JOURNAL OF COMBINATORIAL THEORY, Series A 61, 123–129 (1992)

Note

A Combinatorial Lefschetz Fixed-Point Formula

MAU-HSIANG SHIH

Department of Mathematics, Chung Yuan University, Chung-Li, Taiwan 32023

AND

SHYH-NAN LEE

Institute of Mathematics, National Central University, Chung-Li, Taiwan 32054

Communicated by Victor Klee

Received May 3, 1990; revised July 25, 1991

Let K be any (finite) simplicial complex, and K' a subdivision of K. Let $\varphi: K' \to K$ be a simplicial map, and, for all $j \geqslant 0$, let φ_j denote the algebraical number of j-simplices $\mathscr{S}$ of K' such that $\mathscr{S} \subset \varphi(\mathscr{S})$. From Hopf's alternating trace formula it follows that $\varphi_0 - \varphi_1 + \varphi_2 - \cdots = L(\varphi)$, the Lefschetz number of the simplicial map $\varphi: X \to X$. Here X denotes the space of $|K|$ (*or* $|K'|$). A purely combinatorial proof of the case $K = a$ closed simplex (now $L(\varphi) = 1$) is given, thus solving a problem posed by Ky Fan in 1978.

1

Let K be any (finite) simplicial complex, and K' a subdivision of K. Let $\varphi: K' \to K$ be a simplicial map, and for all $j \geqslant 0$, let φ_j denote the algebraical number of j-simplices $\mathscr{S}$ of K' such that $\mathscr{S} \subset \varphi(\mathscr{S})$. Here "algebraical" means that $\mathscr{S}$ is to be counted $+1$ if the orientation of $\mathscr{S}$ agrees with that of the bigger j-simplex $\varphi(\mathscr{S})$; i.e., if $\mathscr{S} = \{v_0, v_1, ..., v_j\}$ then $v_r = \sum_{s \geqslant 0} a_{rs} \varphi(v_s)$ with $\det(a_{rs}) > 0$, otherwise -1. With the notation explained above, a combinatorial version of Lefschetz's fixed-point formula runs as follows.

THEOREM 1. $\varphi_0 - \varphi_1 + \varphi_2 - \cdots = L(\varphi)$, *the Lefschetz number of the simplicial map* $\varphi: X \to X$.

0097-3165/92 $5.00

582a/61/1-9

Here X denotes the space $|K|$ (or $|K'|$), and, as usual, the *Lefschetz number* is defined to be the alternating trace sum,

$$\sum_{j\geq 0} (-1)^j T_r(\varphi_*: H_j(X; Q) \to H_j(X; Q)),$$

for the induced map in rational homology.

The celebrated 1926 fixed-point theorem of Lefschetz [6] asserts that if $f: X \to X$ is a continuous map and $L(f) \neq 0$, then there exists $\hat{x} \in X$ such that $f(\hat{x}) = \hat{x}$. Lefschetz's fixed-point theorem was proved by Hopf [5], as follows: If $f(x) \neq x$ for all $x \in X$ then we can find a triangulation K' of X, and a simplicial approximation $\varphi: K' \to K$ of f such that $\mathscr{G} \not\subset \varphi(\mathscr{G})$ for all $\mathscr{G} \in K'$. As $f \simeq \varphi$, $L(f) = L(\varphi) = 0$ by the above theorem. (Cf., e.g., Maunder [7, pp. 149–150, 157].)

For the case $X = S^n$, an n-sphere, the above theorem gives a combinatorial formula for the (*Brouwer*) *degree* of φ.

We state an important special case of Theorem 1 explicitly as

THEOREM 2. *Let $K = \mathrm{Cl}(\sigma)$ be a closed n-simplex, K' a subdivision of K and $\varphi: K' \to K$ a simplicial map. Then*

$$\varphi_0 - \varphi_1 + \varphi_2 - \cdots + (-1)^n \varphi_n = 1.$$

Theorem 2 was conjectured by Ky Fan in 1978. We are deeply indebted to Professor Ky Fan for calling attention to this conjecture. A purely combinatorial proof of Theorem 2 is given in Section 2. We wish to thank the referee for pointing out that, in fact, the more general Theorem 1 holds, is implicitly contained in [5], and can be proved by a well-known argument as in Section 3.

