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Discrete Mathematics 293 (2005) 19–28

DISCRETE MATHEMATICS

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Some new Z-cyclic whist tournament designs

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Received 2 July 2003; received in revised form 10 March 2004; accepted 18 August 2004

Available online 13 March 2005

Abstract

Whist tournaments on v players are known to exist for all $v \equiv 0, 1 \pmod 4$. A whist design is said to be Z-cyclic if the players are elements in $Z_m \cup \mathscr{A}$ where $m = v$, $\mathscr{A} = \emptyset$ when $v \equiv 1 \pmod 4$ and $m = v - 1$, $\mathscr{A} = \{\infty\}$ when $v \equiv 0 \pmod 4$ and the rounds of the tournament are arranged so that each round is obtained from the previous round by adding $1 \pmod m$. Despite the fact that the problem of constructing Z-cyclic whist designs has received considerable attention over the past 10–12 years there are many open questions concerning the existence of such designs. A particularly challenging situation is the case wherein 3 divides m. As far back as 1896, E.H. Moore, in his seminal work on whist tournaments, provided a construction that yields Z-cyclic whist designs on $3p + 1$ players for every prime p of the form $p = 4n + 1$. In 1992, nearly a century after the appearance of Moore's paper, the first new results in this challenging problem were obtained by the present authors. These new results were in the form of a generalization of Moore's construction to the case of $3p^n + 1$ players. Since 1992 there have been a few additional advances. Two, in particular, are of considerable interest to the present study. Ge and Zhu (Bull. Inst. Combin. Appl. 32 (2001) 53–62) obtained Z-cyclic solutions for $v = 3s + 1$ for a class of values of $s = 4k + 1$ and Finizio (Discrete Math. 279 (2004) 203–213) obtained Z-cyclic solutions for $v = 3^3 s + 1$ for the same class of s values. A complete generalization of these latter results is established here in that Z-cyclic designs are obtained for $v = 3^{2n+1}t + 1$ for all $n \geqslant 0$ and a class of $t = 4k + 1$ values that includes the class of s values of Ge and Zhu. It is also established that there exists a Z-cyclic solution when $v = 3^{2n+1}w$ for all $n \geqslant 0$ and for a class of $w = 4k + 3$ values. Several other new infinite classes of Z-cyclic whist tournaments are also obtained. Of these, two particular results are the existence of Z-cyclic whist designs for $v = 3^{2n+1} + 1$ for all $n \geqslant 0$, and for $v = 3^{2n}$ for

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doi:10.1016/j.disc.2004.08.019

all $n \geqslant 2$. Furthermore, in the former case the designs are triplewhist tournaments. Our results, as are those of the above-mentioned studies, are constructive in nature.

Keywords: Whist tournaments; Z-cyclic designs; Resolvable BIBDs; Near resolvable BIBDs; Triplewhist designs; Directedwhist designs; Z-cyclic frames

1. Introduction

A whist tournament on v players is a $(v, 4, 3)$ (near) resolvable BIBD. Each block, (a, b, c, d), of the BIBD is called a whist game and represents the fact that the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The design is subject to the (whist) conditions that every player partners every other player exactly once and opposes every other player exactly twice. A whist tournament on v players is denoted by $\mathrm{Wh}(v)$. Each (near) resolution class of the design is called a round of the tournament. It has been known since the 1970s that $\mathrm{Wh}(v)$ exist for all $v \equiv 0, 1 \pmod 4$.

Theorem 1.1 (*Anderson [4]*). *If* $v \equiv 0$ *or* $1 \pmod 4$, *then there exists a* $\mathrm{Wh}(v)$.

