_n$. Suppose $A_{n+1} \cong A_n \otimes_{\{x, y_n\}} B_n$. Then we have $\{x, y_n\} \leqslant B_n$ and there is unique copy of B_n over $\{x, y_n\}$ in A_n, namely $\{x, y_{n-1}, y_n\}$. Now on account of Fact 5.7 since $\mu(B_n/\{x, y_n\}) \geqslant \delta(\{x, y_n\}) = 2$, we have $A_{n+1} \in K_\mu$. Now by compactness we can find set $A \subset \mathfrak{C}$ such that $\{x, y_1, \ldots, y_n, \ldots\}$ and $\{x, y_1, \ldots, y_n\} \cong A_n$. But $A \subset cl^*(x, y_0) = acl(x, y_0)$. Hence $acl(x, y_0)$ is infinite and therefore T_{sem} is not ω-categorical. □

Now since this structure is a SU-rank 1 structure, the algebraic closure defines a pregeometry on $\mathfrak{C}$. The following lemma shows that this pregeometry is not homogeneous. Recall that a pregeometry (X, cl) is called to be homogeneous if for any closed set $Y \subset X$ and $a, b \in X - Y$ there is an automorphism of X (namely, a cl-preserving permutation of X) which fixes Y and $f(a) = b$.

Lemma 5.19. *$(\mathfrak{C}, acl)$ is not homogeneous.*

Proof. Let $A = \{x, y, y', z\}$ be a set with only one relation $R(x, y', z)$. $A \cong \{x, y\} \otimes_{\{x\}} \{x, y', z\}$. Hence $A \in K_\mu$. Now for the singleton $\{x\}$ we have $\{x\} \leqslant^* A$. We embed

x in $\mathfrak{C}$ so that $\{x\} \leqslant^* \mathfrak{C}$. Thus there is a function $g : A \mapsto \mathfrak{C}$ such that $g(x) = x$ and $g(A) \leqslant^* \mathfrak{C}$. We may suppose $g = Id$. Now we claim that $\mathfrak{C} \models \neg(\exists t)R(x, y, t)$. If not, then $\mathfrak{C} \models R(x, y, t)$ for $t \in \mathfrak{C}$. But $t \in cl^*(A) = A$, a contradiction. Now if the pregeometry was homogeneous then there would be a function $h : \mathfrak{C} \mapsto \mathfrak{C}$ preserving the pregeometry such that $h(x) = x$ and $h(y) = y'$. But this not possible as $cl^*(x, y') = \{x, y', z\}$ and $cl^*(x, y) = \{x, y\}$. □

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