A Combinatorial Lefschetz Fixed-Point Formula

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Communicated by Victor Klee
Received May 3, 1990; revised July 25, 1991

Let $K$ be any (finite) simplicial complex, and $K'$ a subdivision of $K$. Let $\varphi: K' \to K$ be a simplicial map, and, for all $j \geq 0$, let $\varphi_j$ denote the algebraical number of $j$-simplices $\mathcal{G}$ of $K'$ such that $\mathcal{G} \subset \varphi(\mathcal{G})$. From Hopf's alternating trace formula it follows that $\varphi_0 - \varphi_1 + \varphi_2 - \cdots = L(\varphi)$, the Lefschetz number of the simplicial map $\varphi: X \to X$. Here $X$ denotes the space of $|K|$ (or $|K'|$). A purely combinatorial proof of the case $K$ is a closed simplex (now $L(\varphi) = 1$) is given, thus solving a problem posed by Ky Fan in 1978.

Let $K$ be any (finite) simplicial complex, and $K'$ a subdivision of $K$. Let $\varphi: K' \to K$ be a simplicial map, and, for all $j \geq 0$, let $\varphi_j$ denote the algebraical number of $j$-simplices $\mathcal{G}$ of $K'$ such that $\mathcal{G} \subset \varphi(\mathcal{G})$. Here “algebraical” means that $\mathcal{G}$ is to be counted +1 if the orientation of $\mathcal{G}$ agrees with that of the bigger $j$-simplex $\varphi(\mathcal{G})$; i.e., if $\mathcal{G} = \{v_0, v_1, \ldots, v_j\}$ then $v_r = \sum_{s \geq 0} a_{rs} \varphi(v_s)$ with $\det(a_{rs}) > 0$, otherwise $-1$. With the notation explained above, a combinatorial version of Lefschetz's fixed-point formula runs as follows.

**Theorem 1.** $\varphi_0 - \varphi_1 + \varphi_2 - \cdots = L(\varphi)$, the Lefschetz number of the simplicial map $\varphi: X \to X$. 

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Here \( X \) denotes the space \(|K|\) (or \(|K'|\)), and, as usual, the \textit{Lefschetz number} is defined to be the alternating trace sum,

\[
\sum_{j \geq 0} (-1)^j \text{Tr}(\varphi_*: H_j(X; \mathbb{Q}) \to H_j(X; \mathbb{Q})),
\]

for the induced map in rational homology.

The celebrated 1926 fixed-point theorem of Lefschetz [6] asserts that if \( f: X \to X \) is a continuous map and \( L(f) \neq 0 \), then there exists \( x \in X \) such that \( f(x) = x \). Lefschetz's fixed-point theorem was proved by Hopf [5], as follows: If \( f(x) \neq x \) for all \( x \in X \) then we can find a triangulation \( K' \) of \( X \), and a simplicial approximation \( \varphi: K' \to K \) of \( f \) such that \( \varphi \notin \varphi(G) \) for all \( G \in K' \). As \( f \simeq \varphi \), \( L(f) = L(\varphi) = 0 \) by the above theorem. (Cf., e.g., Maunder [7, pp. 149–150, 157].)

For the case \( X = S^n \), an \( n \)-sphere, the above theorem gives a combinatorial formula for the (Brouwer) degree of \( \varphi \).

We state an important special case of Theorem 1 explicitly as

**Theorem 2.** Let \( K = \text{Cl}(\sigma) \) be a closed \( n \)-simplex, \( K' \) a subdivision of \( K \) and \( \varphi: K' \to K \) a simplicial map. Then

\[
\varphi_0 - \varphi_1 + \varphi_2 - \cdots + (-1)^n \varphi_n = 1.
\]

Theorem 2 was conjectured by Ky Fan in 1978. We are deeply indebted to Professor Ky Fan for calling attention to this conjecture. A purely combinatorial proof of Theorem 2 is given in Section 2. We wish to thank the referee for pointing out that, in fact, the more general Theorem 1 holds, is implicitly contained in [5], and can be proved by a well-known argument as in Section 3.

