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Alternate proofs of two theorems of Philip Hall on finite p-groups, and some related results

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To the 100th anniversary of Philip Hall (1904–1982)

Abstract

New very detailed proofs of Theorems 2.5 and 2.64 from the seminal paper of Philip Hall [P. Hall, On a theorem of Frobenius, Proc. London Math. Soc. Ser. (2) 40 (1936) 468–501] are given. A number of generalizations of these theorems are proved. For example, we show that if G is a p-group of order $p^{k(p-1)+3}$, $k > 2$, and exponent $> p^k$ with $\Omega_k(G) = G$, then either G is of maximal class or G possesses a normal subgroup H of order p^p and exponent p such that G/H is of maximal class. Counting theorems play important role in this note.

Probably, the following remarkable 'conditionless' structure theorem[1] is one of the deepest consequences of Hall's theory of regular p-groups.

Theorem 1 (P. Hall [6, Theorem 2.5]). *Let $H > \{1\}$ be a normal subgroup of a p-group G. Then there exists in H a chain $\mathcal{C}: \{1\} = L_0 < L_1 < \cdots < L_n = H$ of G-invariant subgroups with the properties* $(i = 1, \dots, n)$:

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[1] As far as I know, this is the first citing the above theorem since its publication in 1936.

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(a) *L_i/L_{i-1} is of order $\leqslant p^{p-1}$ and exponent p, and*
(b) *either the order of L_i is exactly $p^{(p-1)i}$, or else $L_i = \Omega_i(H)$.*

A chain $\mathcal{C}$, having properties (a) and (b) of Theorem 1, is said to be a $(p-1)$-*admissible Hall chain* in H, and this is agrees with the definition of a k-admissible Hall chain following Supplement 2 to Theorem 1. The length of $\mathcal{C}$ is at least $\log_p(\exp(H))$ since $\exp(L_i) \leqslant p^i$ for all i.

It follows from Theorem 1 that if $p > 2$, $e > 1$, $\exp(G) = p^e$, and $|G| \leqslant p^{(p-1)(e-1)}$, then, for some natural number $k < e$, the p-group G has a characteristic subgroup of order $< p^{(p-1)k}$ and exponent p^k. Indeed, let $\mathcal{C}: \{1\} = L_0 < L_1 < \cdots < L_n = G$ be a $(p-1)$-admissible Hall chain in G, which exists by Theorem 1. Since $n \geqslant e$, there exists a natural number $k < e$ such that $|L_k| < p^{(p-1)k}$. In that case, by Theorem 1, $L_k = \Omega_k(G)$, and this subgroup is characteristic in G. It is interesting to give a proof of this assertion independent of Theorem 1.

The original proof of Theorem 1, a skillful and fairly difficult inductive argument, contains a gap. Namely, in [6, p. 481], the number i_1 is defined (this number plays the crucial role in Hall's proof). However, the case in which i_1 does not exist, is overlooked. This gap is easily repaired in part (iii) of our proof of Theorem 1. In view of Remark 2, we do not use the number i_1 at all. All prerequisites for the presented proof of Theorem 1 are contained in §2 of [6] so that proof is a real simplification of the original one. Our proof, especially in part (i), uses some ideas of Hall's proof. As a by-product of this approach, two additional new results, Supplements 1 and 2 to Theorem 1 are presented (these supplements are not consequences of Theorem 1).

Theorem 2.64 in [6] asserts that if a p-group G has order $\leqslant p^{(p-1)k+1}$ and $\Omega_k(G) = G$, then $\exp(G) \leqslant p^k$. This follows immediately from

Theorem 2. *Let $k \in \mathbb{N}$ and let G be a p-group. If G has no subgroup of order $p^{(p-1)k+1}$ and exponent $\leqslant p^k$, then $\exp(\Omega_k(G)) \leqslant p^k$.*

Suppose that G is as in the statement of [6, Theorem 2.64]. Assuming that $\exp(G) > p^k$, we see that G has no subgroup of exponent $\leqslant p^k$ and order $p^{(p-1)k+1} (\geqslant |G|)$. In that case, by Theorem 2, $\exp(\Omega_k(G)) \leqslant p^k$, contrary to the assumption since $\Omega_k(G) = G$.

The proof of Theorem 2 is based on Lemma 3(i). In part (iii) of the proof of Theorem 1 we use a partial case of Theorem 2. Our proof of Theorem 2 is independent of Theorem 1, in contrast to the original proof of [6, Theorem 2.64], hence, it is essentially simpler. As p-groups of maximal class and order $> p^{(p-1)k+1}$ show, Theorem 2 yields the best possible result. Theorem 2 is a partial case of Theorem 4, which is not so elementary since it is based on Blackburn's theory of p-groups of maximal class.

The collecting formula (the so called Hall–Petrescu formula) is used in one place of Hall's proof of Theorem 1 essentially. In our proof, a good substitute for that formula is Theorem 2.

According to Blackburn, a p-group G is said to be *absolutely regular* if $|G : \mho_1(G)| < p^p$. By Hall's regularity criterion (see Lemma 3(c)), absolutely regular p-groups, as their name indicates, are regular. All necessary prerequisites on regular and absolutely regular p-groups, presented in Lemma 3(c),(d).

