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# Mixing regular convex polytopes

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ABSTRACT

The mixing operation for abstract polytopes gives a natural way to construct a minimal common cover of two polytopes. In this paper, we apply this construction to the regular convex polytopes, determining when the mix is again a polytope, and completely determining the structure of the mix in each case.



## 1. Introduction

Abstract polytopes are combinatorial generalizations of the familiar convex polytopes and tessellations of space-forms. Many of the classical geometric constructions carry over to the abstract realm. For example, given an abstract polytope $\mathcal{P}$, we can construct the "pyramid" having $\mathcal{P}$ as a base, and this construction coincides with the usual pyramid construction whenever $\mathcal{P}$ corresponds to a convex polytope. Other constructions on abstract polytopes are new, having no basis in the classical theory. One such example is the mix of two polytopes, introduced in [5]; an analogous construction for maps and hypermaps appears in [2]. The mixing operation is an algebraic construction that finds the minimal natural cover of the automorphism group of two regular polytopes. Since there is a standard way to build a regular abstract polytope from a group, this construction gives rise to the mix of two polytopes.

By applying the mixing construction to the (abstract versions of the) regular convex polytopes, we can find the minimal regular polytopes that cover any subset of the regular convex polytopes. Our goal here is to determine their complete structure; how many faces do they have in each rank, how many flags are there, and what do their facets and vertex-figures look like? Furthermore, we wish to determine which of these new structures are polytopal.

We start by giving some background information on regular abstract polytopes in Section 2. In Section 3, we introduce the mixing operation for directly regular polytopes, and we find several criteria to determine when the mix of two polytopes is again a polytope. Finally, in Section 4, we find the full structure of the mix of any number of regular convex polytopes.

## 2. Polytopes

General background information on abstract polytopes can be found in [4, Chs. 2, 3]. Here we review the concepts essential for this paper.

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### 2.1. Definition of polytopes

Let $\mathcal{P}$ be a ranked partially ordered set whose elements will be called *faces*. The faces of $\mathcal{P}$ will range in rank from $-1$ to $n$, and a face of rank $j$ is called a *j-face*. The 0-faces, 1-faces, and $(n-1)$-faces are also called *vertices*, *edges*, and *facets*, respectively. A *flag* of $\mathcal{P}$ is a maximal chain. We say that two flags are *adjacent* (*j-adjacent*) if they differ in exactly one face (their $j$-face, respectively). If $F$ and $G$ are faces of $\mathcal{P}$ such that $F \le G$, then the *section* $G/F$ consists of those faces $H$ such that $F \le H \le G$.

We say that $\mathcal{P}$ is an *(abstract) polytope of rank n*, also called an *n-polytope*, if it satisfies the following four properties:

(a) There is a unique greatest face $F_n$ of rank $n$ and a unique least face $F_{-1}$ of rank $-1$.
(b) Each flag of $\mathcal{P}$ has $n+2$ faces.
(c) $\mathcal{P}$ is *strongly flag-connected*, meaning that if $\Phi$ and $\Psi$ are two flags of $\mathcal{P}$, then there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that for $i = 0, \ldots, k-1$, the flags $\Phi_i$ and $\Phi_{i+1}$ are adjacent, and each $\Phi_i$ contains $\Phi \cap \Psi$.
(d) (Diamond condition): Whenever $F < G$, where $F$ is a $(j-1)$-face and $G$ is a $(j+1)$-face for some $j$, then there are exactly two $j$-faces $H$ with $F < H < G$.

Note that due to the diamond condition, any flag $\Phi$ has a unique $j$-adjacent flag (denoted $\Phi^j$) for each $j = 0, 1, \ldots, n-1$.

If $F$ is a $j$-face and $G$ is a $k$-face of a polytope with $F \le G$, then the section $G/F$ is a $(k-j-1)$-polytope itself. We can identify a face $F$ with the section $F/F_{-1}$; if $F$ is a $j$-face, then $F/F_{-1}$ is a $j$-polytope. We call the section $F_n/F$ the *co-face at F*. The co-face at a vertex is also called a *vertex-figure*. The section $F_{n-1}/F_0$ of a facet over a vertex is called a *medial section*. Note that the medial section $F_{n-1}/F_0$ is both a facet of the vertex-figure $F_n/F_0$ as well as a vertex-figure of the facet $F_{n-1}/F_{-1}$.

We sometimes need to work with *pre-polytopes*, which are ranked partially ordered sets that satisfy the first, second, and fourth properties above, but not necessarily the third. In this paper, all of the pre-polytopes we encounter will be *flag-connected*, meaning that if $\Phi$ and $\Psi$ are two flags, there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that for $i = 0, \ldots, k-1$, the flags $\Phi_i$ and $\Phi_{i+1}$ are adjacent (but we do not require each flag to contain $\Phi \cap \Psi$). When working with pre-polytopes, we apply all the same terminology as with polytopes.

### 2.2. Regularity

For polytopes $\mathcal{P}$ and $\mathcal{Q}$, an *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an incidence- and rank-preserving bijection on the set of faces. An isomorphism from $\mathcal{P}$ to itself is an *automorphism* of $\mathcal{P}$. We denote the group of all automorphisms of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. There is a natural action of $\Gamma(\mathcal{P})$ on the flags of $\mathcal{P}$, and we say that $\mathcal{P}$ is *regular* if this action is transitive. For convex polytopes, this definition is equivalent to any of the usual definitions of regularity.

Given a regular polytope $\mathcal{P}$, fix a *base flag* $\Phi$. Then the automorphism group $\Gamma(\mathcal{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where $\rho_i$ maps $\Phi$ to the unique flag $\Phi^i$ that is $i$-adjacent to $\Phi$. These generators satisfy $\rho_i^2 = \epsilon$ for all $i$, and $(\rho_i\rho_j)^2 = \epsilon$ for all $i$ and $j$ such that $|i-j| \ge 2$. We say that $\mathcal{P}$ has (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ if for each $i = 1, \ldots, n-1$ the order of $\rho_{i-1}\rho_i$ is $p_i$ (with $2 \le p_i \le \infty$). We also use $\{p_1, \ldots, p_{n-1}\}$ to represent the universal regular polytope of this type, which has an automorphism group with no defining relations other than those mentioned above. Thus $\Gamma(\{p_1, \ldots, p_{n-1}\})$ is a Coxeter group, which we denote by $[p_1, \ldots, p_{n-1}]$. Whenever this universal polytope corresponds to a regular convex polytope, then the name used here is the same as the usual Schläfli symbol for that polytope (see [3]).