Although considerable progress has been made in the last decade or so, much less is known about the existence of Z-cyclic whist tournaments. A whist design is said to be Z-cyclic if the players are elements in $Z_m \cup \mathscr{A}$ where $m = v$, $\mathscr{A} = \emptyset$ when $v \equiv 1 \pmod 4$ and $m = v - 1$, $\mathscr{A} = \{\infty\}$ when $v \equiv 0 \pmod 4$, and where the rounds can be labeled, say, $R_1, R_2, \ldots$ in such a way that R_{j+1} is obtained by adding $+1 \pmod m$ to every element in R_j. When ∞ is present, one has the property that $\infty + 1 = \infty$. Thus an attractive feature of Z-cyclic whist tournaments, both from a theoretical and practical point of view, is that the entire tournament can be described by a single (near) resolution class which is typically called the initial round of the tournament. In the pursuit of Z-cyclic whist tournaments a particularly troublesome case occurs when m is divisible by 3. In 1896, in a paper now considered to be the seminal work on whist tournaments, Moore [13] obtained, among other results, Z-cyclic triplewhist tournaments (defined below) on $3p + 1$ players for all primes p of the form $p = 4n + 1$. After the appearance of Moore's paper nearly 100 years passed before additional infinite families of Z-cyclic whist tournaments were obtained in this troublesome case. The first such new result was obtained by the present authors [6] in 1992. In the years since [6] appeared, there have been some additional successes (see [35] for pertinent references). For the most part the results contained in these latter works can be shown to be included in the results of Ge and Zhu [12]. It is to be emphasized, however, that these works, including that of Ge and Zhu, rely heavily on Moore's materials. In this study we provide several new infinite families of solutions in this troublesome case. In some instances the solutions obtained are for all permissable values of v of a particular form.

In a whist game (a, b, c, d) the opponent pairs $\{a, b\}$, $\{c, d\}$ are called opponents of the first kind and the opponent pairs $\{a, d\}$, $\{b, c\}$ are called opponents of the second kind. A triplewhist tournament on v players, $\mathrm{TWh}(v)$, is a whist tournament with the property that every player opposes every other player exactly once as an opponent of the first kind and exactly once as an opponent of the second kind. One also refers to left hand opponents and right hand opponents in a whist game. These relationships are the obvious ones associated with the players seated at a table with *a* at the North position, *b* at the East position, *c* at the South position and *d* at the West position. A whist tournament is said to be a directed whist tournament on v players, DWh(v), if every player has every other player exactly once as a left-hand opponent and exactly once as a right hand opponent.

We list now some materials that support the constructions and theorems of this paper.

Definition 1.1. A homogeneous $(v, 4, 1)$-DM (i.e. difference matrix) is a $4 \times v$ array such that each row is a copy of Z_v and the set of differences of any two rows equals Z_v.

It is easy to see that if $\gcd(v, 6) = 1$ then there exists a homogeneous $(v, 4, 1)$-DM. Simply take Row i to be i times Z_v, $i = 1, 2, 3, 4$. The homogeneous difference matrices of interest for whist tournaments happen to be those for which v is odd. Thus the only odd numbers for which the existence of a homogeneous $(v, 4, 1)$-DM is in doubt are those that are divisible by 3. The next two results are often helpful.

Theorem 1.2 (*Anderson et al.* [7]). *Let* $v = 4n + 1$. *If there exists a Z-cyclic* TWh(v) *then there exists a homogeneous* $(v, 4, 1)$-DM.

Proof. Define a $4 \times v$ array (a_{ij}) where $a_{1j} = j - 1$, $a_{i1} = 0$, $i = 2, 3, 4$, $a_{2j} = a_{1j}$'s initial round partner, $a_{3j} = a_{1j}$'s initial round opponent of the first kind and $a_{4j} = a_{1j}$'s initial round opponent of the second kind. □

Theorem 1.3 (*Finizio* [8]). *Let* $v = 4n + 1$. *If there exists a Z-cyclic* DWh(v) *then there exists a homogeneous* $(v, 4, 1)$-DM.