We use the standard notation common for papers on p-groups (see [1,2]). Denote $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1\rangle$ and $\mho_n(G) = \langle x^{p^n} \mid x \in G\rangle$. If $A < G$, $\exp(A) \leqslant p^e$ and $k < e$, then $\mho_k(A) \leqslant \Omega_{e-k}(\mho_k(G))$ since $\mho_k(A)$ is generated by elements of order $\leqslant p^{e-k}$. Let

$$\mho^0(G) = G, \qquad \mho^1(G) = \mho_1(G), \qquad \mho^{i+1}(G) = \mho_1\big(\mho^i(G)\big), \quad i = 1, 2, \ldots.$$

Since $\exp(G/\mho^i(G)) \leqslant p^i$, then $\mho_i(G) \leqslant \mho^i(G)$. The subgroups $\mho^i(G)$ are characteristic in G and control the structure of the subgroups $\mho_i(G)$.

In what follows we use the bar convention.

To facilitate the proof of Theorem 1 and all subsequent results, it will be convenient to begin by proving Lemma 3, Theorem 2 and the assertions contained in Remarks 1–4.

Lemma 3. *Let G be a p-group.*

(a) *If G is irregular, it possesses a characteristic subgroup of order $\geqslant p^{p-1}$ and exponent p. In particular, if G is an arbitrary p-group and H, a normal subgroup of G, has a subgroup of order $p^k \leqslant p^{p-1}$ and exponent p, then H possesses a G-invariant subgroup of order p^k and exponent p.*

(b) *Suppose that W, a normal subgroup of G, has a subgroup of order p^{p-1} and exponent p, let $R < W$ be a G-invariant subgroup of order $p^k < p^{p-1}$ and exponent p. Then there exists a G-invariant subgroup $H < W$ of order p^{p-1} and exponent p such that $R < H$. On the other hand, if W has no G-invariant subgroup of order p^{p-1} and exponent p, it is absolutely regular.*

(c) (Hall) (i) *p-groups of class $< p$ (so also groups of order p^p) are regular.* (ii) *Hall regularity criterion* [6, Theorem 2.3]: *absolutely regular p-groups are regular.* (iii) *If G is regular, then* $\exp(\Omega_n(G)) \leqslant p^n$ *and* $|\Omega_n(G)| = |G/\mho_n(G)|$ *for* $n \in \mathbb{N}$.

(d) *Sections of absolutely regular p-groups are absolutely regular.*

(e) *Let H be a normal subgroup of G, where $|H| \leqslant p^{(p-1)e}$ and* $\exp(H) = p^e$. *Then there exists a chain $\{1\} = T_0 < T_1 < \cdots < T_e = H$ of length e of G-invariant subgroups such that*

$$p^{p-1} \geqslant |T_1/T_0| \geqslant |T_2/T_1| \geqslant \cdots \geqslant |T_e/T_{e-1}|, \quad \exp(T_i/T_{i-1}) = p, \quad i = 1, \ldots, e.$$

If, in addition, $|H| = p^{(p-1)e}$, then $|T_i/T_{i-1}| = p^{p-1}$ for all i.

(f) *Let H be a normal subgroup of G, where $|H| = p^{(p-1)e}$ and* $\exp(H) \leqslant p^e$. *Then there exists a chain $\{1\} = T_0 < T_1 < \cdots < T_e = H$ of length e of G-invariant subgroups such that $|T_i/T_{i-1}| = p^{p-1}$ and* $\exp(T_i/T_{i-1}) = p$ *for $i = 1, \ldots, e$.*

(g) *Suppose that W, a normal subgroup of G, is neither absolutely regular nor of maximal class.* (i) [2, Theorem 7.6] *The number of subgroups of order p^p and exponent p in W is* $\equiv 1 \pmod p$. (ii) [3, Corollary 13.3] *If $A < W$ be a G-invariant subgroup of order $p^a < p^p$, then there exists in W a G-invariant subgroup H of order p^p and exponent p containing A.*

(h) (i) *If G is absolutely regular and $|G| > p^{(p-1)k}$, then* $\exp(G) > p^k$. (ii) *If G is of maximal class and $|G| > p^{(p-1)k+1}$, then* $\exp(G) > p^k$. *Any two irregular p-groups of maximal class and the same order have the same exponent.*[2]

(i) *Suppose that $|G| \leqslant p^{pk}$. Then $\mho^{k-1}(G)$ is either absolutely regular or of order p^p and exponent p (the same is true for $\mho_{k-1}(G)(\leqslant \mho^{k-1}(G))$). If, in addition, $|G| < p^{pk}$, the above two subgroups are absolutely regular.*

(j) [2, Theorem 7.4(b)] *Suppose that G is irregular but it is not of maximal class. If G contains a subgroup H of maximal class and index p, then $G/\mathrm{K}_p(G)$ is of order p^{p+1} and exponent p. In that case,* $\exp(G) = \exp(H)$.

