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Weakly open sets in the unit ball of some Banach spaces and the centralizer ☆

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Abstract

We show that every Banach space X whose centralizer is infinite-dimensional satisfies that every non-empty weakly open set in B_Y has diameter 2, where $Y = \widehat{\bigotimes}_{N,s,\pi} X$ (N-fold symmetric projective tensor product of X, endowed with the symmetric projective norm), for every natural number N. We provide examples where the above conclusion holds that includes some spaces of operators and infinite-dimensional C^*-algebras. We also prove that every non-empty weak* open set in the unit ball of the space of N-homogeneous and integral polynomials on X has diameter two, for every natural number N, whenever the Cunningham algebra of X is infinite-dimensional. Here we consider the space of N-homogeneous integral polynomials as the dual of the space $\widehat{\bigotimes}_{N,s,\varepsilon} X$ (N-fold symmetric injective tensor product of X, endowed with the symmetric injective norm). For instance, every infinite-dimensional $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional. We obtain the same result for every non-reflexive L-embedded space, and so for every predual of an infinite-dimensional von Neumann algebra.

Keywords: Banach space; Weakly open set; Symmetric projective tensor product; Symmetric injective tensor product; Homogeneous polynomial; Integral polynomial; Centralizer; Cunningham algebra

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1. Introduction

It is known that a Banach space without the Radon–Nikodým property satisfies that there is a bounded, closed and convex subset such that it does not have slices of small diameter. Several authors proved that for some classical Banach spaces every slice of the unit ball has diameter two. One of the first results along this line is valid for any infinite-dimensional uniform algebra [21]. Spaces with the Daugavet property satisfy that every weak neighborhood of the unit ball has diameter two [24]. For the vector valued spaces $C(K,X)$ and $L_1(\mu,X)$ this phenomenon was characterized in [6]. The interpolation spaces $L_1(\mathbb{R}^+)+L_\infty(\mathbb{R}^+)$ (endowed with two natural norms) and $L_1(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+)$ (endowed with the maximum norm) also satisfy the above property [3]. The class of infinite-dimensional C^*-algebras satisfies that every slice of the unit ball has diameter two [8,7]. Also every non-reflexive Banach space which is M-embedded satisfies that every non-empty weakly open set relative to its unit ball has diameter two [20].

If a Banach space X satisfies the above property, then it is immediate to check that the projective tensor product of X and any other non-trivial Banach space also satisfies the same property. For the symmetric tensor product it is not so clear the behaviour of such property. If either $X=C(K)$ (K is a compact and Hausdorff infinite topological space) or $X=L_1(\mu)$, for any σ-finite and atomless measure μ, then the space $\widehat{\bigotimes}_{N,s,\pi}X$ (the N-fold symmetric projective tensor product of X) satisfies that every slice of the unit ball have diameter two [1]. The same result does also hold for the symmetric projective tensor product of any infinite-dimensional C^*-algebra [2]. The following papers contain isometric results on spaces of polynomials and symmetric tensor products [12,11,15,23].

In this paper we will extend several results previously obtained for concrete classes of spaces. In Section 2, we will prove that a Banach space X whose centralizer is infinite-dimensional space satisfies that every non-empty weakly open set of the unit ball of the space $\widehat{\bigotimes}_{N,s,\pi}X$ has diameter two. In Section 3 we will improve the previous result and obtain the same conclusion for a wider class of spaces that contains some spaces of operators and non-reflexive JB^*-triples, for instance. The last section contains a result of the same type for weak* neighborhoods of the dual of the injective symmetric tensor product of some Banach spaces. More precisely, if X satisfies that its Cunningham algebra is infinite-dimensional, then every non-empty weak* open set of the unit ball of the space of N-homogeneous integral polynomials on X has diameter two. Here we are using that the space of N-homogeneous integral polynomials on X can be identified as the topological dual of the space $\widehat{\bigotimes}_{N,s,\varepsilon}X$ (injective symmetric tensor product of X) and so we consider the weak* topology associated to this identification.

Now we will recall some notions and notation. Throughout the paper, X will be a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). As usual, S_X, B_X and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X.

For a Banach space X and $N\in\mathbb{N}$, we will consider the symmetric projective N-tensor product $\widehat{\bigotimes}_{N,s,\pi}X$ (see [16]). This space is the completion of the linear space generated by $\{x\otimes\overset{N}{\cdots}\otimes x\colon x\in X\}$ under the norm given by

$$\|z\|=\inf\left\{\sum_{i=1}^{N}|\lambda_i|\colon z=\sum_{i=1}^{N}\lambda_i x_i\otimes\overset{N}{\cdots}\otimes x_i,\ \lambda_i\in\mathbb{K},\ x_i\in S_X,\ \forall 1\leqslant i\leqslant N\right\}.$$

Its topological dual can be identified with the space of all N-homogeneous (and bounded) polynomials on X, denoted by $\mathcal{P}^N(X)$ [16, Proposition, Section 2.2]. Every polynomial

$P \in \mathcal{P}^N(X)$ acts as a linear functional $\widehat{P}$ on the N-fold symmetric tensor product through its associated symmetric N-linear form $\overline{P}$ and it is satisfied that $P(x) = \overline{P}(x, \overset{N}{\ldots}, x) = \widehat{P}(x \otimes \overset{N}{\cdots} \otimes x)$ for every element $x \in X$.

The dual norm of the symmetric tensor product is the usual polynomial norm, that is,

$$\|\widehat{P}\| = \|P\| = \sup\{|P(x)|: x \in X, \ \|x\| \leqslant 1\} \quad \big(P \in \mathcal{P}^N(X)\big).$$

A slice of B_X is a subset of the form

$$S\big(B_X, x^*, \alpha\big) := \big\{x \in B_X: \operatorname{Re} x^*(x) > 1 - \alpha\big\},$$

where $x^* \in S_{X^*}$ and $0 < \alpha < 1$.

Given a family $\{X_i\}_{i \in I}$ of Banach spaces, we will denote by $\prod_{i \in I}^{\infty} X_i$ the Banach space of elements $x \in \prod_{i \in I} X_i$ such that $\sup_{i \in I} \|x(i)\| < \infty$, endowed with the sup-norm. We recall that a **function module** is a triple $(K, (X_t)_{t \in K}, X)$, where K is a non-empty compact Hausdorff topological space (called the base space), $(X_t)_{t \in K}$ a family of Banach spaces (called the component spaces), and X a closed $\mathcal{C}(K)$-submodule of the $\mathcal{C}(K)$-module $\prod_{t \in K}^{\infty} X_t$ such that the following conditions are satisfied:

(1) For every $x \in X$, the function $t \to \|x(t)\|$ from K to $\mathbb{R}$ is upper semicontinuous.
(2) For every $t \in K$, we have $X_t = \{x(t): x \in X\}$.
(3) The set $\{t \in K: X_t \neq 0\}$ is dense in K.