Proof. Repeat the construction in the proof of Theorem 1.2 except replace opponent of the first (alt. second) kind by left- (alt. right-) hand opponent. □

Theorem 1.4 (*Finizio* [8]). *For each* $n \geqslant 1$ *there exists a Z-cyclic* TWh(3^{4n}) *and hence there exists a homogeneous* $(3^{4n}, 4, 1)$-DM *for all* $n \geqslant 1$.

Theorem 1.4 is also true if one replaces TWh by DWh [1]. Homogeneous $(v, 4, 1)$-DM are known to exist for $v = 15, 27, 39, 51$. The case $v = 27$ is a very recent result due to Abel and Ge [2]. The others have been known for some time.

Theorem 1.5 (*Anderson et al.* [7]). *If there exist Z-cyclic* Wh(P_i), $i = 1, 2$ *where* $P_i \equiv 1 \pmod 4$, *and if there exists a homogeneous* $(P_1, 4, 1)$-DM, *then there exists a Z-cyclic* Wh$(P_1 P_2)$. *This* Wh$(P_1 P_2)$ *is directed* (*triplewhist*) *if both* Wh(P_i) *are*.

Theorem 1.6 (*Anderson et al.* [7]). *Let* $Q > 3$, $Q \equiv 3 \pmod 4$, $P \equiv 1 \pmod 4$, *where Z-cyclic* Wh$(Q + 1)$ *and* Wh(P) *and a homogeneous* $(Q, 4, 1)$-DM *exist. Then a Z-cyclic* Wh$(QP + 1)$ *exists. Further, if the* Wh$(Q + 1)$ *and the* Wh(P) *are both triplewhist then so is the* Wh$(QP + 1)$.

Definition 1.2. A frame is a group divisible design, $\mathrm{GDD}_\lambda(X, \mathscr{G}, \mathscr{B})$ such that (1) the size of each block is the same, say k, (2) the block set can be partitioned into a family $\mathscr{F}$ of partial resolution classes and (3) each $F_i \in \mathscr{F}$ can be associated with a group $G_j \in \mathscr{G}$ so that F_i contains every point in $X \backslash G_j$ exactly once.

An excellent source of information regarding frames is the book by Furino et al. [9]. When referring to the group type of a frame the exponential notation will be used. For our purposes, if a frame has blocks of size $k=4$ then each block is considered to be a whist game. If the collection of blocks has the property that every pair of elements (players) from distinct groups appear together in exactly three blocks and within these three blocks they appear exactly once as partners then the frame is called a whist frame and is denoted by WhFrame. Each partial resolution class is then called a round of the WhFrame. If the blocks of a WhFrame satisfy any additional conditions such as every pair of players from distinct groups meet exactly once as opponents of the first kind (and, hence, exactly once as opponents of the second kind) then the notation for the frame will reflect this property. Thus one speaks of TWhFrames, DWhFrames, etc. It is also possible to define a *Z*-cyclic WhFrame [12].

Definition 1.3. Suppose $S = Z_m, m = hw$ and Z_m has a subgroup H of order h. Suppose a WhFrame(h^w) has a special round R_1, called the initial round, whose elements form a partition of $S \backslash H$ and is such that it, together with all the other rounds can be arranged in a cyclic order, say $R_1, R_2, \ldots$ so that R_{j+1} can be obtained by adding $+1$ modulo m to every element in R_j then the frame is said to be *Z*-cyclic.

Theorem 1.7. *Suppose* $v - 1 = hw$, $h = 4s + 3$ *and there exists a Z-cyclic* Wh(v) *whose initial round contains* $s+1$ *games* (a_i, b_i, c_i, d_i), $i=1, 2, \ldots, s+1$ *such that* $\{a_i, b_i, c_i, d_i : i=1, \ldots, s+1\}=\{0, w, 2w, \ldots, (h-1)w\} \cup \{\infty\}$ *then there exists a Z-cyclic* WhFrame(h^w).