(k) [2, Theorem 13.21] *Let $A < G$ and suppose that all subgroups of G that contain A as a subgroup of index p, are of maximal class. Then G is also of maximal class.*

(l) (Blackburn, see [2, Theorem 9.6]) *Let G be of maximal class of order $> p^{p+1}$ and exponent p^e. Then G has no normal subgroup of order p^p and exponent p and exactly p maximal subgroups, say $M_1, \dots, M_p$, of G are of maximal class and one of maximal subgroups of G, say G_1, is absolutely regular and* $\exp(G_1) = \exp(G)$. *Next,* $\exp(M_i) < \exp(G)$ *if and only if $|G| = p^{(p-1)e+2}$ and $\mathrm{K}_p(G) = \mho_1(G)$ has exponent p^{e-1}. Regular epimorphic images of G are of exponent p.*

(m) (Blackburn, see [3, Theorems 9.5 and 9.6]) *If G is of maximal class, then $\Omega_2(G) = G$. If, in addition, G is irregular, then its subgroups of orders $> p^p$ are either absolutely regular and contained in G_1 (see* (l)) *or of maximal class. Next, if $|G| > p^p$, then G is irregular and $|G/\mho_1(G)| = p^p$.*

Proof. (a) For a proof of the first assertion, see [2, the paragraph preceding Theorem 7.8]. Let us prove the second assertion. If H is regular, the desired subgroup is contained in the G-invariant subgroup $\Omega_1(H)$ of exponent p (see (c)). Now let H be irregular. Then, by the first assertion, H has a characteristic subgroup K of order $\geqslant p^{p-1}$ and exponent p, and our claim now is obvious.

(b) By (a), there exists a G-invariant subgroup $H \leqslant W$ of order p^{p-1} and exponent p. Set $D = RH$; then D is normal in G. Clearly, $\mathrm{cl}(D) < p$ so D is regular, and we conclude that $\exp(D) = p$ (see (c)) The desired subgroup is any G-invariant subgroup L of order p^{p-1} such that $R < L \leqslant D$. Now suppose that W has no G-invariant subgroup of order p^{p-1} and exponent p. Then, by Lemma 3(g)(i), W is either absolutely regular or irregular of maximal class. In the second case, however, $\Omega_1(\Phi(W))$ is of order p^{p-1} and exponent p, contrary to the assumption.

(d) Since epimorphic images of G are absolutely regular (this is obvious), it suffices to show that any subgroup U of G is absolutely regular. Indeed, by (c),

$$\left|U/\mho_1(U)\right| = \left|\Omega_1(U)\right| \leqslant \left|\Omega_1(G)\right| = \left|G/\mho_1(G)\right| \leqslant p^{p-1},$$

and we are done.

(e) If H is absolutely regular, then $\{1\} < \Omega_1(H) < \cdots < \Omega_e(H) = H$ is the desired chain. Now let H be not absolutely regular. We use induction on $|H|$. The subgroup $L =$

[2] There exist, for $p > 2$, two regular p-groups of maximal class and the same order which have distinct exponents.

$\mho_{e-1}(H)$ is absolutely regular and $\exp(L) = p$, $|L| \leqslant p^{p-1}$ (see (i)). By (b), $L \leqslant U \leqslant H$, where U is G-invariant of order p^{p-1} and exponent p. Since $|H/U| \leqslant p^{(p-1)(e-1)}$ and $\exp(H/U) = p^{e-1}$, there is, by induction, a chain $U/U = T_1/U < \cdots < T_e/U = H/U$ of G-invariant subgroups such that $\exp(T_i/T_{i-1}) = p$ for $i = 1, \ldots, e$ (here $T_0 = \{1\}$) and

$$p^{p-1} \geqslant |T_1/T_0| \geqslant |T_2/T_1| \geqslant \cdots \geqslant |T_e/T_{e-1}|.$$

Then $\{1\} = T_0 < T_1 < \cdots < T_e = H$ is the desired chain.

(f) In view of (e), one may assume that $\exp(H) < p^e$ so H is not absolutely regular, by (h), below. Then H possesses a G-invariant subgroup T_1 of order p^{p-1} and exponent p, by (c). Since $|H/T_1| = p^{(p-1)(e-1)}$ and $\exp(H/T_1) \leqslant \exp(H) \leqslant p^{e-1}$, there is, by induction, a chain $T_1/T_1 < T_2/T_1 < \cdots < T_e/T_1 = H/T_1$ of G-invariant subgroups such that T_{i+1}/T_i is of order p^{p-1} and exponent p, $i = 1, \ldots, e-1$. Then $\{1\} = T_0 < T_1 < \cdots < T_e = H$ is the desired chain.

(h) Parts (i), (ii) follow from (c), (d) and (l).

(i) Assume that $\mho^{k-1}(G)$ is not absolutely regular. One may assume that $k > 1$. We have $|\mho^{i-1} : \mho^i(G)| \geqslant p^p$ for $i = 1, \ldots, k$, by (d). In that case,

$$\left|G : \mho^{k-1}(G)\right| = \prod_{i=1}^{k-1} \left|\mho^{i-1}(G) : \mho^i(G)\right| \geqslant p^{p(k-1)}$$

so $|\mho^{k-1}(G)| \leqslant p^p$. In that case, if $\mho^{k-1}(G)$ is of exponent $> p$, it is absolutely regular. In view of $\mho_{k-1}(G) \leqslant \Omega_1(\mho^{k-1}(G))$, we are done.

(j) We have to check the last equality only. In the case under consideration, $\mathrm{K}_p(G) = \mho_1(H)$ and $\exp(\mho_1(H)) = \frac{1}{p}\exp(H)$ (Lemma 3(l)). Then

$$\exp(G) \leqslant \exp\bigl(\mathrm{K}_p(G)\bigr)\exp\bigl(G/\mathrm{K}_p(G)\bigr) = p \cdot \exp\bigl(\mho_1(H)\bigr) = \exp(H)$$

since H is of maximal class, and we are done since $\exp(G) \geqslant \exp(H)$. □

Remarks. Let G be a p-group and $k, j \in \mathbb{N}$.