For $I \subseteq \{0, 1, \ldots, n-1\}$ and a group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define $\Gamma_I := \langle \rho_i \mid i \in I \rangle$. The strong flag-connectivity of polytopes induces the following *intersection property* in its automorphism group $\Gamma$:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{1}$$

In general, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a group such that each $\rho_i$ has order 2 and such that $(\rho_i\rho_j)^2 = \epsilon$ whenever $|i-j| \ge 2$, then we say that $\Gamma$ is a *string group generated by involutions* (or *sggi*). If $\Gamma$ also satisfies the intersection property given above, then we call $\Gamma$ a *string C-group*. There is a natural way of building a regular polytope $\mathcal{P}(\Gamma)$ from a string $C$-group $\Gamma$ such that $\Gamma(\mathcal{P}(\Gamma)) = \Gamma$ (see [4, Ch. 2E]). Therefore, we get a one-to-one correspondence between (isomorphism classes of) regular $n$-polytopes and string $C$-groups on $n$ specified generators.

Given a regular polytope $\mathcal{P}$ with automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define the *abstract rotations* $\sigma_i := \rho_{i-1}\rho_i$ for $i = 1, \ldots, n-1$. These elements generate the *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$, which has index at most 2. We say that $\mathcal{P}$ is *directly regular* if this index is 2. Note that the regular convex polytopes are all directly regular, and that any section of a directly regular polytope is directly regular.

The rotation subgroup of a directly regular polytope satisfies the relations

$$(\sigma_i \cdots \sigma_j)^2 = \epsilon \quad \text{for } i < j. \tag{2}$$

It also satisfies an intersection property analogous to that for the automorphism groups of regular polytopes. For $1 \le i < j \le n-1$, define $\tau_{i,j} := \sigma_i \cdots \sigma_j$. By convention, we also define $\tau_{i,i} = \sigma_i$, and for $0 \le i \le n$, we define $\tau_{0,i} = \tau_{i,n} = \epsilon$. For $I \subseteq \{0, \ldots, n-1\}$ and $\Gamma^+ := \Gamma^+(\mathcal{P})$, set

$$\Gamma_I^+ := \langle \tau_{i,j} \mid i \le j \text{ and } i-1, j \in I \rangle.$$

Then the *intersection property* for $\Gamma^+$ is given by:

$$\Gamma_I^+ \cap \Gamma_J^+ = \Gamma_{I\cap J}^+ \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{3}$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes (or flag-connected pre-polytopes) of the same rank, not necessarily regular. A function $\gamma : \mathcal{P} \to \mathcal{Q}$ is called a *covering* if it preserves incidence of faces, ranks of faces, and adjacency of flags; then $\gamma$ is necessarily surjective, by the flag-connectedness of $\mathcal{Q}$. We say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there exists a covering $\gamma : \mathcal{P} \to \mathcal{Q}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are directly regular $n$-polytopes, then their rotation groups are both quotients of

$$W^+ := \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } 1 \le i < j \le n-1 \rangle,$$

the rotation subgroup of the Coxeter group $W := [\infty, \ldots, \infty]$ with $n$ generators. Therefore there are normal subgroups $M$ and $K$ of $W^+$ such that $\Gamma^+(\mathcal{P}) = W^+/M$ and $\Gamma^+(\mathcal{Q}) = W^+/K$. Then $\mathcal{P}$ covers $\mathcal{Q}$ if and only if $M \le K$.

## 3. Mixing polytopes

In this section, we will define the mix of two groups (with specified generators), which naturally gives rise to a way to mix polytopes. The mixing operation is analogous to the join of hypermaps [2] and the parallel product of maps [8].

Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x'_1, \ldots, x'_n \rangle$ be groups with $n$ specified generators. Then the elements $z_i = (x_i, x'_i) \in \Gamma \times \Gamma'$ (for $i = 1, \ldots, n$) generate a subgroup of $\Gamma \times \Gamma'$ which we call the *mix* of $\Gamma$ and $\Gamma'$ and denote $\Gamma \diamond \Gamma'$ (see [4, Ch.7A]).

**Proposition 3.1.** *Let $\Gamma$, $\Gamma'$, and $\Gamma''$ be groups with n specified generators. Then*

(a) $\Gamma \diamond \Gamma \simeq \Gamma$
(b) $\Gamma \diamond \Gamma' \simeq \Gamma' \diamond \Gamma$
(c) $(\Gamma \diamond \Gamma') \diamond \Gamma'' \simeq \Gamma \diamond (\Gamma' \diamond \Gamma'')$.

**Proof.** In each case, the function that sends the $n$ generators of the group on the left to the $n$ generators of the group on the right (while preserving the order) is an isomorphism. □

If $\mathcal{P}$ and $\mathcal{Q}$ are directly regular $n$-polytopes, we can mix their automorphism groups or their rotation groups. The theory is essentially the same in either case, but it ends up being easier to use their rotation groups. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and $\Gamma^+(\mathcal{Q}) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$. Let $\beta_i = (\sigma_i, \sigma'_i)$ for $i = 1, \ldots, n-1$. Then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$. Note that for $i < j$, we have $(\beta_i \cdots \beta_j)^2 = \epsilon$, so that the group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ satisfies Eq. (2). In general, however, it will not have the intersection property (Eq. (3)) with respect to its generators $\beta_1, \ldots, \beta_{n-1}$. Nevertheless, it is possible to build a directly regular poset from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ using the method outlined in [6], and we denote that poset $\mathcal{P} \diamond \mathcal{Q}$ and call it the *mix* of $\mathcal{P}$ and $\mathcal{Q}$. (In fact, this poset is always a flag-connected pre-polytope.) Thus $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$. If $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ satisfies the intersection property, then $\mathcal{P} \diamond \mathcal{Q}$ is in fact a polytope.

If the type of $\mathcal{P}$ is $\{p_1, \ldots, p_{n-1}\}$ and the type of $\mathcal{Q}$ is $\{q_1, \ldots, q_{n-1}\}$, then the type of $\mathcal{P} \diamond \mathcal{Q}$ is $\{\ell_1, \ldots, \ell_{n-1}\}$, where $\ell_i$ is the least common multiple of $p_i$ and $q_i$. If the facets of $\mathcal{P}$ are $\mathcal{K}$ and the facets of $\mathcal{Q}$ are $\mathcal{K}'$, then the facets of $\mathcal{P} \diamond \mathcal{Q}$ are $\mathcal{K} \diamond \mathcal{K}'$. The vertex-figures of $\mathcal{P} \diamond \mathcal{Q}$ are obtained similarly.

The following proposition is proved in [1]:

**Proposition 3.2.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular polytopes with $\Gamma^+(\mathcal{P}) = W^+/M$ and $\Gamma^+(\mathcal{Q}) = W^+/K$. Then $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) \simeq W^+/(M \cap K)$.*

In most of the cases we encounter in this paper, $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is in fact equal to $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$. In order to determine when the mix is the entire direct product, it is useful to introduce the *comix* of two groups. If $\Gamma$ has presentation $\langle x_1, \ldots, x_n \mid R \rangle$ and $\Gamma'$ has presentation $\langle x'_1, \ldots, x'_n \mid S \rangle$, then we define the comix of $\Gamma$ and $\Gamma'$, denoted by $\Gamma \Box \Gamma'$, to be the group with presentation

$$\langle x_1, x'_1, \ldots, x_n, x'_n \mid R, S, x_1^{-1}x'_1, \ldots, x_n^{-1}x'_n \rangle.$$

Informally speaking, we can just add the relations from $\Gamma'$ to those of $\Gamma$, rewriting them to use $x_i$ in place of $x'_i$.