We point out that it is satisfied

$$\|x\| = \sup_{t \in K} \|x(t)\|, \quad \forall x \in X.$$

We follow the notation of [10], where the basic results on function modules can be found.

Let X be a Banach space over $\mathbb{K}$ and $L(X)$ the space of all bounded and linear operators on X. By a **multiplier** on X we mean an element $T \in L(X)$ such that every extreme point of B_{X^*} becomes an eigenvector for T^*. Thus, given a multiplier T on X, and an extreme point p of B_{X^*}, there exists a unique number $a_T(p)$ satisfying $T^*(p) = a_T(p)p$. The **centralizer** of X (denoted by $Z(X)$) is defined as the set of those multipliers T on X such that there exists a multiplier S on X satisfying $a_S(p) = \overline{a_T(p)}$ for every extreme point p of B_{X^*}. Thus, if $\mathbb{K} = \mathbb{R}$, then $Z(X)$ coincides with the set of all multipliers on X. In all cases, $Z(X)$ is a closed subalgebra of $L(X)$ isometrically isomorphic to $\mathcal{C}(K_X)$, for some compact Hausdorff topological space K_X (see [10, Proposition 3.10]). Moreover X can be seen as a function module whose base space is precisely K_X, and such that the elements of $Z(X)$ are precisely the operators of multiplication by the elements of $\mathcal{C}(K_X)$ [10, Theorem 4.14].

2. Preliminary results

Lemma 2.1. *(See [9, Lemma 2.1].) Let $(K, (X_t)_{t \in K}, X)$ be a function module, and let x be an extreme point of B_X. Then, for every $t \in K$ we have $\|x(t)\| = 1$.*

Lemma 2.2. *Let X be a Banach space such that B_X contains some extreme point. Assume also that $K := K_X$ is infinite. Then for every $x \in S_X$, $\varepsilon > 0$ and for every weakly null sequence $\{g_n\}$ in $S_{\mathcal{C}(K)}$, there exist $y \in S_X$, $N \in \mathbb{N}$ and $r > 0$, such that $\|x - y\| < \varepsilon$ and $\|g_n y\| > r$ for every $n \geqslant N$.*

Proof. The statement is clear in the case when $\{\|g_n x\|\}$ does not converge to 0. Assume that $\{\|g_n x\|\} \to 0$. By assumption there is an extreme point p in B_X; in view of Lemma 2.1, since $\{\|g_n x\|\} \to 0$ we know that $x + \alpha p \neq 0$ for every scalar α. Hence we can choose $\alpha > 0$ small enough such that $y := \frac{x+\alpha p}{\|x+\alpha p\|}$ satisfies $\|x - y\| < \varepsilon$. Since $\{\|g_n x\|\} \to 0$, we have that $\lim_{n\to\infty} \|g_n y\| = \lim_{n\to\infty} \|\frac{\alpha g_n p}{\|x+\alpha p\|}\|$. By Lemma 2.1, we have that $\|g_n p\| = 1$ for every n in $\mathbb{N}$. We conclude that $\lim_{n\to\infty} \|g_n y\| = \frac{\alpha}{\|x+\alpha p\|}$, so the statement follows easily. $\square$

The following result is standard and a proof was written in detail in [1, Lemma 2.1].

Lemma 2.3. *Let Ω be a locally compact and Hausdorff infinite topological space. Then there are two sequences of non-empty open and relatively compact sets $\{V_n\}$ and $\{U_n\}$ satisfying*

$$\overline{V_n} \subset U_n, \qquad U_n \cap U_m = \emptyset \quad (n \neq m),$$

and sequences of continuous functions $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ in $C_0(\Omega)$ satisfying

$$\{f_n\} \xrightarrow{w} 0, \quad \{g_n\} \xrightarrow{w} 0, \quad \{h_n\} \xrightarrow{w} 0,$$
$$\operatorname{supp} f_n, \operatorname{supp} h_n \subset V_n, \quad f_n(\Omega), h_n(\Omega) \subset [-1, 1],$$
$$\exists s_n, t_n \in V_n\colon \quad f_n(s_n) = f_n(t_n) = 1 = h_n(t_n) = -h_n(s_n), \quad \forall n$$

and

$$0 \leqslant g_n \leqslant 1, \quad \operatorname{supp} g_n \subset U_n, \quad g_n(V_n) = \{1\}, \quad \forall n.$$

Proposition 2.4. *Let X be a Banach space and assume that B_X contains some extreme point and $Z(X)$ is infinite-dimensional. If N is a positive integer and $Y := \widehat{\bigotimes}_{N,s,\pi} X$, then every non-empty open set in (B_Y, w) has diameter two.*

Proof. Let W be a non-empty weakly open set relative to B_Y. From the assumptions, X is infinite-dimensional, so Y is also infinite-dimensional. Then W contains an element in the unit sphere of Y. In fact we can assume that W contains an element y that can be expressed as

$$y = \sum_{i=1}^{m} \alpha_i x_i^N,$$

where $\alpha_i \in \mathbb{K}$, $\sum_{i=1}^{m} |\alpha_i| = 1$, $x_i \in X$, $\|x_i\| = 1$, and we denoted $x^N := x \otimes \overset{N}{\cdots} \otimes x$.

By a previous remark, we can assume that X is a function module with base space equal to some compact $K := K_X$, and such that $Z(X)$ coincide with the set of operators of multiplication by elements of $\mathcal{C}(K)$. Since $Z(X)$ is infinite-dimensional, K is infinite. We can apply now

Lemma 2.3 and so there are weakly null sequences $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ in $\mathcal{C}(K)$ satisfying the conditions stated in the lemma. Since W is a weakly open set in B_Y, it is also open for the relative norm topology. By the assumption we can choose an extreme point p of B_X. Hence, by Lemma 2.2 we may assume that there is $r > 0$ such that $\|g_n x_i\| \geqslant r$ for every $n \in \mathbb{N}$ and $1 \leqslant i \leqslant m$. Hence $\{g_n x_i\}_n$ is equivalent to the usual c_0-basis for each i.