1. Let $\exp(\Omega_k(G)) \leqslant p^k$ and let $G/\Omega_k(G)$ be regular. We claim that $\exp(\Omega_{k+j}(G)) \leqslant p^{p+j}$ and $\Omega_j(G/\Omega_k(G)) = \Omega_{k+j}(G)/\Omega_k(G)$. Indeed, set $H = \Omega_k(G)$ and $F/H = \Omega_j(G/H)$. One may assume that $H < G$; then $\exp(H) = p^k$. If $x \in F$, then $x^{p^j} \in H$ (Lemma 3(c)(iii)) so $o(x) \leqslant p^{k+j}$ and $F \leqslant \Omega_{i+j}(G)$. Now let $y \in G$ with $o(y) \leqslant p^{k+j}$. Then $y^{p^j} \in H$ so $yH \in F/H$ and $y \in F$, and we conclude that $\Omega_{i+j}(G) \leqslant F$.

2. Let H be a normal subgroup of G, $\exp(\Omega_k(H)) = p^k$, $H/\Omega_k(H)$ is absolutely regular and $|\Omega_k(H)| \leqslant p^{(p-1)k}$. Let $\{1\} = L_0 < L_1 < \cdots < L_k = \Omega_k(H)$ be a $(p-1)$-admissible Hall chain in $\Omega_k(H)$ which exists by Lemma 3(e). For a nonnegative integer s, put $L_{k+s}/L_k = \Omega_s(H/L_k)$. We claim that $\{1\} = L_0 < L_1 < \cdots < L_k < L_{k+1} \cdots < H$ is a $(p-1)$-admissible Hall chain in H. Indeed, the factors of the above chain are of order $\leqslant p^{p-1}$ and exponent p and $\Omega_i(\Omega_k(H)) = \Omega_i(H)$ for $i \leqslant k$, and we are done (see Remark 1).

3. Let M be a normal subgroup of G and $\Omega_j(G/M) \leqslant H/M$ for some $H \leqslant G$. Then $\Omega_j(G) \leqslant H$. Indeed, if $x \in G$ with $o(x) \leqslant p^j$, then $o(xM) \leqslant p^j$ so $xM \leqslant \Omega_j(G/M) \leqslant H/M$ and $x \in H$.

4. Let H be a normal subgroup of G and let $F_0 \leqslant H$ be a G-invariant subgroup of order p. Suppose that H/F_0 is of order $p^{(p-1)e}$ and exponent $\leqslant p^e$. We claim that there is in H a $(p-1)$-admissible Hall chain of length $e+1$ with last index $= p$. One may assume that $e > 0$. Set $\bar{G} = G/F_0$. By Lemma 3(f), there is a $(p-1)$-admissible Hall chain $\{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_e = \bar{H}$ in $\bar{H}$. We proceed by induction on e. Suppose that there is a $(p-1)$-admissible Hall chain $\{1\} = L_1 < \cdots < L_{e-1} < F_{e-1}$ in F_{e-1} such that $|F_{e-1}/L_{e-1}| = p$. Then H/L_{e-1} is of order p^p so regular, and H/F_{e-1} is of order p^{p-1} and exponent p. It follows that $\Omega_1(H/L_{e-1})$ is of order $\geqslant p^{p-1}$ and exponent p (Lemma 3(c)(iii)). Let L_e/L_{e-1} be an arbitrary G-invariant subgroup of order p^{p-1} in $\Omega_1(H/L_{e-1})$ (see Lemma 3(c) again). Then $\{1\} = L_0 < L_1 < \cdots < L_{e-1} < L_e < H$ is the desired chain of length $e+1$ in H.

Proof of Theorem 2. We proceed by induction on $|G|$ and k assuming that G is a minimal counterexample. Then $\Omega_k(G) > p^k$ so G is irregular (Lemma 3(c)), and $\exp(G) \geqslant p^{k+1}$.

Let $k = 1$ and let R, a normal subgroup of G, be of exponent p of maximal order. Since G has no subgroup of order $p^{(p-1)1+1} = p^p$ and exponent p, we get $|R| = p^{p-1}$ (Lemma 3(a)). If $x \in G - R$ is of order p, then $S = \langle x, R\rangle$ is of order $p^p = p^{(p-1)1+1}$ and exponent p (Lemma 3(c)), a contradiction. Thus, $R = \Omega_1(G)$ so $\exp(\Omega_1(G)) = p$, and the theorem is true for $k = 1$.