Note that the equality,

\[
\varphi_0 - \varphi_1 + \varphi_2 - \cdots = 1
\]

for all \( \varphi: K' \to K \), holds not only in the above case, but also whenever \( X \) has \textit{trivial rational homology}; e.g., for all triangulations \( K' \) of contractible polyhedra \( X \), or even-dimensional real projective spaces \( X = \mathbb{R}P^{2n} \).

By disregarding orientation in Theorem 2, we see that Sperner's lemma [8] and Ky Fan's lemma [4, pp. 523–524] are direct consequences of Theorem 2.

As an example for the case \( X \) has trivial rational homology in Theorem 1 we have a simplicial complex \( K \) with vertices \( a_0, a_1, \ldots, a_8 \) and the subdivision \( K' \) shown in Fig. 1. For a given vertex \( v \) of \( K' \), we identify \( \varphi(v) = a_i \) with \( \varphi(v) = i \). We have \( \varphi_0 = 5, \varphi_1 = 7, \varphi_2 = 3 \), so that \( \varphi_0 - \varphi_1 + \varphi_2 = 1 \).
We now prove Theorem 2 by a purely combinatorial method. For the proof, we associate to any \( \varphi: K' \to K \), another subdivision \( K'' \) of \( K \), and a simplicial map \( \tau: K'' \to K \), such that

\[
\varphi_0 - \varphi_1 + \varphi_2 - \cdots + (-1)^n \varphi_n = \tau_n.
\]

To define \( K'' \) (see Fig. 2) first obtain a subdivision of \( K \) by deleting \( \sigma \) from an octahedral \( n \)-sphere having \( \sigma \) and \( \bar{\sigma} \) as opposite \( n \)-simplices, then subdivide \( \bar{\sigma} \) further by a copy \( \bar{K} \), and finally subdivide the remaining \( n \)-simplices \( \theta \ast (\sigma \setminus \theta) \) by the \( n \)-simplices \( \mathcal{G} = \mathcal{G} \ast (\sigma \setminus \theta) \), \( \mathcal{G} \in K' \), \( \mathcal{G} \subset \theta \), and their faces. (The symbol "\( \ast \)" denotes the join operation.) Let

\[
\tau(v) = v \quad \text{if} \quad v \text{ is a vertex of } \sigma;
\]

\[
\tau(\bar{v}) = \varphi(v) \quad \text{if} \quad \bar{v} \text{ is a vertex of } \bar{K}.
\]

Clearly \( \mathcal{G} \subset \varphi(\mathcal{G}) = \theta \) iff \( \tau(\mathcal{G}) = \sigma \). By the oriented Sperner lemma (which was proved by Brown and Cairns [1]), we have \( \tau_n = 1 \). (A passing remark: Brown and Cairns' theorem can be proved by a "path-following" algorithm, see Cohen [2] and Ky Fan [3]). The required equality now follows by verifying directly that for \( \text{dim}(\mathcal{G}) \) even (resp., odd) the orientation of \( \mathcal{G} \)
agrees with that of $\varphi(\mathcal{F})$ iff that of $\mathcal{F}$ agrees (resp., disagrees) with that of $\tau(\mathcal{F})$. To see this, let

$$\sigma = \{a_0, a_1, \ldots, a_n\}, \quad \theta = \{a_{i_0}, a_{i_1}, \ldots, a_{i_j}\}, \quad \sigma \setminus \theta = \{a_{i_{j+1}}, \ldots, a_{i_n}\},$$

$$\mathcal{G} = \{v_0, v_1, \ldots, v_j\}, \quad \varphi(v_r) = a_{i_r}, \quad r = 0, 1, \ldots, j.$$

Then

$$\bar{\theta} \ast (\sigma \setminus \theta) = \{\bar{a}_{i_0}, \bar{a}_{i_1}, \ldots, \bar{a}_{i_j}, a_{i_{j+1}}, \ldots, a_{i_n}\},$$

$$\bar{\mathcal{G}} \ast (\sigma \setminus \theta) = \{ar{v}_0, \bar{v}_1, \ldots, \bar{v}_j, a_{i_{j+1}}, \ldots, a_{i_n}\}.$$