Proof. In the initial round of the *Z*-cyclic Wh(v) remove the $s + 1$ games (a_i, b_i, c_i, d_i). The remaining games form the initial round for a *Z*-cyclic WhFrame(h^w) having groups $\{0, w, 2w, \ldots, (h - 1)w\} + 0, 1, 2, \ldots, w - 1$. □

Remark 1.8. It is to be noted that the statements in the next five theorems are modified versions of the corresponding theorems found in [1112]. In each of these latter references, TWhFrame appears where we have WhFrame and TWh appears where we have Wh. Careful scrutiny of the proofs of these theorems indicates that the final design inherits any property that is common to all of the input designs. Consequently not only can one obtain triplewhist results from these theorems but also directed whist results, etc. The format chosen here is intended to take advantage of this flexibility.

Theorem 1.9 (*Ge and Zhu [12]*). *Suppose there exists a Z-cyclic* WhFrame$(h^{v/h})$ *and a Z-cyclic* WhFrame$(u^{h/u})$ *then there exists a Z-cyclic* WhFrame$(u^{v/u})$.

Theorem 1.10 (*Ge and Zhu [12]*). *If there exists a Z-cyclic* WhFrame(h^w) *and if there exists a homogeneous* $(g, 4, 1)$-DM *then there exists a Z-cyclic* WhFrame$((hg)^w)$.

In Theorem 1.10 the process is known as an inflation by g.

Theorem 1.11 (*Ge and Zhu* [12]). *Suppose there exists a Z-cyclic* WhFrame(h^w) *and a Z-cyclic* Wh(h), $h \equiv 1 \pmod 4$. *Then there exists a Z-cyclic* Wh(hw).

Theorem 1.12 (*Ge and Zhu* [12]). *Suppose there exists a Z-cyclic* WhFrame(h^w) *and a Z-cyclic* Wh$(h+1)$, $h \equiv 3 \pmod 4$. *Then there exists a Z-cyclic* Wh$(hw+1)$.

Corollary 1.13. *Suppose* $s = 4t + 1$ *is such that there exists a Z-cyclic* WhFrame *of type* 3^s, *then there exists a Z-cyclic* Wh$(3s + 1)$.

Theorem 1.14 (*Ge and Ling* [11]). *Suppose there exists a Z-cyclic K*-GDD *of group type* g^n. *If there exists a Z-cyclic* WhFrame(h^k) *for each* $k \in K$, *then there exists a Z-cyclic* WhFrame$((hg)^n)$.

Two frames that are important for our constructions are presented below in Examples 1.1 and 1.2. The TWhFrame of group type 3^{81} was built using the construction associated with Theorem 1.14 [11].

Example 1.1. A *Z*-cyclic WhFrame(3^9). Groups are $\{0, 9, 18\} + 0, 1, 2, \ldots, 8$. The initial round is given by the six games: (1, 12, 2, 24), (8, 21, 19, 4), (13, 23, 16, 15), (3, 6, 5, 10), (25, 17, 11, 22), (20, 7, 26, 14).

Example 1.2. The initial round of a *Z*-cyclic TWhFrame(3^{81}) is given by the 60 games listed below. The groups are $\{0, 81, 162\} + 0, 1, \ldots, 80$.