Now we let $k > 1$. Then G has a noncyclic subgroup of order p^{k+1} (otherwise G is cyclic) so $p > 2$. If $M < G$ is maximal, then $\exp(M) \geqslant p^k$ since $\exp(G) > p^k$. Let $A < G$ be a subgroup of maximal order among subgroups of exponent $\leqslant p^k$; then $|A| \leqslant p^{(p-1)k}$, by hypothesis, and $A < G$ since $\exp(A) < \exp(G)$. Let $A \leqslant M < G$, where $|G:M| = p$; then $A \leqslant \Omega_k(M)$. By induction, $\exp(\Omega_k(M)) = p^k$ so $\Omega_k(M) = A$, whence A is normal in G and $\exp(A) = p^k$ since $\exp(M) \geqslant p^k$. By assumption, there is $g \in G - A$ with $o(g) \leqslant p^k$. Then $g^p \in M$ so $g^p \in \Omega_{k-1}(M) \leqslant A$. Set $B = \langle g, A\rangle$; then $|B| = p|A| > |A|$. If $B \leqslant F < G$, where $|G : F| = p$, then $B \leqslant \Omega_k(F)$ has exponent p^k (here we use induction), contrary to the choice of A. Thus, F does not exist so $B = G$ and $|G : A| = p$, $|G| \leqslant p^{(p-1)k+1} < p^{pk}$ and $\exp(G) = p^{k+1}$. Therefore, by Lemma 3(i), $\mho_{k-1}(G)$ is absolutely regular since it has an element of order p^2. Let $\Omega_1(\mho_1(G)) \leqslant H$, where H is a G-invariant subgroup of order p^{p-1} and exponent p (H exists, by Lemma 3(b)). Then $\mho_{k-1}(A) \leqslant \Omega_1(\mho_{k-1}(G)) \leqslant H$ since $\mho_{k-1}(A)$ is generated by elements of order p. If $H \nleqslant A$, then $G = AH$ and $G/(H \cap A) = (A/(H \cap A)) \times (H/(H \cap A))$ is of exponent p^{k-1}. In that case, $\exp(G) = p^k$, a contradiction. Thus, $H \leqslant A$. Set $\bar{G} = G/H$. Let $x \in G - A$ be such that $o(x)$ is as small as possible; then $o(x) \leqslant p^k$. In that case, $\bar{G} = \langle \bar{x}, \bar{A}\rangle$, $\exp(\bar{A}) = p^{k-1}$, $|\bar{A}| \leqslant p^{(p-1)(k-1)}$ and $o(\bar{x}) \leqslant p^{k-1}$ since $x^{p^{k-1}} \leqslant \Omega_1(\mho_{k-1}(G)) \leqslant H$. We also have $|\bar{G}| \leqslant p^{(p-1)(k-1)+1}$ and $\Omega_{k-1}(\bar{G}) = \bar{G}$. The group $\bar{G}$ has no subgroup of order $p^{(p-1)(k-1)+1}$ ($\geqslant |\bar{G}|$) and exponent $\leqslant p^{k-1}$. Therefore, by induction, $\exp(\Omega_{k-1}(\bar{G})) = p^{k-1}$ so $\exp(\bar{G}) = p^{k-1}$, and we have $\exp(G) \leqslant p^k$, a final contradiction. □

Remarks. Let G be a p-group and $k \in \mathbb{N}$.

5. Let $\Omega_k(G) = G$. If A is maximal among proper subgroups X of G satisfying $\Omega_k(X) = X$, then $|G : A| = p$. Indeed, assume that A is not normal in G. Take $x \in$

$G - \mathrm{N}_G(A)$. Let $A < M < G$, where M is maximal in G. Then $A \neq A^x \leqslant M$, $A < H = \langle A, A^x \rangle \leqslant M < G$ and $\Omega_k(H) = H$, contrary to the choice of A. Thus, A is normal in G. Let $y \in G - A$ be of minimal order; then $o(y) \leqslant p^k$, $y^p \in A$, $\Omega_k(\langle y, A \rangle) = \langle y, A \rangle > A$ so $G = \langle y, A \rangle > A$ and $|G : A| = p$, as was to be shown.

6. Let $A < G$ be maximal among subgroups of G of exponent $\leqslant p^k$. We claim that if $|A| \leqslant p^{(p-1)k}$, then $A = \Omega_k(G)$. Assume that this is false; then $A < G$. Set $N = \mathrm{N}_G(A)$. Assume that $N < G$. Then, by induction, $A = \Omega_k(N)$ is characteristic in N so $N = G$ and $A = \Omega_k(G)$, i.e., G is not a counterexample. Thus, A is normal in G. Let $y \in G - A$ be of minimal order; then $o(y) \leqslant p^k$ and $y^p \in A$. Set $B = \langle y, A \rangle$. Then $|B| \leqslant p^{(p-1)k+1}$ and $\Omega_k(B) = B$. It follows from Theorem 2 that $\exp(B) = p^k$, contrary to the choice of A. Thus, $A = \Omega_k(G)$, as was to be shown. (Compare with Theorem 2.)

If, in Remark 6, A is of order $p^{(p-1)k+1}$, it is not necessarily normal in G (let G be a p-group of maximal class and order $\geqslant p^{(p-1)(k-1)+3}$; if G contains a subgroup A of order $p^{(p-1)k+1}$ and exponent p^k, it is maximal among subgroups of G of exponent p^k, but $A < \Omega_k(G)$). See, however, Theorem 6.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Set $\exp(H) = p^e$. We may assume that $e > 1$, $p > 2$ and H is not absolutely regular. Indeed, if $e = 1$, then any chain satisfying condition (a), is a Hall chain. If H is absolutely regular, then $\{1\} < \Omega_1(H) < \cdots < \Omega_e(H) = H$ is *the unique* $(p-1)$-admissible Hall chain in H. Next, if $p = 2$, then any part of a chief series of G, lying below H, is a Hall chain in H.

We proceed by induction on $|H|$.