For simplicity, let (see Fig. 2)

$$\bar{a}_r = \sum_{s \geq 0} b_{rs} a_s, \quad r = 0, 1, \ldots, n; \quad (b_{rs}) = \frac{1}{2n+1} \begin{pmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 \end{pmatrix}.$$
Let

\[
\begin{pmatrix}
v_0 \\ v_1 \\ \vdots \\ v_j
\end{pmatrix} = B_1 \begin{pmatrix}
\varphi(v_0) \\ \varphi(v_1) \\ \vdots \\ \varphi(v_j)
\end{pmatrix}, \quad \begin{pmatrix}
\varphi(v_0) \\ \varphi(v_1) \\ \vdots \\ a_{ij+1} \\ a_{in}
\end{pmatrix} = B_2 \begin{pmatrix}
\tau(v_0) \\ \tau(v_1) \\ \vdots \\ \tau(v_j) \\ \tau(a_{i+1}) \\ \vdots \\ \tau(a_{n})
\end{pmatrix}.
\]

A computation shows that

\[
B_2 = \begin{pmatrix}
B_1 & 0 \\ 0 & I_{n-j}
\end{pmatrix} \begin{pmatrix}
B' & \tilde{2} \\ \tilde{2} & I_{n-j}
\end{pmatrix},
\]

where

\[
B' = \frac{1}{2n+1} \begin{pmatrix}
1 & 2 & \cdots & 2 \\
2 & 1 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
2 & 2 & \cdots & 1
\end{pmatrix}_{(j+1) \times (j+1)}
\]

and

\[
\tilde{2} = \begin{pmatrix}
2 & \cdots & 2 \\
\vdots & \ddots & \vdots \\
2 & \cdots & 2
\end{pmatrix}_{(j+1) \times (n-j)}
\]

Thus

\[
\det B_2 = (-1)^j (2n+1)^{-j-1} (2j + 1) \det B_1
\]

and, with this identity, the proof of Theorem 2 is complete.

For the proof of Theorem 1, let

\[
s: C_j(K; Q) \to C_j(K'; Q), \quad j \geq 0,
\]

denote the canonical chain subdivision. Composing it with the simplicial maps

\[
\varphi: C_j(K'; Q) \to C_j(K; Q),
\]
one obtains the chain endomorphisms

$$\varphi s: C_j(K; Q) \to C_j(K; Q).$$

Inspection shows that

$$\varphi_j = T_r(\varphi s: C_j(K; Q) \to C_j(K; Q)).$$

So the well-known alternating sum formula

$$\sum_{j \geq 0} (-1)^j T_r(\varphi s: C_j(K; Q) \to C_j(K; Q))$$

$$= \sum_{j \geq 0} (-1)^j T_r((\varphi s)_*: H_j(K; Q) \to H_j(K; Q))$$

implies the required equality. This completes the proof of Theorem 1.

Recall that the chain subdivision induces isomorphisms in homology,

$$s_*: H_j(K; Q) \cong H_j(K'; Q).$$

This follows from the fact that if $\tau: K' \to K$ is a simplicial map which images each vertex of $K'$ to a vertex of the simplex of $K$ containing it, then

$$\tau s = \text{Id}_K. \quad (\ast)$$

As usual we have identified the homologies of $K$ and $K'$ under $s_*$ and denoted either by $H_j(X; Q)$.

Finally, we mention that formula $(\ast)$ is equivalent to saying that for all $\sigma \in K$ the algebraical number of simplices $\mathcal{G}$ of $K'$ such that $\tau(\mathcal{G}) = \sigma$ equals 1. This oriented Sperner lemma [1] follows also, by induction on $n$, from Theorem 1 because it amounts to saying that for a closed $n$-simplex $K$ and $\tau: K' \to K$ as in $(\ast)$ one has $\tau_n = 1$.

REFERENCES