Just as the mix of two rotation groups has a simple description in terms of quotients of $W^+$, so does the comix of two rotation groups.

**Proposition 3.3.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular polytopes with $\Gamma^+(\mathcal{P}) = W^+/M$ and $\Gamma^+(\mathcal{Q}) = W^+/K$. Then $\Gamma^+(\mathcal{P}) \Box \Gamma^+(\mathcal{Q}) \simeq W^+/MK$.*

**Proof.** Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \mid R \rangle$, and let $\Gamma^+(\mathcal{Q}) = \langle \sigma_1, \ldots, \sigma_{n-1} \mid S \rangle$, where $R$ and $S$ are sets of relators in $W^+$. Then $M$ is the normal closure of $R$ in $W^+$ and $K$ is the normal closure of $S$ in $W^+$. We can write $\Gamma^+(\mathcal{P}) \Box \Gamma^+(\mathcal{Q}) = \langle \sigma_1, \ldots, \sigma_{n-1} \mid R \cup S \rangle$, so we want to show that $MK$ is the normal closure of $R \cup S$ in $W^+$. It is clear that $MK$ contains $R \cup S$, and since $M$ and $K$ are normal, $MK$ is normal, and so it contains the normal closure of $R \cup S$. To show that $MK$ is contained in the normal closure of $R \cup S$, it suffices to show that if $N$ is a normal subgroup of $W^+$ that contains $R \cup S$, then it must also contain $MK$. Clearly, such an $N$ must contain the normal closure $M$ of $R$ and the normal closure $K$ of $S$. Therefore, $N$ contains $MK$, as desired. □

### 3.1. Size of the mix

Now we can determine how the size of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is related to the size of $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$.

**Proposition 3.4.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite directly regular n-polytopes. Then*

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})| \cdot |\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})| = |\Gamma^+(\mathcal{P})| \cdot |\Gamma^+(\mathcal{Q})|.$$

**Proof.** Let $\Gamma^+(\mathcal{P}) = W^+/M$ and $\Gamma^+(\mathcal{Q}) = W^+/K$. Then by Proposition 3.2, $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = W^+/(M \cap K)$, and by Proposition 3.3, $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q}) = W^+/MK$. Let $\pi_1 : \Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$ and $\pi_2 : \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$ be the natural epimorphisms. Then $\ker \pi_1 \simeq M/(M \cap K)$ and $\ker \pi_2 \simeq MK/K \simeq M/(M \cap K)$. Therefore, we have that

$$|\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})| = |\Gamma^+(\mathcal{P})||\ker \pi_1| = |\Gamma^+(\mathcal{P})||\ker \pi_2| = |\Gamma^+(\mathcal{P})||\Gamma^+(\mathcal{Q})|/|\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})|,$$

and the result follows. □

The following corollary is immediate.

**Corollary 3.5.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite directly regular n-polytopes such that $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$ is trivial. Then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$. Furthermore, if $\mathcal{P}$ has g flags and $\mathcal{Q}$ has h flags, then $\mathcal{P} \diamond \mathcal{Q}$ has gh/2 flags (assuming that $\mathcal{P} \diamond \mathcal{Q}$ is a polytope).*

We conclude this section by determining some cases for which $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$ is indeed trivial.

**Proposition 3.6.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular n-polytopes. Let $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q}) = \langle \sigma_1, \dots, \sigma_{n-1}\rangle$. Suppose $\sigma_i$ is trivial for some i. If $i \geq 2$, then $\sigma_{i-1}$ has order 1 or 2, and if $i \leq n-2$, then $\sigma_{i+1}$ has order 1 or 2.*

**Proof.** If $\sigma_i = \epsilon$ for $i \geq 2$, then the relation $(\sigma_{i-1}\sigma_i)^2 = \epsilon$ reduces to $\sigma_{i-1}^2 = \epsilon$, and thus $\sigma_{i-1}$ has order 1 or 2. The proof for $\sigma_{i+1}$ is the same. □

**Corollary 3.7.** *Let $\mathcal{P}$ be a directly regular polytope of type $\{p_1, \dots, p_n\}$ and let $\mathcal{Q}$ be a directly regular polytope of type $\{q_1, \dots, q_n\}$. Suppose that $\gcd(p_i, q_i)$ is odd for all i, and that it is 1 for at least one i. Then $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$ is trivial, and thus $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.*

**Proof.** Fix a $k$ such that $\gcd(p_k, q_k) = 1$. Then $\sigma_k$ is trivial in $\Gamma^+(\mathcal{P})\Box\Gamma^+(\mathcal{Q})$. Now, by Proposition 3.6, if $k \geq 2$, then $\sigma_{k-1}$ has order 1 or 2. On the other hand, $\sigma_{k-1}$ also has order dividing $\gcd(p_{k-1}, q_{k-1})$, which is odd. Therefore, $\sigma_{k-1}$ is trivial. Proceeding in this manner, we conclude that all of the generators $\sigma_i$ for $i < k$ are trivial. Similarly, if $k \leq n-2$, then $\sigma_{k+1}$ must be trivial by the same reasoning, and we find that all the generators are trivial. □

### 3.2. Polytopality of the mix

In order for the mix of $\mathcal{P}$ and $\mathcal{Q}$ to be a polytope, the group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ must satisfy the intersection property (Eq. (3)). We naturally would like to have simple conditions that determine when this is the case. Here is one broadly applicable result:

**Proposition 3.8.** *Let $\mathcal{P}$ be a directly regular n-polytope with facets isomorphic to $\mathcal{K}$. Let $\mathcal{Q}$ be a directly regular flag-connected n-pre-polytope with facets isomorphic to $\mathcal{K}'$. If $\mathcal{K}$ covers $\mathcal{K}'$, then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal.*

**Proof.** Since $\mathcal{K}$ covers $\mathcal{K}'$, the facets of $\mathcal{P} \diamond \mathcal{Q}$ are isomorphic to $\mathcal{K}$. Therefore, the canonical projection from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$ is one-to-one on the subgroup of the facets, and by [1, Lemma 3.2], the group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ has the intersection property. Therefore, $\mathcal{P} \diamond \mathcal{Q}$ is a polytope. □

Another useful result is Theorem 9.1 from [1]; we reproduce it below.