Let F_0 be a G-invariant subgroup of order p in H and set $\bar{G} = G/F_0$. Then, by induction, there is in $\bar{H}$ a $(p-1)$-admissible Hall chain

$$\{\bar{1}\} = \bar{F}_0 < \bar{F}_1 < \cdots < \bar{F}_n = \bar{H}.$$

Obviously, $\exp(\bar{F}_i) \leqslant p^i$ so $\exp(F_i) \leqslant p^{i+1}$ for all i. Let i_0 be the greatest value of i such that $|\bar{F}_i| = p^{(p-1)i}$. In view of Remark 4, one may assume that $i_0 < n$; then $p > 2$ and $|\bar{F}_{i_0+1}| < p^{(p-1)(i_0+1)}$ so $\Omega_{i_0+1}(\bar{H}) = \bar{F}_{i_0+1}$ since the displayed chain satisfies condition (b) of the theorem. It follows that $\Omega_{i_0+1}(H) \leqslant F_{i_0+1}$ (Remark 3) so

$(*)$ $\Omega_{i_0+1}(H) = \Omega_{i_0+1}(F_{i_0+1})$.

Since $|F_{i_0+1}| \leqslant p^{(p-1)(i_0+1)}$, it follows from Theorem 2 that $\exp(\Omega_{i_0+1}(F_{i_0+1})) \leqslant p^{i_0+1}$ or, what is the same,

$(**)$ $\exp(\Omega_{i_0+1}(H)) \leqslant p^{i_0+1}$.

Next, by the choice of i_0, we have $|\bar{F}_{i_0+1}/\bar{F}_{i_0}| < p^{p-1}$, and we conclude that $\bar{H}/\bar{F}_{i_0}$ has no G-invariant subgroup of order p^{p-1} and exponent p (indeed, if $\bar{U}/\bar{F}_{i_0}$ is a such subgroup, then $\exp(\bar{U}) \leqslant p^{i_0+1}$ so $\bar{U} \leqslant \Omega_{i_0+1}(\bar{H}) = \bar{F}_{i_0+1}$, which is a contradiction).

Thus, $\bar{F}_{i_0+1}/\bar{F}_{i_0} = \Omega_1(\bar{H}/\bar{F}_{i_0})$ whence $\bar{H}/\bar{F}_{i_0}$ is absolutely regular (Lemma 3(b)) so $H/F_{i_0}(\cong \bar{H}/\bar{F}_{i_0})$ is also absolutely regular.

Assume that $i_0 = 0$. Then $|\bar{F}_1| < p^{p-1}$ so $\Omega_1(\bar{H}) = \bar{F}_1$, by (b), and $|F_1| = |F_0||\bar{F}_1| \leqslant p^{p-1}$. In that case, F_1 must be of order p^{p-1} and exponent p (otherwise, H is absolutely regular, by Lemma 3(b)). Then $\Omega_1(H) = F_1$ and $H/\Omega_1(H)$ is absolutely regular (see the previous paragraph). By Remark 2, there is a $(p-1)$-admissible Hall chain in H.

In what follows let $i_0 > 0$; then $\bar{F}_1$ is of order p^{p-1} and exponent p so $|F_1| = p^p$ and $\exp(F_1) \leqslant p^2$. We also have $\exp(F_{i_0+1}) \leqslant |F_0| \exp(\bar{F}_{i_0+1}) \leqslant p \cdot p^{i_0+1} = p^{i_0+2}$ and, according to this, we have to consider separately the following three possibilities:

(i) $\exp(F_{i_0+1}) < p^{i_0+1}$,
(ii) $\exp(F_{i_0+1}) = p^{i_0+1}$, and
(iii) $\exp(F_{i_0+1}) = p^{i_0+2}$.

(i) Suppose that $\exp(F_{i_0+1}) < p^{i_0+1}$; then, by $(*)$, $F_{i_0+1} = \Omega_{i_0+1}(F_{i_0+1}) = \Omega_{i_0+1}(H)$. It follows from the last equality that $\exp(H) < p^{i_0+1}$ so $\Omega_{i_0+1}(H) = H$ and hence $F_{i_0+1} = H$. By Remark 4, there exists in F_{i_0} a $(p-1)$-admissible Hall chain

$$\{1\} = L_0 < L_1 < \cdots < L_{i_0} < F_{i_0}$$

satisfying

$$|F_{i_0}\colon L_{i_0}| = p, \qquad |L_{i_0}| = p^{(p-1)i_0}, \qquad |H/L_{i_0}| = |H/F_{i_0}||F_{i_0}/L_{i_0}| \leqslant p^{p-2} \cdot p = p^{p-1}$$

so H/L_{i_0} is regular of exponent $\leqslant p^2$.

If $\exp(H/L_{i_0}) = p$, then $\{1\} = L_0 < L_1 < \cdots < L_{i_0} < H$ is the desired chain.

Now we let $\exp(H/L_{i_0}) = p^2$. By Lemma 3(c)(iii), $U/L_{i_0} = \Omega_1(H/L_{i_0})$ is of exponent p and index $|F_{i_0}/L_{i_0}| = p$ in H/L_{i_0} since $\exp(H/F_{i_0}) = p$. Therefore,