In the complex case we can assume that $\alpha_i \in \mathbb{R}^+$. In this case, we define the sequences $\{u_n\}$ and $\{v_n\}$ in Y by

$$u_n = \sum_{i=1}^{m} \alpha_i \bigl((1-g_n)x_i + h_n p\bigr)^N, \qquad v_n = \sum_{i=1}^{m} \alpha_i \bigl((1-g_n)x_i + w h_n p\bigr)^N,$$

where w is a complex number satisfying $w^N = -1$. Let us notice that for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and every $s \in K$, depending on $s \in V_n$ or $s \notin V_n$, one has

$$\begin{aligned} \bigl\|\bigl((1-g_n)x_i + \lambda h_n p\bigr)(s)\bigr\| &= \bigl\|h_n(s)p(s)\bigr\| \leqslant 1, \\ \bigl\|\bigl((1-g_n)x_i + \lambda h_n p\bigr)(s)\bigr\| &= \bigl\|\bigl((1-g_n)x_i\bigr)(s)\bigr\| \leqslant 1, \end{aligned}$$

hence $\|u_n\|, \|v_n\| \leqslant 1$.

The real case, for N odd can be handled in the same way as the complex case.

Finally, if the space is real and N is even, we consider the elements

$$\begin{aligned} u_n &= \sum_{i \in P} \alpha_i \bigl((1-g_n)x_i + f_n p\bigr)^N + \sum_{i \in R} \alpha_i \bigl((1-g_n)x_i + h_n p\bigr)^N, \\ v_n &= \sum_{i \in P} \alpha_i \bigl((1-g_n)x_i + h_n p\bigr)^N + \sum_{i \in R} \alpha_i \bigl((1-g_n)x_i + f_n p\bigr)^N \end{aligned}$$

where $P := \{i\colon\ \alpha_i \geqslant 0\}$ and $R := \{i\colon\ \alpha_i < 0\}$. It is immediate to check that u_n, v_n belong to B_X.

Since p is an extreme point in B_X, in view of Lemma 2.1, there are functionals $\varphi_{t_n} \in (X_{t_n})^*$, $\psi_{s_n} \in (X_{s_n})^*$ such that $\|\varphi_{t_n}\| = \|\psi_{s_n}\| = 1 = \varphi_{t_n}(p(t_n)) = \psi_{s_n}(p(s_n))$. Then the functionals on X defined by

$$\varphi_n(x) = \varphi_{t_n}\bigl(x(t_n)\bigr), \qquad \psi_n(x) = \psi_{s_n}\bigl(x(s_n)\bigr) \quad (x \in X)$$

belong to B_{X^*}.

In this case, we consider the N-homogeneous polynomial given by

$$Q_n = \psi_n^{N-1} \varphi_n.$$

It is clear that $\|Q_n\| \leqslant 1$ and

$$\begin{aligned} \widehat{Q}_n(u_n) - \widehat{Q}_n(v_n) = {} & \sum_{i \in P} \alpha_i f_n(s_n)^{N-1} f_n(t_n) + \sum_{i \in R} \alpha_i h_n(s_n)^{N-1} h_n(t_n) \\ & - \sum_{i \in P} \alpha_i h_n(s_n)^{N-1} h_n(t_n) - \sum_{i \in R} \alpha_i f_n(s_n)^{N-1} f_n(t_n) \end{aligned}$$

$$= \sum_{i\in P}\alpha_i + \sum_{i\in R}\alpha_i(-1)^{N-1} - \sum_{i\in P}\alpha_i(-1)^{N-1} - \sum_{i\in R}\alpha_i$$

$$= 2\sum_{i=1}^{s}|\alpha_i| = 2.$$

Since p is an extreme point of B_X, in view of Lemma 2.1, we know that $\|h_n p\| = \|f_n p\| = 1$ for every n. Hence $\{g_n x_i\}$, $\{h_n p\}$ and $\{f_n p\}$ are equivalent to the usual basis of c_0. Hence, the subspace $M := \overline{\mathrm{Lin}}\{x_i, g_n x_i, h_n p, f_n p\colon n \in \mathbb{N},\ i \in \{1,\dots,m\}\}$ of X is isomorphic to c_0. If $\{f_n\}$ is weakly null in $\mathcal{C}(K)$ and $x \in X$, then $\{f_n x\}$ is weakly null in X. Since M has the Dunford–Pettis property, in view of [22, Theorem 2.1], then it has the polynomial Dunford–Pettis property. That is, if $\{z_n\}$ converges weakly to z in M, then $\{Q(z_n)\}$ converges to $Q(z)$ for every $Q \in \mathcal{P}^N(M)$. Since the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ used above are weakly null in $\mathcal{C}(K)$, then we deduce that the sequences $\{u_n\}$ and $\{v_n\}$ converge weakly to $y = \sum_{i=1}^m \alpha_i x_i^N$. Hence u_n and $v_n \in W$ for n large enough. In any case, it holds

$$\left|\widehat{Q}_n(u_n) - \widehat{Q}_n(v_n)\right| = 2,$$

and so $2 \leqslant \|u_n - v_n\| \leqslant \operatorname{diam} W$. □

Remark 2.5. Let us notice that in the previous result we proved that for any weakly open set W in (B_Y, w) it is satisfied that

$$2 = \operatorname{diam} W = \sup\left\{ \operatorname{Re}\widehat{Q}(y_1 - y_2)\colon\ y_1, y_2 \in W,\ \exists x_i^* \in S_{X^*}\ (1 \leqslant i \leqslant N),\ Q(x) = \prod_{i=1}^{N} x_i^*(x),\ \forall x \in X \right\}.$$

We will use the well-known fact that every N-homogeneous polynomial P on X can be extended in a canonical way to an N-homogeneous polynomial $P^{(2}$ on X^{**} [4]. $P^{(2}$ is called the Aron–Berner extension of P and it satisfies $\|P^{(2}\| = \|P\|$.

We will improve Proposition 2.4 by removing one of the assumptions. The following statement also generalizes previous results obtained before for concrete classes of spaces (see [1, Proposition 2.2]).

Corollary 2.6. *Let X be a Banach space such that $Z(X)$ is infinite-dimensional. If N is a positive integer and $Y := \widehat{\bigotimes}_{N,s,\pi} X$, then every non-empty open set in (B_Y, w) has diameter two. Indeed the diameter of every weakly open set can be computed by using N-homogeneous polynomials which are products of N elements of X^*.*

Proof. Assume that W is a non-empty weakly open set in (B_Y, w). The assumption implies that X is infinite-dimensional, so is Y. Then W contains an element $y_0 \in S_Y$. We can clearly assume that there is $\eta > 0$ and $P_1, \dots, P_k$ in $\mathcal{P}^N(X)$ such that

$$W := \left\{ y \in B_Y\colon\ \left|\widehat{P_i}(y) - \widehat{P_i}(y_0)\right| < \eta,\ \forall 1 \leqslant i \leqslant k \right\}.$$

Denote by Q_i the Aron–Berner extension of the N-homogeneous polynomial P_i to X^{**}, for each i. The set

$$\widetilde{W} := \left\{ z \in B_{\widehat{\bigotimes}_{N,s,\pi} X^{**}} \colon \left| \widehat{Q}_i(z) - \widehat{P}_i(y_0) \right| < \eta \right\}$$

is a non-empty open set in $(B_{\widehat{\bigotimes}_{N,s,\pi} X^{**}}, w)$.