**Proposition 3.9.** *Let $\mathcal{P}$ be a directly regular n-polytope of type $\{p_1, \dots, p_{n-1}\}$, and let $\mathcal{Q}$ be a directly regular n-polytope of type $\{q_1, \dots, q_{n-1}\}$. If $p_i$ and $q_i$ are relatively prime for each $i = 1, \dots, n-1$, then $\mathcal{P} \diamond \mathcal{Q}$ is a directly regular n-polytope of type $\{p_1q_1, \dots, p_{n-1}q_{n-1}\}$, and $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.*

In general, when we mix $\mathcal{P}$ and $\mathcal{Q}$, we have to verify the full intersection property. But as we shall see, some parts of the intersection property are automatic. Recall that for a subset $I$ of $\{0, \dots, n-1\}$ and a rotation group $\Gamma^+ = \langle \sigma_1, \dots, \sigma_{n-1}\rangle$, we define

$$\Gamma_I^+ = \langle \tau_{i,j} \mid i \leq j \text{ and } i-1, j \in I\rangle,$$

where $\tau_{i,j} = \sigma_i \cdots \sigma_j$.

**Proposition 3.10.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular n-polytopes, and let $I, J \subseteq \{0, \dots, n-1\}$. Let $\Lambda = \Gamma^+(\mathcal{P})$, $\Delta = \Gamma^+(\mathcal{Q})$, and $\Gamma^+ = \Lambda \diamond \Delta$. Then $\Gamma_I^+ \cap \Gamma_J^+ \leq \Lambda_{I\cap J} \times \Delta_{I\cap J}$. Furthermore, if $\Gamma_I^+ = \Lambda_I \times \Delta_I$ and $\Gamma_J^+ = \Lambda_J \times \Delta_J$, then $\Gamma_I^+ \cap \Gamma_J^+ = \Lambda_{I\cap J} \times \Delta_{I\cap J}$.*

**Proof.** Since $\Gamma_I^+ \leq \Lambda_I \times \Delta_I$ and $\Gamma_J^+ \leq \Lambda_J \times \Delta_J$, we have

$$\begin{aligned}\Gamma_I^+ \cap \Gamma_J^+ &\leq (\Lambda_I \times \Delta_I) \cap (\Lambda_J \times \Delta_J)\\ &= (\Lambda_I \cap \Lambda_J) \times (\Delta_I \cap \Delta_J)\\ &= \Lambda_{I\cap J} \times \Delta_{I\cap J},\end{aligned}$$

where the last line follows from the polytopality of $\mathcal{P}$ and $\mathcal{Q}$. This proves the first part. For the second part, we note that if $\Gamma_I^+ = \Lambda_I \times \Delta_I$ and $\Gamma_J^+ = \Lambda_J \times \Delta_J$, then we get equality in the first line. $\square$

**Corollary 3.11.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular $n$-polytopes, and let $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1}\rangle$. Let $1 \leq i, j \leq n-1$. Then*

$$\langle \beta_1, \ldots, \beta_i\rangle \cap \langle \beta_j, \ldots, \beta_{n-1}\rangle \leq \langle \beta_j, \ldots, \beta_i\rangle.$$

*In particular, if $j > i$ then the given intersection is trivial.*

**Proof.** The claim follows directly from Proposition 3.10 by taking $I = \{0, \ldots, i\}$ and $J = \{j-1, \ldots, n-1\}$. $\square$

**Corollary 3.12.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be directly regular polyhedra. Then $\mathcal{P} \diamond \mathcal{Q}$ is a directly regular polyhedron.*

**Proof.** In order for $\mathcal{P} \diamond \mathcal{Q}$ to be a polyhedron (and not just a pre-polyhedron), it must satisfy the intersection property. For polyhedra, the only requirement is that $\langle \beta_1\rangle \cap \langle \beta_2\rangle = \langle \epsilon\rangle$, which holds by Corollary 3.11. $\square$

Corollary 3.12 is extremely useful. In addition to telling us that the mix of any two polyhedra is a polyhedron, it makes it simpler to verify the polytopality of the mix of 4-polytopes, since the facets and vertex-figures of the mix are guaranteed to be polytopal.

We now prove some general results that work for polytopes in any rank. We start with a refinement of [6, Lemma 10].

**Proposition 3.13.** *Let $n \geq 4$, and let $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1}\rangle$ be a group satisfying Eq. (2). Suppose that both $\Gamma_0 := \langle \sigma_2, \ldots, \sigma_{n-1}\rangle$ and $\Gamma_{n-1} := \langle \sigma_1, \ldots, \sigma_{n-2}\rangle$ satisfy the intersection property, and that*

$$\langle \sigma_1, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_2, \ldots, \sigma_{n-1}\rangle = \langle \sigma_2, \ldots, \sigma_{n-2}\rangle.$$

*Then $\Gamma$ satisfies the intersection property.*

**Proof.** By [6, Lemma 10], it suffices to show that for $2 \leq i \leq n-1$ we have

$$\langle \sigma_1, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_i, \ldots, \sigma_{n-1}\rangle = \langle \sigma_i, \ldots, \sigma_{n-2}\rangle.$$

Now, we have

$$\begin{aligned}\langle \sigma_1, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_i, \ldots, \sigma_{n-1}\rangle &= \langle \sigma_1, \ldots, \sigma_{n-2}\rangle \cap (\langle \sigma_2, \ldots, \sigma_{n-1}\rangle \cap \langle \sigma_i, \ldots, \sigma_{n-1}\rangle)\\ &= (\langle \sigma_1, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_2, \ldots, \sigma_{n-1}\rangle) \cap \langle \sigma_i, \ldots, \sigma_{n-1}\rangle\\ &= \langle \sigma_2, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_i, \ldots, \sigma_{n-1}\rangle\\ &= \langle \sigma_i, \ldots, \sigma_{n-2}\rangle,\end{aligned}$$

where the last equality follows from the fact that $\Gamma_0$ satisfies the intersection property. $\square$

In the following results, we shall say that $\mathcal{P}$ has *type* $\{\mathcal{K}, \mathcal{L}\}$ if all facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$ and all vertex-figures of $\mathcal{P}$ are isomorphic to $\mathcal{L}$.