$$\{1\} = L_0 < L_1 < \cdots < L_{i_0} < U$$

is a $(p-1)$-admissible Hall chain in U. Let $W/L_{i_0} = \mho_1(H/L_{i_0})$; then $|W/L_{i_0}| = |(H/L_{i_0}) : (U/L_{i_0})| = p$ (part (iii) of Lemma 3(c)). Since $\exp(H/W) = p$ and $|H/W| < p^{p-1}$, we get $\mho_1(H) < W$ ($<$ since $|H/\mho_1(H)| \geqslant p^p$: H is not absolutely regular). Therefore, there exists a G-invariant subgroup T_{i_0} satisfying $\mho_1(H) < T_{i_0} < W$ and $|T_{i_0}| = p^{(p-1)i_0}$ (recall that $p^{(p-1)i_0} = |L_{i_0}| < |H|$ and $|H : \mho_1(H)| \geqslant p^p > |H : W|$). We have $\exp(T_{i_0}) \leqslant \exp(H) \leqslant p^{i_0}$, so there exists in T_{i_0} a $(p-1)$-admissible Hall chain $\{1\} = T_0 < T_1 < \cdots < T_{i_0}$ and all indices of that chain are equal to p^{p-1} (Lemma 3(f)). Since H/T_{i_0} is of order $\leqslant p^{p-1}$ and exponent p, $\{1\} = T_0 < T_1 < \cdots < T_{i_0} < H$ is the desired chain.

(ii) Suppose that $\exp(F_{i_0+1}) = p^{i_0+1}$; then $F_{i_0+1} = \Omega_{i_0+1}(H)$, by $(*)$. Since $H/\Omega_{i_0+1}(H)$, as an epimorphic image of H/F_{i_0}, is absolutely regular and $|\Omega_{i_0+1}(H)| \leqslant p^{(p-1)(i_0+1)}$, there is a $(p-1)$-admissible Hall chain in H, by Lemma 3(e) and Remark 2.

(iii) Suppose that $\exp(F_{i_0+1}) = p^{i_0+2}$. Then, by $(**)$, $\exp(\Omega_{i_0+1}(H)) = p^{i_0+1}$. We have $F_{i_0} \leqslant \Omega_{i_0+1}(H)$ and H/F_{i_0} is absolutely regular so $H/\Omega_{i_0+1}(H)$ is also absolutely regular and, in addition, $|\Omega_{i_0+1}(H)| \leqslant p^{(p-1)(i_0+1)}$. Therefore, there is a $(p-1)$-admissible Hall chain in H, by Remark 2.

The proof is complete. □

Let H be a normal subgroup of order p^m and exponent p^e of a p-group G and let $\mathcal{C}: \{1\} = L_0 < L_1 < \cdots < L_n = H$ be a $(p-1)$-admissible Hall chain in H.

(A) Suppose, in addition, that $m \geqslant (p-1)e$. Assume that for some $i \leqslant e$, we have $|L_i| < p^{(p-1)i}$; then $n > e$ since $m < (p-1)n$. In that case, by Theorem 1, $L_e = \Omega_e(G) = H$ so $n \leqslant e$, a contradiction. Thus, for all $i \leqslant e$, we must have $|L_i| = p^{(p-1)i}$.
(B) Suppose that, for some $i < n$, we have $|L_i| < p^{(p-1)i}$ (here we do not assume that $m \leqslant (p-1)e$). Then $L_i = \Omega_i(H) < H$ so $\exp(L_i) = p^i$. It follows that $\exp(L_j) = p^j$ for all $j \leqslant n$ so $n = e$.
(C) Let i_0 be the maximal value of i satisfying $|L_i| = p^{(p-1)i}$. Then, by Theorem 1, the members L_j, $j > i_0$, of the chain $\mathcal{C}$ are determined uniquely by the equality $L_j = \Omega_j(H)$.

Supplement 1 to Theorem 1. *Let $k < p$ be a natural number and let H be a normal subgroup of a p-group G. Then there is in H a chain $\{1\} = L_0 < L_1 < \cdots < L_n = H$ of G-invariant subgroups with the properties* $(i = 1, \dots, n)$:

(a) *L_i/L_{i-1} is of order $\leqslant p^k$ and exponent p, and*
(b) *either the order of L_i is exactly p^{ik}, or else $L_i = \Omega_i(H)$.*

Setting, in Supplement 1, $k = p-1$, we get Theorem 1.

Supplement 2 to Theorem 1. *Let H be a regular normal subgroup of a p-group G and let $k \in \mathbb{N}$. Then there is in H a chain $\{1\} = L_0 < L_1 < \cdots < L_n = H$ of G-invariant subgroups with the properties* $(i = 1, \dots, n)$:

(a) *L_i/L_{i-1} is of order $\leqslant p^k$ and exponent p, and*
(b) *either the order of L_i is exactly p^{ik}, or else $L_i = \Omega_i(H)$.*

A chain $\mathcal{C}$ satisfying conditions (a) and (b) of any of the above supplements, is said to be a *k-admissible Hall chain* in H (independently of the structure of H). To prove the above supplements, it suffices to repeat, word for word, the proof of Theorem 1. Hall's proof of Theorem 1 for regular H is not easier than in general case. In the second supplement one can replace regularity by the following condition: whenever U is a section of H, then $|\Omega_n(U)| = |U/\mho_n(U)|$ for all $n \in \mathbb{N}$. In that case, according to [7], we also have $\exp(\Omega_k(U)) \leqslant p^k$ and $\mho_k(U) = \{x^{p^k} \mid x \in U\}$ for all sections U of H. Following Mann, such groups are called $\mathcal{P}$-groups. By Lemma 3(c), regular p-groups are $\mathcal{P}$-groups.