By assumption, $Z(X)$ is infinite-dimensional, so $Z(X^{**})$ is also infinite-dimensional by [17, Corollary I.3.15]. Since $B_{X^{**}}$ has extreme points, we can apply Proposition 2.4 and Remark 2.5. Hence for every $\varepsilon > 0$ there are elements $\varphi_1, \dots, \varphi_N \in S_{X^{***}}$ such that the polynomial $Q = \prod_{i=1}^{N} \varphi_i$ satisfies

$$2 - \varepsilon < \operatorname{Re} \widehat{Q}(\tilde{u} - \tilde{v}),$$

for some $\tilde{u}, \tilde{v} \in \widetilde{W}$. We can assume that there are positive integers n and m, real numbers t_i, s_j $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ and elements $u_i, v_j \in B_{X^{**}}$ such that

$$\sum_{i=1}^{n} |t_i| = \sum_{j=1}^{m} |s_j| = 1, \qquad \tilde{u} = \sum_{i=1}^{n} t_i u_i^N, \qquad \tilde{v} = \sum_{j=1}^{m} s_j v_j^N$$

and

$$\operatorname{Re}\left(\sum_{i=1}^{n} t_i Q(u_i) - \sum_{j=1}^{m} s_j Q(v_j) \right) > 2 - \varepsilon. \tag{2.1}$$

Since B_{X^*} is w^*-dense in $B_{X^{***}}$, there are functionals $x_1^*, \dots, x_N^* \in S_{X^*}$ such that the polynomial $P(x) = \prod_{i=1}^{N} x_i^*(x)$ satisfies that

$$\operatorname{Re}\left(\sum_{i=1}^{n} t_i \widetilde{P}(u_i) - \sum_{j=1}^{m} s_j \widetilde{P}(v_j) \right) > 2 - \varepsilon, \tag{2.2}$$

where $\widetilde{P}(u) := \prod_{i=1}^{N} u(x_i^*)$ for each $u \in X^{**}$.

Since $\tilde{u}, \tilde{v} \in \widetilde{W}$, in view of (2.2) and [13, Theorem 2], there are $x_i, y_j \in B_X$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ such that $x := \sum_{i=1}^{n} t_i x_i^N$, $y := \sum_{j=1}^{m} s_j y_j^N$ satisfy that $x, y \in W$ and also

$$\operatorname{Re}\left(\sum_{i=1}^{n} t_i P(x_i) - \sum_{j=1}^{m} s_j P(y_j) \right) > 2 - \varepsilon.$$

Since $P \in B_{\mathcal{P}^N(X)}$, $\operatorname{diam} W \geqslant \operatorname{Re} \widehat{P}(x - y) > 2 - \varepsilon$, so $\operatorname{diam} W = 2$, as we wanted to show. □

3. The main result

Given a Banach space X, we consider the increasing sequence of its even duals

$$X \subseteq X^{**} \subseteq X^{(4} \subseteq \cdots \subseteq X^{(2n} \subseteq \cdots,$$

and we define $X^{(\infty}$ as the completion of the normed space $\bigcup_{n=0}^{\infty} X^{(2n}$.

Proposition 3.1. *Let X be a Banach space. Then $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$.*

Proof. We begin by noticing the existence of a natural embedding $f \to \tilde{f}$ from $(X^*)^{2n}$ to $(X^{(\infty)})^*$ for every $n \in \mathbb{N}$. Indeed, let f be in $(X^*)^{(2n}$. Given $\alpha \in \bigcup_{n=0}^{\infty} X^{(2n}$, there exists $m \in \mathbb{N}$ such that α belongs to $X^{(2m}$, so that, regarding f as an element of $(X^*)^{(2m} = (X^{(2m)})^*$, the symbol $f(\alpha)$ has a meaning which does not depend on m. In this way we are provided with a natural Hahn–Banach extension of f to $\bigcup_{n=0}^{\infty} X^{(2n}$, which extends uniquely by continuity to $X^{(\infty}$, giving rise to an element $\tilde{f}$ of $(X^{(\infty)})^*$. Hence $(X^*)^{(\infty}$ is linearly isometric to a closed subspace of $(X^{(\infty)})^*$. Since $B_{(X^*)^{(\infty}}$ is convex and the subset $B = \bigcup_n B_{X^{2n+1}} \subset B_{(X^{(\infty)})^*}$ satisfies that $\|f\| = \sup_{b \in B} |f(b)|$ for every $f \in X^{(\infty}$, then $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$. $\square$

For every Banach space X and $N \in \mathbb{N}$, we will show that there is a natural embedding $P \to \widetilde{P}$ from $\mathcal{P}^N(X)$ to $\mathcal{P}^N(X^{(\infty)})$. We denote by $P^{(2}$ the Aron–Berner extension of an element $P \in \mathcal{P}^N(X)$ to X^{**}. We know that this canonical extension satisfies $\|P^{(2}\| = \|P\|$. We denote by $P^{(2n}$ the Aron–Berner extension of $P^{(2n-2}$ to $X^{(2n}$. We have that $\|P^{(2n}\| = \|P\|$ for all $n \in \mathbb{N}$. Indeed, let P be in $\mathcal{P}^N(X)$. Given $\alpha \in \bigcup_{n=0}^{\infty} X^{(2n}$, there exists $m \in \mathbb{N}$ such that α belongs to $X^{(2m}$, allowing us to consider the element $P^{(2m}(\alpha)$, which is well defined. In this way we are provided with a natural extension of P to $\bigcup_{n=0}^{\infty} X^{(2n}$, which extends uniquely by continuity to $X^{(\infty}$, giving rise to an element $\widetilde{P}$ of $\mathcal{P}^N(X^{(\infty)})$.

In this way we have the following chain of embeddings

$$\mathcal{P}^N(X) \hookrightarrow \mathcal{P}^N\big(X^{**}\big) \hookrightarrow \mathcal{P}^N\big(X^{(4)}\big) \hookrightarrow \cdots \hookrightarrow \mathcal{P}^N\big(X^{(2n)}\big) \hookrightarrow \cdots,$$

where each arrow means Aron–Berner extension.