**Proposition 3.14.** *Let $\mathcal{P}$ be a directly regular $n$-polytope of type $\{\mathcal{K}, \mathcal{L}\}$ with medial sections $\mathcal{M}$, and let $\mathcal{Q}$ be a directly regular $n$-polytope of type $\{\mathcal{K}', \mathcal{L}'\}$ with medial sections $\mathcal{M}'$. Suppose that $\mathcal{K} \diamond \mathcal{K}'$ and $\mathcal{L} \diamond \mathcal{L}'$ are polytopal. If $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal.*

**Proof.** Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1}\rangle$ and let $\Gamma^+(\mathcal{Q}) = \langle \sigma_1', \ldots, \sigma_{n-1}'\rangle$. Let $\beta_i = (\sigma_i, \sigma_i')$, so that $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1}\rangle$. The facets of $\mathcal{P} \diamond \mathcal{Q}$ are $\mathcal{K} \diamond \mathcal{K}'$, and the vertex figures are $\mathcal{L} \diamond \mathcal{L}'$, both of which are polytopal. Thus, Proposition 3.13 tells us that $\mathcal{P} \diamond \mathcal{Q}$ is a polytope if $\langle \beta_1, \ldots, \beta_{n-2}\rangle \cap \langle \beta_2, \ldots, \beta_{n-1}\rangle = \langle \beta_2, \ldots, \beta_{n-2}\rangle$. From Proposition 3.10 we get that

$$\langle \beta_1, \ldots, \beta_{n-2}\rangle \cap \langle \beta_2, \ldots, \beta_{n-1}\rangle \leq \langle \sigma_2, \ldots, \sigma_{n-2}\rangle \times \langle \sigma_2', \ldots, \sigma_{n-2}'\rangle.$$

The right hand side is just $\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, and since this is equal to $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \langle \beta_2, \ldots, \beta_{n-2}\rangle$, the result follows. $\square$

**Theorem 3.15.** *Let $\mathcal{P}$ be a directly regular $n$-polytope of type $\{\mathcal{K}, \mathcal{L}\}$ with medial sections $\mathcal{M}$, and let $\mathcal{Q}$ be a directly regular $n$-polytope of type $\{\mathcal{K}', \mathcal{L}'\}$ with medial sections $\mathcal{M}'$. Suppose that $\mathcal{K} \diamond \mathcal{K}'$ and $\mathcal{L} \diamond \mathcal{L}'$ are polytopal, and suppose that $\Gamma^+(\mathcal{K} \diamond \mathcal{K}') = \Gamma^+(\mathcal{K}) \times \Gamma^+(\mathcal{K}')$ and that $\Gamma^+(\mathcal{L} \diamond \mathcal{L}') = \Gamma^+(\mathcal{L}) \times \Gamma^+(\mathcal{L}')$. Then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal if and only if $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$.*

**Table 1**
The mix of regular convex polyhedra.

| Polyhedron | $f_0$ | $f_1$ | $f_2$ | $g$ |
|---|---|---|---|---|
| $\{3, 3\} \diamond \{3, 4\}$ | 24 | 144 | 96 | 576 |
| $\{3, 3\} \diamond \{3, 5\}$ | 48 | 360 | 240 | 1 440 |
| $\{3, 3\} \diamond \{4, 3\}$ | 96 | 144 | 24 | 576 |
| $\{3, 3\} \diamond \{5, 3\}$ | 240 | 360 | 48 | 1 440 |
| $\{3, 4\} \diamond \{3, 5\}$ | 72 | 720 | 480 | 2 880 |
| $\{3, 4\} \diamond \{4, 3\}$ | 48 | 288 | 48 | 1 152 |
| $\{3, 4\} \diamond \{5, 3\}$ | 120 | 720 | 96 | 2 880 |
| $\{3, 5\} \diamond \{4, 3\}$ | 96 | 720 | 120 | 2 880 |
| $\{3, 5\} \diamond \{5, 3\}$ | 240 | 1 800 | 240 | 7 200 |
| $\{4, 3\} \diamond \{5, 3\}$ | 480 | 720 | 72 | 2 880 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{3, 5\}$ | 288 | 8 640 | 5 760 | 34 560 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{4, 3\}$ | 576 | 3 456 | 576 | 13 824 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{5, 3\}$ | 1 440 | 8 640 | 1 152 | 34 560 |
| $\{3, 3\} \diamond \{3, 5\} \diamond \{4, 3\}$ | 1 152 | 8 640 | 1 440 | 34 560 |
| $\{3, 3\} \diamond \{3, 5\} \diamond \{5, 3\}$ | 2 880 | 21 600 | 2 880 | 86 400 |
| $\{3, 3\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 5 760 | 8 640 | 288 | 34 560 |
| $\{3, 4\} \diamond \{3, 5\} \diamond \{4, 3\}$ | 576 | 17 280 | 2 880 | 69 120 |
| $\{3, 4\} \diamond \{3, 5\} \diamond \{5, 3\}$ | 1 440 | 43 200 | 5 760 | 172 800 |
| $\{3, 4\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 2 880 | 17 280 | 576 | 69 120 |
| $\{3, 5\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 5 760 | 43 200 | 1 440 | 172 800 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{3, 5\} \diamond \{4, 3\}$ | 6 912 | 207 360 | 34 560 | 829 440 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{3, 5\} \diamond \{5, 3\}$ | 17 280 | 518 400 | 69 120 | 2 073 600 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 34 560 | 207 360 | 6 912 | 829 440 |
| $\{3, 3\} \diamond \{3, 5\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 69 120 | 518 400 | 17 280 | 2 073 600 |
| $\{3, 4\} \diamond \{3, 5\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 34 560 | 1 036 800 | 34 560 | 4 147 200 |
| $\{3, 3\} \diamond \{3, 4\} \diamond \{3, 5\} \diamond \{4, 3\} \diamond \{5, 3\}$ | 414 720 | 12 441 600 | 414 720 | 49 766 400 |

**Proof.** Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and let $\Gamma^+(\mathcal{Q}) = \langle \sigma_1', \ldots, \sigma_{n-1}' \rangle$. Let $\beta_i = (\sigma_i, \sigma_i')$, so that $\Gamma^+(\mathcal{P} \diamond \mathcal{Q}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$. Proposition 3.14 proves that if $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, then $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. Conversely, suppose $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. Since $\mathcal{P} \diamond \mathcal{Q}$ is polytopal, we have that

$$\langle \beta_2, \ldots, \beta_{n-2} \rangle = \langle \beta_1, \ldots, \beta_{n-2} \rangle \cap \langle \beta_2, \ldots, \beta_{n-1} \rangle.$$

Now, the right-hand side is $\Gamma^+(\mathcal{K} \diamond \mathcal{K}') \cap \Gamma^+(\mathcal{L} \diamond \mathcal{L}')$. Since $\Gamma^+(\mathcal{K} \diamond \mathcal{K}') = \Gamma^+(\mathcal{K}) \times \Gamma^+(\mathcal{K}')$ and $\Gamma^+(\mathcal{L} \diamond \mathcal{L}') = \Gamma^+(\mathcal{L}) \times \Gamma^+(\mathcal{L}')$, we can apply Proposition 3.10 to see that the right hand side is equal to $\langle \sigma_2, \ldots, \sigma_{n-2} \rangle \times \langle \sigma_2', \ldots \sigma_{n-2}' \rangle$, which is $\Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$. Since the left-hand side is equal to $\Gamma^+(\mathcal{M} \diamond \mathcal{M}')$, we get that $\Gamma^+(\mathcal{M} \diamond \mathcal{M}') = \Gamma^+(\mathcal{M}) \times \Gamma^+(\mathcal{M}')$, as desired. $\square$

## 4. The mix of the regular convex polytopes

Now we will actually mix the regular convex polytopes. In each case, we will determine the number of flags, the number of faces of each rank, and whether the mix is polytopal. The Schläfli type of the mix is easily obtained by taking the least common multiple of the corresponding entries in all the component polytopes. In most cases, the results have been found in two ways: by judicious use of the preceding results, and by direct calculation using GAP [7].