Note that the equality,

$$\varphi_0 - \varphi_1 + \varphi_2 - \cdots = 1$$

for all $\varphi: K' \to K$, holds not only in the above case, but also whenever X has *trivial rational homology*; e.g., for all triangulations K' of contractible polyhedra X, or even-dimensional real projective spaces $X = RP^{2n}$.

By disregarding orientation in Theorem 2, we see that Sperner's lemma [8] and Ky Fan's lemma [4, pp. 523–524] are direct consequences of Theorem 2.

As an example for the case X has trivial rational homology in Theorem 1 we have a simplicial complex K with vertices $a_0, a_1, ..., a_8$ and the subdivision K' shown in Fig. 1. For a given vertex v of K', we identify $\varphi(v) = a_i$ with $\varphi(v) = i$. We have $\varphi_0 = 5$, $\varphi_1 = 7$, $\varphi_2 = 3$, so that $\varphi_0 - \varphi_1 + \varphi_2 = 1$.

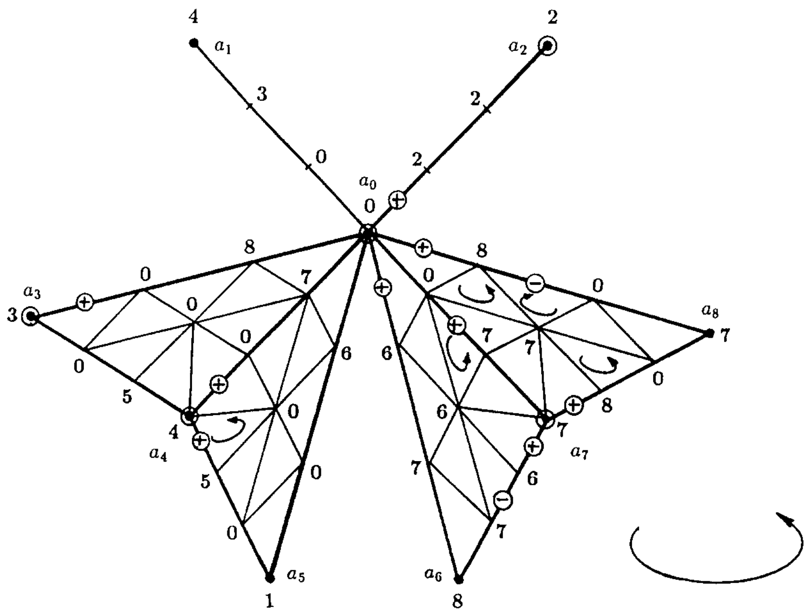

FIGURE 1

We now prove Theorem 2 by a purely combinatorial method. For the proof, we associate to any $\varphi\colon K' \to K$, another subdivision K'' of K, and a simplicial map $\tau\colon K'' \to K$, such that

$$\varphi_0 - \varphi_1 + \varphi_2 - \cdots + (-1)^n \varphi_n = \tau_n.$$

To define K'' (see Fig. 2) first obtain a subdivision of K by deleting σ from an octahedral n-sphere having σ and $\bar{\sigma}$ as opposite n-simplices, then subdivide $\bar{\sigma}$ further by a copy $\overline{K'}$, and finally subdivide the remaining n-simplices $\bar{\theta} * (\sigma\backslash\theta)$ by the n-simplices $\tilde{\mathscr{G}} = \bar{\mathscr{G}} * (\sigma\backslash\theta)$, $\mathscr{G} \in K'$, $\mathscr{G} \subset \theta$, and their faces. (The symbol "$*$" denotes the join operation.) Let

$$\tau(v) = v \qquad \text{if } v \text{ is a vertex of } \sigma;$$

$$\tau(\bar{v}) = \varphi(v) \qquad \text{if } \bar{v} \text{ is a vertex of } \overline{K'}.$$