Hence we can complete the above chain as follows

$$\mathcal{P}^N(X) \hookrightarrow \mathcal{P}^N\big(X^{**}\big) \hookrightarrow \mathcal{P}^N\big(X^{(4)}\big) \hookrightarrow \cdots \hookrightarrow \mathcal{P}^N\big(X^{(2n)}\big) \hookrightarrow \cdots \hookrightarrow \mathcal{P}^N\big(X^{(\infty)}\big),$$

and the embedding $P \to \widetilde{P}$ from $\mathcal{P}^N(X)$ to $\mathcal{P}^N(X^{(\infty)})$ is isometric.

As we will see later, the next result improves Corollary 2.6.

Theorem 3.2. *Let X be a Banach space such that $Z(X^{(\infty)})$ is infinite-dimensional. Then for every N, every non-empty weakly open set relative to the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$ has diameter two.*

Proof. Let W be a non-empty weakly open set relative to the unit ball of $Y := \widehat{\bigotimes}_{N,s,\pi} X$. By the assumptions W contains an element $y_0 \in S_Y$. Since W is a weakly open set, we can assume that there is $\eta > 0$ and $P_i \in \mathcal{P}^N(X)$ $(1 \leqslant i \leqslant k)$ such that

$$W := \{y \in B_Y: \ |\widehat{P_i}(y) - \widehat{P_i}(y_0)| < \eta, \ \forall 1 \leqslant i \leqslant k\}.$$

We denote by Q_i the canonical extensions of the polynomials P_i $(1 \leqslant i \leqslant k)$ to $X^{(\infty}$ and define

$$\widetilde{W} := \{\tilde{u} \in B_{\widehat{\bigotimes}_{N,s,\pi} X^{(\infty}}: \ |\widehat{Q}_i(\tilde{u}) - \widehat{P}_i(y_0)| < \eta, \ \forall 1 \leqslant i \leqslant k\}.$$

Hence $\widetilde{W}$ is a weakly open set relative to the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X^{(\infty}$ and it is non-empty since it does contain W. By Corollary 2.6, $\widetilde{W}$ has diameter two. By the proof of Corollary 2.6, for every $\varepsilon > 0$ there are functionals $\varphi_j \in B_{(X^{(\infty})^*}$ $(1 \leqslant j \leqslant N)$ and elements $\tilde{u}, \tilde{v} \in \widetilde{W}$ such that the N-homogeneous polynomial Q given by $Q = \prod_{j=1}^N \varphi_j$ satisfies

$$\widehat{Q}(\tilde{u} - \tilde{v}) > 2 - \varepsilon.$$

Since $\widetilde{W}$ is open relative to the norm topology, by using also the definition of $X^{(\infty}$ and Proposition 3.1, we can assume that there is a natural number m such that $\tilde{u}, \tilde{v} \in \widehat{\bigotimes}_{N,s,\pi} X^{(2m}$ and $\varphi_j \in X^{(2m+1}$ for each j. Now we proceed as in the last part of the proof of Corollary 2.6 and find elements $u, v \in \widetilde{W} \cap B_{\widehat{\bigotimes}_{N,s,\pi} X^{(2m-2}}$ and functionals $\psi_j \in X^{(2m-1}$ $(1 \leqslant j \leqslant N)$ such that the polynomial $P := \prod_{j=1}^N \psi_j$ satisfies that

$$\widehat{P}(u - v) > 2 - \varepsilon.$$

After a finite number of steps, we deduce that $\operatorname{diam} W \geqslant 2 - \varepsilon$. Since ε is any positive number we conclude that $\operatorname{diam} W = 2$. $\square$

Now we will provide more examples of spaces where the previous results can be applied.

It is known that for every T in $Z(X)$, T^{**} lies in $Z(X^{**})$ [17, Corollary I.3.15]. Since T^{**} is an isometric extension of T, there exists a natural embedding from $L(X)$ to $L(X^{(\infty})$. The image under this embedding of an operator $T \in L(X)$ is the operator such that its restriction to $X^{(2m}$ is obtained by transposing T $2m$ times. Indeed, it is known that the image of $Z(X)$ under this embedding is contained in $Z(X^{(\infty})$ (see [9, Proposition 4.3]).

For a Banach space X, an L-**projection** on X is a (linear) projection $P : X \to X$ satisfying $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$. In such a case, we will say that the subspace $P(X)$ is an L-**summand** of X. X is L-embedded if it is an L-summand of X^{**}.

Let us notice that the composition of two L-projections on X is an L-projection [10, Proposition 1.7], so the closed linear subspace of $L(X)$ generated by all L-projections on X is a subalgebra of $L(X)$. This algebra, denoted by $C(X)$, is called the **Cunningham algebra** of X. It is known that $C(X)$ is linearly isometric to $Z(X^*)$ [10, Theorems 5.7 and 5.9].

We will provide examples of Banach spaces X such that $Z(X^{(\infty})$ is infinite-dimensional. The following Banach spaces satisfy this condition:

(1) A Banach space X satisfying that $Z(X^{(2n})$ is infinite-dimensional for some natural number n, and so every infinite-dimensional predual of an L_1-space.

(2) X is a real or complex non-reflexive Banach space that X^{**} is a JB^*-triple (see [19,18] for the definition) [9, Corollary 5.2]. For instance, every real or complex infinite-dimensional C^*-algebra.
(3) The space $\mathcal{C}(K,(X,\tau))$ where K is an infinite compact topological space, X is a non-null Banach space and τ is a topology such that the weak topology is contained in τ and the norm topology is finer than τ [9, Proposition 3.2].
(4) $L(X,Y)$ (the space of all bounded and linear operators from X to Y) for every Banach spaces X, Y such that either $C(X)$ is infinite-dimensional or $Z(Y)$ is infinite-dimensional (see [17, Lemma VI.1.1]). For instance, any infinite-dimensional space $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional.

Proposition 3.3. *Let X be a non-reflexive Banach space such that X^* is L-embedded. Then $Z(X^{(\infty)})$ is infinite-dimensional.*

Proof. We have that $X^{***} = (X^* \oplus N)_{\ell_1}$ for some subspace N of X^{***}, and hence $X^{4)}$ is isometric to $(X^{**} \oplus N^*)_{\ell_\infty}$. Since X is non-reflexive, then $N \neq \{0\}$ and so we have that $\dim Z(X^{(4)}) \geqslant \dim Z(X^{**}) + 1 \geqslant 1$ [9, Lemma 2.2]. By using the same argument and induction, we deduce that $\dim Z(X^{(2n)}) \geqslant n-1$ for every natural number n. Since we already know that $Z(X^{(\infty)})$ contains $Z(X^{(2n)})$ for every n, then $Z(X^{(\infty)})$ is infinite-dimensional. □

Let us remark that non-reflexive JB^*-triples are spaces satisfying the above assumption in view of [14, Corollary 11] and [5, Proposition 3.4].