### *4.1. Rank 3*

The five regular convex polyhedra are the tetrahedron $\{3, 3\}$, the octahedron $\{3, 4\}$, the icosahedron $\{3, 5\}$, the cube $\{4, 3\}$, and the dodecahedron $\{5, 3\}$. By Corollary 3.12, the mix of any number of these is a polyhedron (i.e., polytopal). For the mix of two regular convex polyhedra $\mathcal{P}$ and $\mathcal{Q}$, Corollary 3.7 shows that in every case, we get $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = \Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$. By carefully grouping polytopes, we can use Corollary 3.7 for the mix of three or more regular convex polyhedra as well. For example, since $\{3, 3\} \diamond \{3, 4\}$ is of type $\{3, 12\}$, we can apply Corollary 3.7 to $(\{3, 3\} \diamond \{3, 4\}) \diamond \{3, 5\}$.

Due to the above considerations, we can always apply Corollary 3.5 to find the number of flags in the mix. To find the remaining information, we note that if $\mathcal{P}$ is a regular polyhedron of type $\{p, q\}$ with $g$ flags, then $\mathcal{P}$ has $g/(2q)$ vertices, $g/4$ edges, and $g/(2p)$ facets.

Information about the mix of the regular convex polyhedra is summarized in Table 1, where $f_0$ is the number of vertices, $f_1$ is the number of edges, $f_2$ is the number of 2-faces, and $g$ is the size of the automorphism group (which is also the number of flags). Since $\mathcal{P} \diamond \mathcal{P} = \mathcal{P}$ for any polytope $\mathcal{P}$, there are only finitely many mixes of the regular convex polyhedra. Note that whenever one of the mixes has the same number of vertices and facets, the polytope is in fact self-dual.

### 4.2. Rank 4

Next we present the mix of the regular convex 4-polytopes. There are six regular convex polytopes: $\mathcal{P}_1 = \{3, 3, 3\}$, $\mathcal{P}_2 = \{3, 3, 4\}$, $\mathcal{P}_3 = \{3, 3, 5\}$, $\mathcal{P}_4 = \{3, 4, 3\}$, $\mathcal{P}_5 = \{4, 3, 3\}$, and $\mathcal{P}_6 = \{5, 3, 3\}$. For $I = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, 6\}$ we define $\mathcal{P}_I = \mathcal{P}_{i_1} \diamond \cdots \diamond \mathcal{P}_{i_k}$. As was the case in rank 3, the mix of the rotation groups always turns out to be the direct product of the components. Finding the number of flags of the mix is then simple. To find the number $f_3$ of cells (i.e. facets) of the mix, we divide the number of flags by the number of flags in the facet, which we find by using Table 1. Similar calculations give the number of vertices, edges, and 2-faces.

Now we address the concern of polytopality. Unlike the mix of regular convex polyhedra, there are mixes of regular convex 4-polytopes that are not polytopal. The smallest such example (by group order) is $\{4, 3, 3\} \diamond \{3, 3, 4\}$. For the mix of any two regular convex 4-polytopes, we can appeal directly to Proposition 3.8 or Theorem 3.15 to determine polytopality, and this settles every case. For the mix of three or more regular convex 4-polytopes, we can always group them in such a way as to again apply one of these results. For example, the vertex-figures of $\{3, 3, 3\} \diamond \{3, 3, 4\}$ are $\{3, 3\} \diamond \{3, 4\}$, and so they cover $\{3, 3\}$. Therefore, the mix $\{3, 3, 3\} \diamond \{3, 3, 4\} \diamond \{4, 3, 3\}$ is polytopal by Proposition 3.8. Table 2 summarize our results.

### 4.3. Ranks 5 and higher

In ranks 5 and higher, the only regular convex polytopes are the $n$-simplex $T^n := \{3^{n-1}\}$, the $n$-cube $B^n := \{4, 3^{n-2}\}$, and the $n$-cross-polytope $C^n := \{3^{n-2}, 4\}$. We will determine the group of their mix and which of the mixes are polytopal, as well as the number of faces in each rank.

**Theorem 4.1.** *For $n \geq 3$, we have*

$$\Gamma^+(T^n \diamond B^n) = \Gamma^+(T^n) \times \Gamma^+(B^n)$$
$$\Gamma^+(T^n \diamond C^n) = \Gamma^+(T^n) \times \Gamma^+(C^n)$$
$$\Gamma^+(B^n \diamond C^n) = \Gamma^+(B^n) \times \Gamma^+(C^n)$$
$$\Gamma^+(T^n \diamond B^n \diamond C^n) = \Gamma^+(T^n) \times \Gamma^+(B^n) \times \Gamma^+(C^n).$$

**Proof.** The first three follow immediately from Corollary 3.7. For the last one, we note that $T^n \diamond B^n$ is of type $\{12, 3^{n-2}\}$ while $C^n$ is of type $\{3^{n-2}, 4\}$, so Corollary 3.7 applies again. □

**Theorem 4.2.** *For $n \geq 4$, $T^n \diamond B^n$, $T^n \diamond C^n$, and $T^n \diamond B^n \diamond C^n$ are all polytopal, and $B^n \diamond C^n$ is not.*

**Proof.** Since $T^n$ has the same vertex-figures as $B^n$, and the same facets as $C^n$, then the mix with either of these polytopes is polytopal by Proposition 3.8. Now, consider the facets of $T^n \diamond B^n$. These facets are of type $T^{n-1} \diamond B^{n-1}$; in particular, they cover $T^{n-1}$. Therefore, the facets of $T^n \diamond B^n$ cover the facets of $C^n$, and thus the mix $T^n \diamond B^n \diamond C^n$ is polytopal. For the final case, we note that the facets of $B^n \diamond C^n$ are $B^{n-1} \diamond T^{n-1}$ and that the vertex-figures are $T^{n-1} \diamond C^{n-1}$. By Theorem 4.1, we see that the facets and the vertex-figures are both direct products of their components. Then by Theorem 3.15, we see that $B^n \diamond C^n$ is polytopal if and only if the medial sections also mix as the direct product. But the medial sections of $B^n$ and of $C^n$ are both $T^{n-2}$, and thus the mix of the medial sections is also $T^{n-2}$. Therefore, $B^n \diamond C^n$ is not polytopal. □

Now we calculate the number $g$ of flags and the numbers $f_k$ of $k$-faces of each of the mixes. Table 3 summarizes this information for $T^n$, $B^n$, and $C^n$.