Clearly $\mathscr{G} \subset \varphi(\mathscr{G}) = \theta$ iff $\tau(\tilde{\mathscr{G}}) = \sigma$. By the *oriented Sperner lemma* (which was proved by Brown and Cairns [1]), we have $\tau_n = 1$. (A passing remark: Brown and Cairns' theorem can be proved by a "path-following" algorithm, see Cohen [2] and Ky Fan [3]). The required equality now follows by verifying directly that for $\dim(\mathscr{G})$ even (resp., odd) the orientation of $\mathscr{G}$

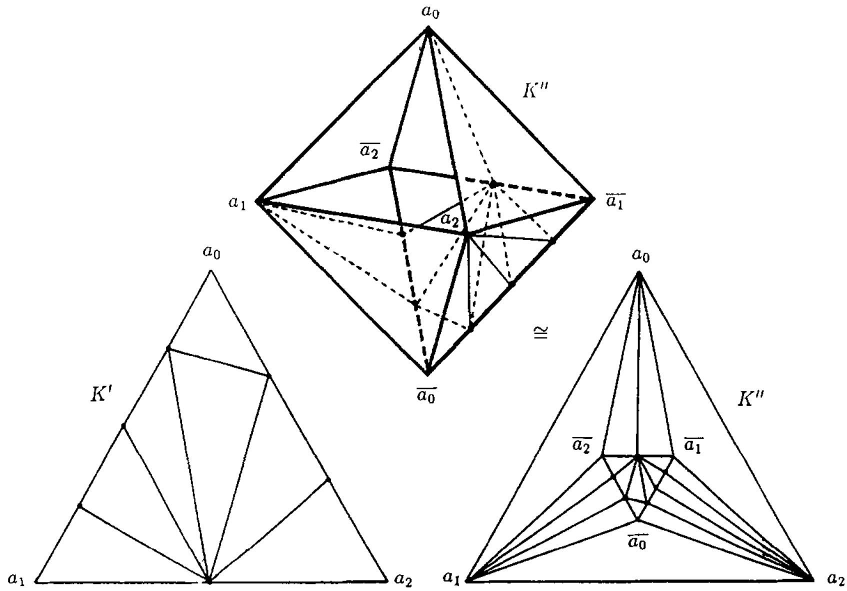

FIGURE 2

agrees with that of $\varphi(\mathscr{G})$ iff that of $\tilde{\mathscr{G}}$ agrees (resp., disagrees) with that of $\tau(\tilde{\mathscr{G}})$. To see this, let

$$\sigma = \{a_0, a_1, ..., a_n\}, \qquad \theta = \{a_{i_0}, a_{i_1}, ..., a_{i_j}\}, \qquad \sigma \backslash \theta = \{a_{i_{j+1}}, ..., a_{i_n}\},$$

$$\mathscr{G} = \{v_0, v_1, ..., v_j\}, \qquad \varphi(v_r) = a_{i_r}, \qquad r = 0, 1, ..., j.$$

Then

$$\bar{\theta} * (\sigma \backslash \theta) = \{\overline{a_{i_0}}, \overline{a_{i_1}}, ..., \overline{a_{i_j}}, a_{i_{j+1}}, ..., a_{i_n}\},$$

$$\bar{\mathscr{G}} * (\sigma \backslash \theta) = \{\overline{v_0}, \overline{v_1}, ..., \overline{v_j}, a_{i_{j+1}}, ..., a_{i_n}\}.$$

For simplicity, let (see Fig. 2)

$$\overline{a_r} = \sum_{s \geq 0} b_{rs} a_s, \; r = 0, 1, ..., n; \qquad (b_{rs}) = \frac{1}{2n+1} \begin{pmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 \end{pmatrix}.$$