Proposition 3.4. *Let X be a non-reflexive Banach space that is L-embedded, then $C(X^{(\infty)})$ is infinite-dimensional.*

Proof. If $T \in L(X)$ is an L-projection, then T^{**} is also an L-projection in X^{**}. Hence every L-projection on X can be extended to an L-projection on $X^{(\infty}$, and so $C(X^{(\infty)})$ contains $C(X^{(2n)})$ for every n.

By assumption we have that $X^{**} = (X \oplus N)_{\ell_1}$ for some subspace $N \neq \{0\}$ of X^{**}, and hence X^{***} is isometric to $(X^* \oplus N^*)_{\ell_\infty}$. Keeping in mind the proof of the above result, we conclude that $\dim Z(X^{(***)}) \geqslant \dim Z(X^*) + 1$. By induction we deduce that

$$\dim Z\big(X^{(2n+1)}\big) \geqslant \dim Z\big(X^{(2n-1)}\big) + 1 \geqslant n.$$

Since $C(Y)$ is linearly isometric to $Z(Y^*)$ for every Banach space Y [10, Theorems 5.7 and 5.9], then we conclude that $\dim C(X^{(2n)}) \geqslant n$ and so $C(X^{(\infty)})$ is infinite-dimensional. □

4. The space of integral polynomials

A polynomial P is said to be **integral** [16] if there is a regular Borel measure μ on (B_{X^*}, w^*) such that

$$P(x) = \int_{B_{X^*}} \big(x^*(x)\big)^n \, d\mu\big(x^*\big), \quad \forall x \in X. \tag{4.1}$$

As usual, denote by $\mathcal{P}_I^N(X)$ the space of all N-**homogeneous integral polynomials on** X. We recall that the integral norm of an integral polynomial P, $\|P\|_I$, is the infimum of the set

$$\{\|\mu\|\colon \mu \text{ satisfies (4.1)}\}.$$

Endowed with the integral norm $\mathcal{P}_I^N(X)$ becomes a Banach space. For every Banach space X we have $\mathcal{P}_I^N(X) \subseteq \mathcal{P}^N(X)$ and it is clear that $\|P\| \leqslant \|P\|_I$ for every $P \in \mathcal{P}_I^N(X)$. For $x^* \in S_{X^*}$ it is easily seen that $(x^*)^N$ is an integral polynomial (associated to the measure δ_{x^*}) and $\|(x^*)^N\|_I = 1$. Apart from the projective norm, there are other reasonable norms that can be placed on $\bigotimes_{N,s} X$. Given an element $\sum_{i=1}^k \lambda_i x_i \otimes x_i \otimes \cdots \otimes x_i$ in the N-fold symmetric tensor product on X, we recall that its injective norm is given by

$$\sup_{x \in B_{X^*}} \left| \sum_{i=1}^{k} \lambda_i \bigl(x^*(x_i)\bigr)^N \right|.$$

This may also be regarded as the norm inherited from $\mathcal{P}^N(X^*)$. We denote the completion of $\bigotimes_{N,s} X$ with respect to this norm by $\widehat{\bigotimes}_{N,s,\varepsilon} X$.

It is shown in [16, Theorem, Section 3.4] that the dual of $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is isometrically isomorphic to $(\mathcal{P}_I^N(X), \|.\|_I)$.

We will show that the Cunningham algebra plays the role of the centralizer in the third section in order to obtain results of similar nature for the space of integral polynomials.

Proposition 4.1. *Let X be a Banach space such that $C(X)$ is infinite-dimensional. Then for every positive integer N, every weak-$*$ neighborhood relative to the unit ball of $\mathcal{P}_I^N(X)$ has diameter two.*

Proof. Let $N \in \mathbb{N}$ and W be a non-empty weak-$*$ open set relative to the unit ball of $\mathcal{P}_I^N(X)$. By [12, Proposition 1] (see also [11]) the set of extreme points of the unit ball of $\mathcal{P}_I^N(X)$ is contained in $\{\pm\varphi^N\colon \varphi \in S_{X^*}\}$.

It follows from the Krein–Milman Theorem that there exist $k \in \mathbb{N}$, $t_j \in [-1, 1]$ and $\varphi_j \in S_{X^*}$ $(1 \leqslant j \leqslant k)$ with $\sum_{j=1}^k |t_j| = 1$ such that $P := \sum_{j=1}^k t_j \varphi_j^N \in W$. By [10, Theorems 5.7 and 5.9] $C(X)$ is isomorphic to $Z(X^*)$, so $K := K_{X^*}$ is infinite. By Lemma 2.3, there are sequences of disjoint open sets, $\{V_n\}$ and $\{U_n\}$ in K and sequences $\{g_n\}$ and $\{h_n\}$ in $\mathcal{C}(K)$ satisfying that

$$\begin{gathered} 0 \leqslant g_n \leqslant 1, \quad -1 \leqslant h_n \leqslant 1, \quad \|h_n\|_\infty = 1, \quad \forall n \in \mathbb{N}, \\ \{g_n\} \xrightarrow{w} 0, \quad \{h_n\} \xrightarrow{w} 0, \qquad (4.2) \\ \operatorname{supp} h_n \subset V_n, \quad \operatorname{supp} g_n \subset U_n, \quad g_n(V_n) = \{1\}. \qquad (4.3) \end{gathered}$$

Let p be an extreme point of B_{X^*}. For every n and $j \leqslant k$, for each $t \in K$, depending on $t \in V_n$ or $t \notin V_n$ we have

$$\begin{aligned} \bigl\|\bigl((1-g_n)\varphi_j \pm h_n p\bigr)(t)\bigr\| &= \bigl\|(h_n p)(t)\bigr\| \leqslant 1, \\ \bigl\|\bigl((1-g_n)\varphi_j \pm h_n p\bigr)(t)\bigr\| &= \bigl\|\bigl((1-g_n)\varphi_j\bigr)(t)\bigr\| \leqslant 1, \end{aligned}$$

hence

$$\big\|(1-g_n)\varphi_j \pm h_n p\big\| \leqslant 1. \tag{4.4}$$

We write $M := \{j \in \{1,\dots,k\}\colon t_j > 0\}$ and $R = \{1,\dots,k\}\backslash M$. If N is odd, for every natural number n we consider the integral polynomials P_n and Q_n given by

$$P_n := \sum_{j\in M} t_j\big((1-g_n)\varphi_j + h_n p\big)^N + \sum_{j\in N} t_j\big((1-g_n)\varphi_j - h_n p\big)^N,$$
$$Q_n := \sum_{j\in M} t_j\big((1-g_n)\varphi_j - h_n p\big)^N + \sum_{j\in N} t_j\big((1-g_n)\varphi_j + h_n p\big)^N.$$

It follows from (4.4) that $P_n, Q_n \in B_{\mathcal{P}_I^N(X)}$. In view of (4.2) $\{(1-g_n)\varphi_j \pm h_n p\}$ converges weakly to φ_j in X^* for every $j \leqslant k$. As a consequence, $\{P_n\}$ and $\{Q_n\}$ converge to P in the weak-$*$ topology of $\mathcal{P}_I^N(X)$. Since $P \in W$ and W is a weak-$*$ open set in $B_{\mathcal{P}_I^N(X)}$, then for m large enough, we have that P_m and Q_m belong to W. From now on, we fix m large enough such that the previous conditions hold.