Let us calculate the number of flags in each mix. For $n = 1$, all of the mixes are just segments, with 2 flags. For $n = 2$, we get that $T^n \diamond B^n = T^n \diamond C^n = T^n \diamond B^n \diamond C^n = \{12\}$, with 24 flags, while $B^n \diamond C^n = B^n$, with 8 flags. Now, for $n \geq 3$, Theorem 4.1 tells us that the mixes are all full direct products of the component groups. Then by Corollary 3.5, we see that $T^n \diamond B^n$ and $T^n \diamond C^n$ both have $2^{n-1}n!(n+1)!$ flags, that $B^n \diamond C^n$ has $2^{2n-1}(n!)^2$ flags, and that $T^n \diamond B^n \diamond C^n$ has $2^{2n-2}(n!)^2(n+1)!$ flags.

For any polytope $\mathcal{P}$, let $g(\mathcal{P})$ be the number of flags of $\mathcal{P}$, and let $f_k(\mathcal{P})$ be the number of $k$-faces of $\mathcal{P}$. Then

$$g(\mathcal{P}) = f_k(\mathcal{P})g(F_k/F_{-1})g(F_n/F_k).$$

We can use this to calculate $f_k(\mathcal{P})$. First, in the case $T^n \diamond B^n$, we see that the $k$-faces are $T^k \diamond B^k$, and the co-$k$-faces are $T^{n-1-k} \diamond T^{n-1-k} = T^{n-1-k}$. Thus we have

$$f_k(T^n \diamond B^n) = \frac{2^{n-1}n!(n+1)!}{g(T^k \diamond B^k)g(T^{n-1-k})}.$$

Now, for $k = 0$, the mix $T^k \diamond B^k$ has a single flag; for $k \geq 1$, it has $2^{k-1}k!(k+1)!$ flags. So we see that the number of vertices of $T^n \diamond B^n$ is

$$f_0(T^n \diamond B^n) = \frac{2^{n-1}n!(n+1)!}{(1)(n!)} = 2^{n-1}(n+1)!,$$

**Table 2**
The mix of regular convex 4-polytopes.

| $I$ | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $g$ | Polytopal? |
|---|---|---|---|---|---|---|
| {1, 2} | 40 | 480 | 1 920 | 960 | 23 040 | Y |
| {1, 3} | 600 | 14 400 | 72 000 | 36 000 | 864 000 | Y |
| {1, 4} | 120 | 5 760 | 5 760 | 120 | 69 120 | Y |
| {1, 5} | 960 | 1 920 | 480 | 40 | 23 040 | Y |
| {1, 6} | 36 000 | 72 000 | 14 400 | 600 | 864 000 | Y |
| {2, 3} | 960 | 34 560 | 230 400 | 115 200 | 2 764 800 | Y |
| {2, 4} | 192 | 4 608 | 18 432 | 384 | 221 184 | Y |
| {2, 5} | 128 | 1 536 | 1 536 | 128 | 73 728 | N |
| {2, 6} | 4 800 | 57 600 | 46 080 | 1 920 | 2 764 800 | N |
| {3, 4} | 2 880 | 138 240 | 691 200 | 14 400 | 8 294 400 | Y |
| {3, 5} | 1 920 | 46 080 | 57 600 | 4 800 | 2 764 800 | N |
| {3, 6} | 72 000 | 1 728 000 | 1 728 000 | 72 000 | 103 680 000 | N |
| {4, 5} | 384 | 18 432 | 4 608 | 192 | 221 184 | Y |
| {4, 6} | 14 400 | 691 200 | 138 240 | 2 880 | 8 294 400 | Y |
| {5, 6} | 115 200 | 230 400 | 34 560 | 960 | 2 764 800 | Y |
| {1, 2, 3} | 4 800 | 691 200 | 13 824 000 | 6 912 000 | 165 888 000 | Y |
| {1, 2, 4} | 960 | 276 480 | 1 105 920 | 23 040 | 13 271 040 | Y |
| {1, 2, 5} | 7 680 | 92 160 | 92 160 | 7 680 | 4 423 680 | Y |
| {1, 2, 6} | 288 000 | 3 456 000 | 2 764 800 | 115 200 | 165 888 000 | Y |
| {1, 3, 4} | 14 400 | 8 294 400 | 41 472 000 | 864 000 | 497 664 000 | Y |
| {1, 3, 5} | 115 200 | 2 764 800 | 3 456 000 | 288 000 | 165 888 000 | Y |
| {1, 3, 6} | 4 320 000 | 103 680 000 | 103 680 000 | 4 320 000 | 6 220 800 000 | Y |
| {1, 4, 5} | 23 040 | 1 105 920 | 276 480 | 960 | 13 271 040 | Y |
| {1, 4, 6} | 864 000 | 41 472 000 | 8 294 400 | 14 400 | 497 664 000 | Y |
| {1, 5, 6} | 6 912 000 | 13 824 000 | 691 200 | 4 800 | 165 888 000 | Y |
| {2, 3, 4} | 23 040 | 6 635 520 | 132 710 400 | 2 764 800 | 1 592 524 800 | Y |
| {2, 3, 5} | 15 360 | 2 211 840 | 11 059 200 | 921 600 | 530 841 600 | N |
| {2, 3, 6} | 576 000 | 82 944 000 | 331 776 000 | 13 824 000 | 19 906 560 000 | N |
| {2, 4, 5} | 3 072 | 884 736 | 884 736 | 3 072 | 42 467 328 | N |
| {2, 4, 6} | 115 200 | 33 177 600 | 26 542 080 | 46 080 | 1 592 524 800 | N |
| {2, 5, 6} | 921 600 | 11 059 200 | 2 211 840 | 15 360 | 530 841 600 | N |
| {3, 4, 5} | 46 080 | 26 542 080 | 33 177 600 | 115 200 | 1 592 524 800 | N |
| {3, 4, 6} | 1 728 000 | 995 328 000 | 995 328 000 | 1 728 000 | 59 719 680 000 | N |
| {3, 5, 6} | 13 824 000 | 331 776 000 | 82 944 000 | 576 000 | 19 906 560 000 | N |
| {4, 5, 6} | 2 764 800 | 132 710 400 | 6 635 520 | 23 040 | 1 592 524 800 | Y |
| {1, 2, 3, 4} | 115 200 | 398 131 200 | 7 962 624 000 | 165 888 000 | 95 551 488 000 | Y |
| {1, 2, 3, 5} | 921 600 | 132 710 400 | 663 552 000 | 55 296 000 | 31 850 496 000 | Y |
| {1, 2, 3, 6} | 34 560 000 | 4 976 640 000 | 19 906 560 000 | 829 440 000 | 1 194 393 600 000 | Y |
| {1, 2, 4, 5} | 184 320 | 53 084 160 | 53 084 160 | 184 320 | 2 548 039 680 | Y |
| {1, 2, 4, 6} | 6 912 000 | 1 990 656 000 | 1 592 524 800 | 27 648 000 | 95 551 488 000 | Y |
| {1, 2, 5, 6} | 55 296 000 | 663 552 000 | 132 710 400 | 921 600 | 31 850 496 000 | Y |
| {1, 3, 4, 5} | 2 764 800 | 1 592 524 800 | 1 990 656 000 | 6 912 000 | 95 551 488 000 | Y |
| {1, 3, 4, 6} | 103 680 000 | 59 719 680 000 | 59 719 680 000 | 103 680 000 | 3 583 180 800 000 | Y |
| {1, 3, 5, 6} | 829 440 000 | 19 906 560 000 | 4 976 640 000 | 34 560 000 | 1 194 393 600 000 | Y |
| {1, 4, 5, 6} | 165 888 000 | 7 962 624 000 | 398 131 200 | 115 200 | 95 551 488 000 | Y |
| {2, 3, 4, 5} | 368 640 | 1 274 019 840 | 6 370 099 200 | 22 118 400 | 305 764 761 600 | N |
| {2, 3, 4, 6} | 13 824 000 | 47 775 744 000 | 191 102 976 000 | 331 776 000 | 11 466 178 560 000 | N |
| {2, 3, 5, 6} | 110 592 000 | 15 925 248 000 | 15 925 248 000 | 110 592 000 | 3 822 059 520 000 | N |
| {2, 4, 5, 6} | 22 118 400 | 6 370 099 200 | 1 274 019 840 | 368 640 | 305 764 761 600 | N |
| {3, 4, 5, 6} | 331 776 000 | 191 102 976 000 | 47 775 744 000 | 13 824 000 | 11 466 178 560 000 | N |
| {1, 2, 3, 4, 5} | 22 118 400 | 76 441 190 400 | 382 205 952 000 | 1 327 104 000 | 18 345 885 696 000 | Y |
| {1, 2, 3, 4, 6} | 829 440 000 | 2 866 544 640 000 | 11 466 178 560 000 | 19 906 560 000 | 687 970 713 600 000 | Y |
| {1, 2, 3, 5, 6} | 6 635 520 000 | 955 514 880 000 | 955 514 880 000 | 6 635 520 000 | 229 323 571 200 000 | Y |
| {1, 2, 4, 5, 6} | 1 327 104 000 | 382 205 952 000 | 76 441 190 400 | 22 118 400 | 18 345 885 696 000 | Y |
| {1, 3, 4, 5, 6} | 19 906 560 000 | 11 466 178 560 000 | 2 866 544 640 000 | 829 440 000 | 687 970 713 600 000 | Y |
| {2, 3, 4, 5, 6} | 2 654 208 000 | 9 172 942 848 000 | 9 172 942 848 000 | 2 654 208 000 | 2 201 506 283 520 000 | N |
| {1, 2, 3, 4, 5, 6} | 159 252 480 000 | 550 376 570 880 000 | 550 376 570 880 000 | 159 252 480 000 | 132 090 377 011 200 000 | Y |