(1, 86, 93, 26),	(2, 91, 121, 64),	(3, 99, 128, 53),
(4, 92, 106, 242),	(5, 12, 163, 188),	(6, 34, 37, 214),
(7, 102, 76, 239),	(8, 119, 143, 241),	(9, 113, 129, 68),
(10, 40, 164, 226),	(11, 25, 166, 161),	(13, 126, 61, 103),
(14, 69, 70, 236),	(15, 125, 131, 240),	(16, 117, 49, 220),
(17, 108, 57, 100),	(18, 47, 165, 215),	(19, 179, 219, 27),
(20, 33, 42, 227),	(21, 238, 169, 158),	(22, 175, 223, 45),
(23, 120, 140, 234),	(24, 41, 43, 213),	(28, 127, 75, 112),
(29, 116, 63, 228),	(30, 135, 71, 235),	(31, 190, 237, 46),
(32, 48, 171, 230),	(35, 225, 191, 147),	(36, 211, 178, 139),
(38, 62, 170, 160),	(39, 59, 185, 153),	(44, 50, 177, 159),
(51, 122, 105, 124),	(52, 115, 87, 118),	(54, 233, 192, 154),
(55, 141, 67, 137),	(56, 217, 229, 60),	(58, 198, 97, 130),
(65, 114, 101, 123),	(66, 197, 110, 144),	(72, 201, 104, 221),
(73, 216, 111, 152),	(74, 150, 95, 151),	(77, 183, 88, 157),
(78, 206, 96, 212),	(79, 200, 89, 224),	(80, 173, 85, 187),
(82, 174, 107, 167),	(83, 202, 145, 172),	(84, 209, 134, 180),
(90, 210, 149, 194),	(94, 142, 184, 207),	(98, 138, 181, 189),
(109, 156, 193, 208),	(132, 186, 205, 203),	(133, 168, 199, 196),
(136, 148, 218, 222),	(146, 182, 204, 195),	(155, 176, 232, 231).

Example 1.3. As an application of Theorem 1.12, the initial round of a *Z*-cyclic TWh(244) can be constructed by adjoining the game $(\infty, 81, 0, 162)$ to the initial round games of the

TWhFrame(3^{81}) of Example 1.2. Since $h=3$ the required TWh($h+1$) is the classic TWh(4) whose initial round is given by the single whist game $(\infty, 1, 0, 2)$.

We quote now the results of Ge and Zhu and those of Finizio. The set R is the union of three sets: (1) the set of all primes of the form $4t+1$, (2) the set of q^2 such that q is a prime with $3<q<500$ and q of the form $q=4t+3$ and (3) the set $\{21, 77, 133, 161, 781\}$.

Theorem 1.15 (*Ge and Zhu [12]*). *Let v be an arbitrary product of elements in R. Then there exists a Z-cyclic* TWh($3v+1$).

Theorem 1.16 (*Finizio [8]*). *Let v be an arbitrary product of elements in R. Then there exists a Z-cyclic* Wh(3^3v+1).

2. New Z-cyclic designs

For ease of reference the following sets are defined.

$\mathrm{DM} = \{s : \text{there exists a homogeneous } (s, 4, 1)\text{-DM}\}$,
$A = \{s = 4k+3 : \text{there exists a } Z\text{-cyclic Wh}(s+1)\}$,
$P = \{s = 4k+1 : \text{there exists a } Z\text{-cyclic Wh}(s)\}$.
$L = \{s = 4k+3 : \text{there exists a } Z\text{-cyclic Wh}(s^2)\}$,
$\mathrm{GZ} = \{s = 4k+1 : \text{there exists a } Z\text{-cyclic Wh}(3s+1)\}$,
$\mathrm{FM} = \{s = 4k+3 : \text{there exists a } Z\text{-cyclic Wh}(3s)\}$,
$\mathscr{G}\mathscr{Z} = \mathrm{DM} \cap \mathrm{GZ}$,
$\mathscr{F}\mathscr{M} = \mathrm{DM} \cap \mathrm{FM}$.

Theorem 2.1. *There exists a Z-cyclic* TWhFrame($3^{3^{4n}}$) *for all $n \geqslant 1$.*

Proof. The proof is by induction on n. For $n=1$ there is the Z-cyclic TWhFrame(3^{81}) of Example 1.2. Assume the theorem true for $n=k$ and consider the case $n=k+1$, $k\geqslant 1$. Begin with the Z-cyclic TWhFrame($3^{(81)^k}$) of the induction hypothesis and inflate this frame by 81 (see Theorem 1.4) to obtain, via Theorem 1.10, a Z-cyclic TWhFrame($(3.81)^{(81)^k}$). Consider this latter frame to have group type $h^{v/h}$ and consider the Z-cyclic TWhFrame(3^{81}) to have group type $u^{h/u}$. An application of Theorem 1.9 produces a Z-cyclic TWhFrame($3^{(81)^{k+1}}$). $\square$