By Lemma 2.3 we know that there is an element $t_m \in V_m$ satisfying $h_m(t_m) = 1$. Since p is an extreme point of B_{X^*}, by Lemma 2.1, $\|p(t)\| = 1$ for every $t \in K$. Let α_{t_m} be an element of $B_{(X^*_{t_m})^*}$, such that $\alpha_{t_m}(p(t_m)) = 1$. Then the functional defined by $\alpha_m(\varphi) := \alpha_{t_m}(\varphi(t_m))$ belongs to B_{X^*}. We have that $\alpha_m(\varphi_j(1-g_m) + h_m p) = 1$ and $\alpha_m(\varphi_j(1-g_m) - h_m p) = -1$ for all $1 \leqslant j \leqslant k$. Since B_X is w^*-dense in $B_{X^{**}}$, there exists $\{x_n\} \in B_X$ such that

$$\lim_n \big\{\big((1-g_m)\varphi_j + h_m p\big)(x_n)\big\} = 1 = -\lim_n \big\{\big((1-g_m)\varphi_j - h_m p\big)\big\}, \quad \forall 1 \leqslant j \leqslant k$$

and for m fixed.

By using that N is odd and that for each n the element $x_n \otimes \overset{N}{\cdots} \otimes x_n$ belongs to $B_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ we have

$$\begin{aligned}\|P_m - Q_m\|_I &\geqslant \big|(P_m - Q_m)(x_n)^N\big| \\ &= \sum_{j\in M} t_j\big(\big(\varphi_j(1-g_m) + h_m p\big)(x_n)\big)^N + \sum_{j\in R} t_j\big(\big(\varphi_j(1-g_m) - h_m p\big)(x_n)\big)^N \\ &\quad - \sum_{j\in M} t_j\big(\big(\varphi_j(1-g_m) - h_m p\big)(x_n)\big)^N - \sum_{j\in R} t_j\big(\big(\varphi_j(1-g_m) + h_m p\big)(x_n)\big)^N.\end{aligned}$$

By taking limit ($n \to \infty$), we deduce that $\operatorname{diam} W \geqslant \|P_m - Q_m\|_I \geqslant 2\sum_{i=1}^k |t_i| = 2$.

Now we will prove the same fact in the case that N is even. By Lemma 2.3 we know that for every n there are elements $t_n, s_n \in V_n$ and functions f_n, h_n satisfying

$$f_n(s_n) = f_n(t_n) = 1 = h_n(t_n) = -h_n(s_n).$$

In this case we consider the polynomials given by

$$P_n = \sum_{j\in M} t_j\big((1-g_n)\varphi_j + f_n p\big)^N + \sum_{j\in R} t_j\big((1-g_n)\varphi_j + h_n p\big)^N,$$
$$Q_n = \sum_{j\in M} t_j\big((1-g_n)\varphi_j + h_n p\big)^N + \sum_{j\in R} t_j\big((1-g_n)\varphi_j + f_n p\big)^N.$$

By using the same argument of the odd case, $P_n, Q_n \in B_{\mathcal{P}_I^N(X)}$ and the sequences $\{P_n\}$ and $\{Q_n\}$ converge to P in the w^*-topology of $\mathcal{P}_I^N(X)$. Hence for m large enough, $P_m, Q_m \in W$. We fix m from now on.

Since p is an extreme point of B_{X^*}, by Lemma 2.1, $\|p(t)\| = 1$ for every $t \in K$. Hence there are functionals $\alpha_{t_m} \in (X_{t_m}^*)^*$, $\beta_{s_m} \in (X_{s_m}^*)^*$ such that $\|\alpha_{t_m}\| = \|\beta_{s_m}\| = 1 = \alpha_{t_m}(p(t_m)) = \beta_{s_m}(p(s_m))$. Then the functionals on X^* defined by

$$\alpha_m\big(x^*\big) = \alpha_{t_m}\big(x^*(t_m)\big), \qquad \beta_m\big(x^*\big) = \beta_{s_m}\big(x^*(s_m)\big) \quad \big(x^* \in X^*\big)$$

belong to $B_{X^{**}}$. We have that

$$\alpha_m\big((1-g_m)\varphi_j + h_m p\big) = 1 = \alpha_m\big((1-g_m)\varphi_j + f_m p\big) = 1, \quad \forall 1 \leqslant j \leqslant k$$

and

$$\beta_m\big((1-g_m)\varphi_j + f_m p\big) = 1 = -\beta_m\big((1-g_m)\varphi_j + h_m p\big), \quad \forall 1 \leqslant j \leqslant k.$$

Since B_X is w^*-dense in $B_{X^{**}}$, for each m there exist $\{x_n\}, \{y_n\} \in B_X$ such that

$$\lim_n\big\{\big((1-g_m)\varphi_j + h_m p\big)(x_n)\big\} = 1 = \lim_n\big\{\big((1-g_m)\varphi_j + f_m p\big)(x_n)\big\}, \quad \forall 1 \leqslant j \leqslant k,$$
$$\lim_n\big\{\big((1-g_m)\varphi_j + f_m p\big)(y_n)\big\} = 1 = -\lim_n\big\{\big((1-g_m)\varphi_j + h_m p\big)(y_n)\big\}, \quad \forall 1 \leqslant j \leqslant k.$$