**Table 3**
The regular convex $n$-polytopes for $n \geq 5$.

| $\mathcal{P}$ | $f_k$ | $g$ |
|---|---|---|
| $T^n$ | $\binom{n+1}{k+1}$ | $(n+1)!$ |
| $B^n$ | $2^{n-k}\binom{n}{k}$ | $2^n n!$ |
| $C^n$ | $2^{k+1}\binom{n}{k+1}$ | $2^n n!$ |

**Table 4**
The mix of the regular convex $n$-polytopes for $n \geq 5$.

| Mix | $f_0$ | $f_{n-1}$ | $f_k(1 \leq k \leq n-2)$ | $g$ |
|---|---|---|---|---|
| $T^n \diamond B^n$ | $2^{n-1}(n+1)!$ | $2n(n+1)$ | $2^{n-k}(n-k)!\binom{n+1}{k+1}\binom{n}{k}$ | $2^{n-1}n!(n+1)!$ |
| $T^n \diamond C^n$ | $2n(n+1)$ | $2^{n-1}(n+1)!$ | $2^{k+1}(k+1)!\binom{n+1}{k+1}\binom{n}{k+1}$ | $2^{n-1}n!(n+1)!$ |
| $B^n \diamond C^n$ | $2^{n+1}n$ | $2^{n+1}n$ | $2^{n+2}\binom{n}{k}\binom{n}{k+1}$ | $2^{2n-1}(n!)^2$ |
| $T^n \diamond B^n \diamond C^n$ | $2^n n(n+1)!$ | $2^n n(n+1)!$ | $2^{n+1}(n+1)!\binom{n}{k}\binom{n}{k+1}$ | $2^{2n-2}(n!)^2(n+1)!$ |

and that the number of $k$-faces for $k \geq 1$ is

$$f_k(T^n \diamond B^n) = \frac{2^{n-1}n!(n+1)!}{2^{k-1}k!(k+1)!(n-k)!} = 2^{n-k}(n-k)!\binom{n+1}{k+1}\binom{n}{k}.$$

Since $T^n \diamond C^n$ is dual to this, we get that the number of facets is $2^{n-1}(n+1)!$, and the number of $k$-faces for $k \leq n-2$ is $2^{k+1}(k+1)!\binom{n+1}{k+1}\binom{n}{k+1}$.

Moving on to $B^n \diamond C^n$, we have that the $k$-faces are $B^k \diamond T^k$, and the co-$k$-faces are $T^{n-1-k} \diamond C^{n-1-k}$. The number of vertices is equal to the number of facets, which is equal to

$$\frac{2^{2n-1}(n!)^2}{(1)2^{n-2}(n-1)!n!} = 2^{n+1}n.$$

The number of $k$-faces for $1 \leq k \leq n-2$ is

$$\frac{2^{2n-1}(n!)^2}{2^{k-1}k!(k+1)!2^{n-k-2}(n-k-1)!(n-k)!} = 2^{n+2}\binom{n}{k}\binom{n}{k+1}.$$

Finally, we consider the mix $T^n \diamond B^n \diamond C^n$. The $k$-faces are $T^k \diamond B^k$ and the co-$k$-faces are $T^{n-k-1} \diamond C^{n-k-1}$. The number of vertices is equal to the number of facets, which is equal to

$$\frac{2^{2n-2}(n!)^2(n+1)!}{(1)2^{n-2}(n-1)!n!} = 2^n n(n+1)!.$$

The number of $k$-faces for $1 \leq k \leq n-2$ is

$$\frac{2^{2n-2}(n!)^2(n+1)!}{2^{k-1}k!(k+1)!2^{n-k-2}(n-k-1)!(n-k)!} = 2^{n+1}(n+1)!\binom{n}{k}\binom{n}{k+1}.$$

We summarize these results in Table 4.

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