Corollary 2.2. *There exists a Z-cyclic* TWh($3^{4n+1}+1$) *for all $n \geqslant 0$.*

Proof. For $n=0$ there is the classic TWh(4) (see Example 1.3). For $n\geqslant 1$ combine Remark 1.8, Theorem 2.1 and Corollary 1.13. $\square$

Corollary 2.3. *There exists a Z-cyclic* Wh($3^{4n+1}s+1$) *for all $s \in \mathscr{G}\mathscr{Z}$ and for all $n \geqslant 0$. The solution is a triplewhist design if there exists a Z-cyclic* TWh($3s+1$).

Proof. Inflate the frame of Theorem 2.1 by s and invoke Theorem 1.12. $\square$

Corollary 2.4. *There exists a Z-cyclic* $\mathrm{Wh}(3^{4n+1}w)$ *for all* $w \in \mathscr{FM}$ *and for all* $n \geqslant 0$. *The solution is a triplewhist design if there exists a* $\mathrm{TWh}(3w)$.

Proof. Inflate the frame of Theorem 2.1 by w and apply Theorem 1.11. □

Theorem 2.5. *There exists a Z-cyclic* $\mathrm{WhFrame}(3^{3^{4n+2}})$ *for all* $n \geqslant 0$.

Proof. The proof is by induction on n. For $n = 0$ there is the Z-cyclic $\mathrm{WhFrame}(3^9)$ of Example 1.1. Assume the theorem true for $n = k$ and consider $n = k + 1$, $k \geqslant 0$. Begin with the Z-cyclic $\mathrm{WhFrame}(3^{9(81)^k})$ of the induction hypothesis and inflate by 81 to obtain a Z-cyclic $\mathrm{WhFrame}((3.81)^{9(81)^k})$. Consider this latter frame to have group type $h^{v/h}$ and consider the Z-cyclic $\mathrm{WhFrame}(3^{81})$ to have group type $u^{h/u}$. An application of Theorem 1.9 yields a Z-cyclic $\mathrm{WhFrame}(3^{9(81)^{k+1}})$. □

Corollary 2.6. *There exists a Z-cyclic* $\mathrm{Wh}(3^{4n+3} + 1)$ *for all* $n \geqslant 0$.

Proof. Combine Theorem 2.5 with Theorem 1.12. □

Corollary 2.7. *There exists a Z-cyclic* $\mathrm{Wh}(3^{4n+3}s + 1)$ *for all* $s \in \mathscr{GL}$ *and for all* $n \geqslant 0$.

Proof. Inflate the frame of Theorem 2.5 by s and apply Theorem 1.12. □

Corollary 2.8. *There exists a Z-cyclic* $\mathrm{Wh}(3^{4n+3}w)$ *for all* $w \in \mathscr{FM}$ *and for all* $n \geqslant 0$.

Proof. Inflate the frame of Theorem 2.5 by w and apply Theorem 1.11. □

Theorem 2.9. *There exists a Z-cyclic* $\mathrm{Wh}(3^{2n+1} + 1)$ *for all* $n \geqslant 0$. *The solution is a triplewhist design when n is even*.

Proof. Combine Corollaries 2.2 and 2.6. □

Theorem 2.10. (a) *There exists a Z-cyclic* $\mathrm{Wh}(3^{2n+1}s + 1)$ *for all* $s \in \mathscr{GL}$ *and for all* $n \geqslant 0$.

(b) *There exists a Z-cyclic* $\mathrm{Wh}(3^{2n+1}w)$ *for all* $w \in \mathscr{FM}$ *and for all* $n \geqslant 0$.