Given $w := x_1 \otimes \overset{N}{\cdots} \otimes x_N$ in $\bigotimes_N X$, we recall that the element $w_s := \frac{1}{n!}\sum_{\sigma\in\Pi_N} x_{\sigma(1)} \otimes \overset{N}{\cdots} \otimes x_{\sigma(N)}$ belongs to $\widehat{\bigotimes}_{N,s,\varepsilon} X$ (Π_N is the set of all permutations on $\{1, 2, \ldots, N\}$). It is immediate that $w_s \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ if the elements $x_i \in B_X$ for every $1 \leqslant i \leqslant N$. If for each n, we take $w = y_n \otimes \overset{N-1}{\cdots} \otimes y_n \otimes x_n$, then $w_s \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ and we obtain that

$$\begin{aligned}
\|P_m - Q_m\|_I &\geqslant \big|(P_m - Q_m)(w_s)\big| \\
&= \Bigg|\sum_{j\in M} t_j\big((1-g_m)(\varphi_j + f_m p)(y_n)\big)^{N-1}\big((1-g_m)(\varphi_j + f_m p)(x_n)\big) \\
&\quad + \sum_{j\in R} t_j\big((1-g_m)(\varphi_j + h_m p)(y_n)\big)^{N-1}\big((1-g_m)(\varphi_j + h_m p)(x_n)\big) \\
&\quad - \sum_{j\in M} t_j\big((1-g_m)(\varphi_j + h_m p)(y_n)\big)^{N-1}\big((1-g_m)(\varphi_j + h_m p)(x_n)\big) \\
&\quad - \sum_{j\in R} t_j\big((1-g_m)(\varphi_j + f_m p)(y_n)\big)^{N-1}\big((1-g_m)(\varphi_j + f_m p)(x_n)\big)\Bigg|.
\end{aligned}$$

By taking limit ($n \to \infty$), we deduce that $\operatorname{diam} W \geqslant \|P_m - Q_m\|_I \geqslant 2\sum_{i=1}^k |t_i| = 2$. In any case, we have proved that $\operatorname{diam} W = 2$. $\square$

Theorem 4.2. *Let X be a Banach space such that $C(X^{(\infty})$ is infinite-dimensional. Then for every positive integer N, every weak-$*$ neighborhood relative to the unit ball of $\mathcal{P}_I^N(X)$ has diameter two.*

Proof. Let $N \in \mathbb{N}$ and W be a non-empty weak-$*$ open set relative to the unit ball of $\mathcal{P}_I^N(X)$. By [12, Proposition 1] the set of extreme points of the unit ball of $\mathcal{P}_I^N(X)$ is contained in $\{\pm\varphi^N\colon \varphi \in S_{X^*}\}$. By using Krein–Milman Theorem there exist $k \in \mathbb{N}$, $t_j \in [-1, 1]$ and $f_j \in S_{X^*}$ $(1 \leqslant j \leqslant k)$ with $\sum_{j=1}^k |t_j| = 1$ such that $P_0 := \sum_{j=1}^k t_j (f_j)^N \in W$.

Let $Y := \widehat{\bigotimes}_{N,s,\varepsilon} X$. Since W is a weakly-$*$ open set, we can assume that there is $\eta > 0$ and $y_i \in S_Y$ $(1 \leqslant i \leqslant m)$ such that

$$W := \bigl\{P \in B_{\mathcal{P}_I^N(X)}\colon \bigl|P(y_i) - P_0(y_i)\bigr| < \eta,\ \forall 1 \leqslant i \leqslant m\bigr\}.$$

It is clear that $y_i \in S_{\widehat{\bigotimes}_{N,s,\varepsilon} X^{(\infty}}$ for all $1 \leqslant i \leqslant m$. By the remarks of Section 3, P_0 can be seen as an N-homogeneous polynomial on $X^{(\infty}$ and it is also clear that it is also an element in $B_{\mathcal{P}_I^N(X^{(\infty)})}$. We define

$$\widetilde{W} := \bigl\{P \in B_{\mathcal{P}_I^N(X^{(\infty)})}\colon \bigl|P(y_i) - P_0(y_i)\bigr| < \eta,\ \forall 1 \leqslant i \leqslant m\bigr\}.$$

Hence $\widetilde{W}$ is a weak* neighborhood of P_0 in the unit ball of $\mathcal{P}_I^N(X^{(\infty)})$. Given $\varepsilon > 0$, by Proposition 4.1, there are $s_j \in [-1, 1]$ and $\varphi_j, \psi_j \in S_{(X^{(\infty)})^*}$ $(1 \leqslant j \leqslant k)$ with $\sum_{j=1}^k |s_j| = 1$ such that $P := \sum_{j=1}^k s_j \varphi_j^N$, $Q := \sum_{j=1}^k s_j \psi_j^N \in \widetilde{W}$ and there is $w \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X^{(\infty}}$ satisfying $|(P - Q)(w)| > 2 - \varepsilon$.

It is clear that for every net $\{f_\alpha\}$ that converges weakly-$*$ to f in $(X^{(\infty)})^*$, we have that $\{f_\alpha^N\}$ converges weakly-$*$ to f^N in $\mathcal{P}_I^N(X^{(\infty)})$. By Proposition 3.1 $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$. Since $\widetilde{W}$ is weakly-$*$ open, we can assume that $\varphi_j, \psi_j \in S_{(X^*)^{(\infty}}$ for all $1 \leqslant j \leqslant k$.

By using also the definitions of $X^{(\infty}$ and $(X^*)^{(\infty}$, given $\varepsilon > 0$ we can assume that there is a natural number n such that, $\varphi_j, \psi_j \in S_{X^{(2n+1}}$ for all $1 \leqslant j \leqslant k$, $w \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X^{(2n}}$.

Now we proceed as in the proof of Corollary 2.6. Since the unit ball of $X^{(2n-1}$ is w^*-dense in the unit ball of $X^{(2n+1}$, we can assume that $\varphi_j, \psi_j \in S_{X^{(2n-1}}$ for all $1 \leqslant j \leqslant k$. Since $B_{X^{(2n-2}}$ is w^*-dense in $B_{X^{(2n}}$, we can also assume that $w \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X^{(2n-2}}$. After a finite number of steps, we deduce that $\operatorname{diam} W \geqslant 2 - \varepsilon$. Since ε is any positive number we conclude that $\operatorname{diam} W = 2$. $\square$

Let us remark that the above theorem extends Proposition 4.1, since every L-projection can be extended to an L-projection on $X^{(\infty}$ and so $C(X)$ is always contained in $C(X^{(\infty)})$.

A predual X of a non-reflexive JBW^*-triple satisfies that $C(X^{(\infty})$ is infinite-dimensional, and so the above result can be applied to the dual of any infinite-dimensional C^*-algebra (see [14, Corollary 11] and [5, Proposition 3.4]). We recall that the Cunningham algebra of $X^{(\infty}$ is infinite-dimensional for every non-reflexive L-embedded space X in view of Proposition 3.4.

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