Let

$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_j \end{pmatrix} = B_1 \begin{pmatrix} \varphi(v_0) \\ \varphi(v_1) \\ \vdots \\ \varphi(v_j) \end{pmatrix}, \qquad \begin{pmatrix} \overline{v_0} \\ \overline{v_1} \\ \vdots \\ \overline{v_j} \\ a_{i_{j+1}} \\ \vdots \\ a_{i_n} \end{pmatrix} = B_2 \begin{pmatrix} \tau(\overline{v_0}) \\ \tau(\overline{v_1}) \\ \vdots \\ \tau(\overline{v_j}) \\ \tau(a_{i_{j+1}}) \\ \vdots \\ \tau(a_{i_n}) \end{pmatrix}.$$

A computation shows that

$$B_2 = \begin{pmatrix} B_1 & 0 \\ 0 & I_{n-j} \end{pmatrix} \begin{pmatrix} B' & \tilde{2} \\ 0 & I_{n-j} \end{pmatrix},$$

where

$$B' = \frac{1}{2n+1} \begin{pmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 \end{pmatrix}_{(j+1)\times(j+1)}$$

and

$$\tilde{2} = \begin{pmatrix} 2 & \cdots & 2 \\ \vdots & & \vdots \\ 2 & \cdots & 2 \end{pmatrix}_{(j+1)\times(n-j)}$$

Thus

$$\det B_2 = (-1)^j\,(2n+1)^{-(j+1)}\,(2j+1) \det B_1$$

and, with this identity, the proof of Theorem 2 is complete.

3

For the proof of Theorem 1, let

$$s\colon C_j(K; Q) \to C_j(K'; Q), \qquad j \geqslant 0,$$

denote the canonical chain subdivision. Composing it with the simplicial maps

$$\varphi\colon C_j(K'; Q) \to C_j(K; Q),$$

one obtains the chain endomorphisms

$$\varphi s\colon C_j(K; Q) \to C_j(K; Q).$$

Inspection shows that

$$\varphi_j = T_r(\varphi s\colon C_j(K; Q) \to C_j(K; Q)).$$

So the well-known alternating sum formula

$$\sum_{j \geq 0} (-1)^j T_r(\varphi s\colon C_j(K; Q) \to C_j(K; Q))$$
$$= \sum_{j \geq 0} (-1)^j T_r((\varphi s)_*\colon H_j(K; Q) \to H_j(K; Q))$$

implies the required equality. This completes the proof of Theorem 1.

Recall that the chain subdivision induces isomorphisms in homology,

$$s_*\colon H_j(K; Q) \xrightarrow{\cong} H_j(K'; Q).$$

This follows from the fact that if $\tau\colon K' \to K$ is a simplicial map which images each vertex of K' to a vertex of the simplex of K containing it, then

$$\tau s = \mathrm{Id}_K. \qquad (*)$$

As usual we have identified the homologies of K and K' under s_* and denoted either by $H_j(X; Q)$.

Finally, we mention that formula $(*)$ is equivalent to saying that for all $\sigma \in K$ the algebraical number of simplices $\mathscr{G}$ of K' such that $\tau(\mathscr{G}) = \sigma$ equals 1. This *oriented Sperner lemma* [1] follows also, by induction on n, from Theorem 1 because it amounts to saying that for a closed n-simplex K and $\tau\colon K' \to K$ as in $(*)$ one has $\tau_n = 1$.

References

1. A. B. Brown and S. S. Cairns, Strengthening of Sperner's lemma applied to homology, *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), 113–114.
2. D. I. A. Cohen, On the Sperner lemma, *J. Combin. Theory* **2** (1967), 585–587.
3. K. Fan, Simplicial maps from an orientable n-pseudomanifold into S^m with the octahedral triangulation, *J. Combin. Theory* **2** (1967), 588–602.

4. K. FAN, Some properties of convex sets related to fixed point theorems, *Math. Ann.* **266** (1984), 519–537.
5. H. HOPF, A new proof of the Lefschetz formula on invariant points, *Proc. Nat. Acad. Sci. U.S.A.* **14** (1928), 149–153.
6. S. LEFSCHETZ, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.
7. C. R. F. MAUNDER, "Algebraic Topology," Cambridge Univ. Press, London, 1970.
8. E. SPERNER, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6** (1928), 265–272.