Proof. (a) Combine Corollaries 2.3 and 2.7. (b) Combine Corollaries 2.4 and 2.8. □

Theorem 2.9 can be improved in the following manner. There is a known Z-cyclic $\mathrm{TWh}(28)$ [3], i.e. the case $n=1$ of Theorem 2.9. Thus, using the homogeneous (27, 4, 1)-DM [2] and setting $Q = 3^3$, $P = 3^{4n}$ in Theorem 1.6 one obtains the following theorem.

Theorem 2.11. *There exists a Z-cyclic* $\mathrm{TWh}(3^{2s+1} + 1)$ *for all odd values of* s.

As a consequence, Corollary 2.2 combined with Theorem 2.11 provides a proof of the following theorem.

Theorem 2.12. *There exists a Z-cyclic* $\mathrm{TWh}(3^{2n+1}+1)$ *for all* $n \geqslant 0$.

The homogeneous (27, 4, 1)-DM is also helpful in the construction of a Z-cyclic $\mathrm{Wh}(729)$, a previously unknown design.

Example 2.1. To obtain the initial round of a Z-cyclic $\mathrm{Wh}(3^6)$ inflate the frame of Example 1.1 by 27 and apply Theorem 1.11.

The Z-cyclic $\mathrm{Wh}(3^6)$ of Example 2.1 combined with Theorem 1.5 enable us to prove the existence of Z-cyclic $\mathrm{Wh}(3^{2n})$ for all $n \geqslant 2$.

Theorem 2.13. *There exists a Z-cyclic* $\mathrm{Wh}(3^{2n})$ *for all* $n \geqslant 2$.

Proof. Theorem 1.4 asserts the existence of a Z-cyclic $\mathrm{TWh}(3^{4m})$ for all $m \geqslant 1$. Hence the theorem is true for all even n. Since these latter designs are triplewhist designs it follows that there exists a homogeneous $(3^{4m}, 4, 1)$-DM for all $m \geqslant 1$. An application of Theorem 1.5 with $P_1 = 3^{4m}$ and $P_2 = 3^6$ yields a Z-cyclic $\mathrm{Wh}(3^{4m+6})$, for all $m \geqslant 1$. This establishes the theorem for all odd $n \geqslant 3$. $\square$

3. Extending the solution set of Ge and Zhu

The set R of Theorem 1.15 represents the class of $s = 4k + 1$ for which Ge and Zhu [12] obtained Z-cyclic triplewhist designs. In their paper Ge and Zhu [12] demonstrate that the existence of a (group) cyclic ordered whist tournament (see [1] for the definition) on $v = 4t + 1$ players (with the players being elements in an Abelian group G of order v) leads to a TWhFrame of type 3^v. The construction is over $Z_3 \times G$. Hence the resulting TWhFrame is not Z-cyclic unless $G = Z_v$ and $\gcd(3, v) = 1$. The theorem of Ge and Ling [11], Theorem 1.14, combined with the materials of Section 2 and difference families found in [1] enable us to extend the solution set of Ge and Zhu [12].

Theorem 3.1. *Let* $s = 4t + 1$ *be such that* (1) $s \in$ DM *and* (2) *there exists a Z-cyclic* (T)WhFrame *of type* 3^s. *Then there exists a Z-cyclic* $(T)\mathrm{Wh}(3^{4n+1}s + 1)$, *for all* $n \geqslant 0$.

Proof. Apply Corollaries 1.13 and 2.3. $\square$

The following theorem is found in [1].

Theorem 3.2. *Let* $v = 20t + 1$ *with* $t \leqslant 50$ *then there exists a* $(v, 5, 1)$-DF *over* Z_v *except, possibly, for* $v \in \{321, 501, 621, 681, 901\}$. *Additionally there is a* $(1141, 5, 1)$-DF *over* Z_{1141}.

Theorem 3.3. *Let* $S = S_1 \cup S_2$, *where* $S_1 = \{77, 161, 301, 581, 721, 961, 1141\}$ *and* $S_2 = \{141, 201, 261, 381, 441, 861, 921, 981\}$, *then each* $s \in S$ *satisfies the hypotheses of Theorem* 3.1.

Proof. For each $s \in S$ the existence of the homogeneous $(s, 4, 1)$-DM follows from the fact that a Z-cyclic $\mathrm{DWh}(s)$ exists [1]. For each $s \in S$, $s \neq 77$ there is a Z-cyclic TWhFrame of type 3^s via Theorem 1.14 noting that the required CGDD is generated from the $(s, 5, 1)$-DF of Theorem 3.2. For $s = 77$ the TWhFrame of type 3^s follows from the Ge–Zhu construction mentioned above and the fact that there exists a Z-cyclic ordered whist design on 77 players [1]. □

Corollary 3.4. *Let S be the set introduced in Theorem* 3.3 *then* $S \subset \mathscr{G}\mathscr{Z}$.

Corollary 3.5. *Let* v *denote an arbitrary product of elements in S then* $v \in \mathscr{G}\mathscr{Z}$.

Proof. The proof is much the same as the corresponding construction found in [12]. Let s_1, s_2 denote any two elements from S (it is not required that s_1 be distinct from s_2). Inflate the Z-cyclic TWhFrame of type 3^{s_1} by s_2. Considering this inflated frame to have group type $h^{v/h}$ and the uninflated frame to have group type $u^{h/u}$ an application of Theorem 1.9 produces a Z-cyclic TWhFrame of type $3^{s_1 s_2}$. Consequently, there exists a Z-cyclic TWh$(3s_1s_2 + 1)$. The existence of a homogeneous $(s_1s_2, 4, 1)$-DM follows from the fact that there exists a Z-cyclic DWh(s_1s_2) via Theorem 1.5. The theorem now follows by recursively applying this result. □

4. Additional results

The materials of Sections 1 and 2 can be utilized to obtain some new Z-cyclic results for cases in which $q|m$ where q is a prime of the form $q = 4t + 3$, $t \geqslant 1$.

Theorem 4.1. *Let* $q = 4t + 3$, $t \geqslant 1$ *be a prime. There exists a Z-cyclic* WhFrame$(q^{q^{2n}})$ *for all* $q \in L \cap A$ *and for all* $n \geqslant 1$.

Proof. For $n = 1$ one can apply Theorem 1.6 with $Q = q$, $P = q^2$ to obtain a Z-cyclic Wh$(q^3 + 1)$ whose initial round has a structure that allows for an application of Theorem 1.7, with $h = q$, and consequently the desired frame. Assume the theorem true for $n = k$ and consider the case $n = k + 1$, $k \geqslant 1$. Inflate the $q^{q^{2k}}$ frame of the induction hypothesis by q^2 to obtain a frame with group type $h^{v/h}$ with $h = q^3$, $v = q^{2k+3}$. Considering the frame of the case $n = 1$, to have group type $u^{h/u}$ with $u = q$, an application of Theorem 1.9 produces a frame with group type $q^{q^{2k+2}}$. The proof is now complete by induction. □

Corollary 4.2. *Let* $q = 4t + 3$, $t \geqslant 1$ *be a prime. There exists a Z-cyclic* Wh$(q^{2n+1}s + 1)$ *for all* $q \in L \cap A$, *for all* $s \in P \cap$ DM *and for all* $n \geqslant 0$.

Proof. Inflate the frame of Theorem 4.1 by s, and incorporating Theorem 1.6, apply Theorem 1.12. □

Corollary 4.3. *Let* $q = 4t + 3$, $t \geqslant 1$ *be a prime. Let* $q \in L \cap A$ *and let* $s = 4k + 3$ *be such that* $s \in$ DM *and* $qs \in P$ *then there exists a Z-cyclic* Wh$(q^{2n+1}s)$ *for all* $n \geqslant 0$.

Proof. Inflate the frame of Theorem 4.1 by s and apply Theorem 1.11. □

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Further reading

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