Remark. 7. An irregular p-group G of maximal class is a $\mathcal{P}$-group if and only if $|G| = p^{p+1}$ and $|\Omega_1(G)| = p^p$. Indeed, $|\Omega_1(G)| = |G/\mho_1(G)|$ and the right-hand side of the last equality equals p^p (Lemma 3(m)). In that case, $\Omega_1(G)$ is a normal subgroup of G of order p^p and exponent p. Then $|G| = p^{p+1}$, by Lemma 3(l); in that case, as it is easy to check, we must have $p > 2$. On the other hand, if G is of maximal class and order p^{p+1} with $|\Omega_1(G)| = p^p$, it is a $\mathcal{P}$-group since all its proper sections are regular. Mann gave an example of irregular group G of order p^{p+1}, $p > 2$, such that $|\Omega_1(G)| = p^p$. (It is easy to show that if all subgroups of order p^{p+1} of an irregular p-group G of maximal class are $\mathcal{P}$-groups, then $|G| = p^{p+1}$.)

There exist p-groups without p-admissible Hall chains. Indeed, a p-group of maximal class and order $\geqslant p^{2p}$ has no p-admissible Hall chain.

As the proof of Theorem 1 shows, if $\{1\} = L_0 < L_1 < \cdots < L_n = H$ is a $(p-1)$-admissible Hall chain in $H \trianglelefteq G$, then $|L_1 : L_0| \geqslant |L_2 : L_1| \geqslant |L_n : L_{n-1}|$. The similar assertion is not true for p-admissible Hall chains as the group $H = G = \langle x, y \mid x^8 = 1, y^4 = x^4, x^y = x^{-1}\rangle$ shows (indices of the unique 2-admissible Hall chain in G are 4, 2, 4).

Let G be a p-group and let H, a normal subgroup in G, be of order p^{kp} and exponent $\leqslant p^k$. Then there exists a p-admissible Hall chain in H of length k. Indeed, the claim is trivial for $k = 1$. Assuming $k > 1$, we proceed by induction on k. By Lemma 3(h), H is neither absolutely regular nor irregular of maximal class. The subgroup $\mho_{k-1}(H)$ is of order $\leqslant p^p$ and exponent p (Lemma 3(i)). Then, by Lemma 3(g)(ii), $\mho_{k-1}(H) \leqslant F < H$, where F is a G-invariant subgroup of order p^p and exponent p. We have $|H/F| = p^{p(k-1)}$ and $\exp(H/F) \leqslant p^{k-1}$ so there is a p-admissible Hall chain $F_1/F = F/F < F_2/F < \cdots < F_k/F$ in H/F, by induction; then $\{1\} = F_0 < F_1 < \cdots < F_k = H$ is the desired chain.

It appears that the same approach as in the proof of Theorem 2, allows us to give the new proof of the following

Theorem 4 (= [1, Theorem 4]). *Let $k > 1$. Suppose that a p-group G has no subgroup of order $p^{(p-1)k+2}$ and exponent $\leqslant p^k$. Then either* $\exp(\Omega_k(G)) \leqslant p^k$ *or G is of maximal class and*[3] *of order $\geqslant p^{(p-1)k+2}$.*

Theorem 2 follows from Theorem 4 immediately. The proof of Theorem 4 is not so elementary: it based of the theory of p-groups of maximal class.

To facilitate the proof of Theorem 4, we first prove the following

Lemma 5. *Suppose that G is a group of order $p^{(p-1)k+2}$ and $\Omega_k(G) = G$. Then either* $\exp(G) \leqslant p^k$ *or G is of maximal class.*

Proof. We are working by induction on $|G|$ and k assuming that G is a minimal counterexample. Then $\exp(\Omega_k(G)) > p^k$ (in that case, G is irregular) and G is not of maximal class. Therefore, by Theorem 2, G possesses a subgroup A of order $p^{(p-1)k+1}$ and exponent $\leqslant p^k$. It follows from $\exp(G) > p^k$ that $\exp(A) = p^k$ since $|G : A| = p$, and then

[3] It is asserted in [1, Theorem 4] that if G is of maximal class, then $|G| = p^{(p-1)k+2}$. In fact, there is no restriction on the order of G in this case, as Theorem 4(b) shows.

$\exp(G) = p^{k+1}$. By Lemma 3(h), A is not absolutely regular. Since G is not of maximal class, $k > 1$ (see Lemma 3(c)).

Assume that A is of maximal class; then A is irregular (Lemma 3(m)) and $\exp(G) = \exp(A) = p^k$ (Lemma 3(j)), a contradiction.

Since $|G| = p^{(p-1)k+2} \leqslant p^{pk}$, the subgroup $\mho_{k-1}(G)$ is absolutely regular since it has an element of order p^2 (Lemma 3(i)). Since $\exp(A) = p^k$, the subgroup $\mho_{k-1}(A)$ is generated by elements of order p so it is contained in $\Omega_1(\mho_{k-1}(G))$; in that case, $\exp(\mho_{k-1}(A)) = p$ and $|\mho_{k-1}(A)| \leqslant p^{p-1}$ (Lemma 3(c)(iii)). By Lemma 3(b), $\Omega_1(\mho_{k-1}(G)) \leqslant U$, where U is a G-invariant subgroup of order p^p and exponent p. Assume that $U \nleqslant A$. Then $G = UA$ and $G/U \cong A/(U \cap A)$ is of exponent p^{k-1} since $\mho_{k-1}(A) \leqslant U$; in that case, $\exp(G) \leqslant \exp(G/U)\exp(U) = p^k$, a contradiction. Thus, $U < A$. Write $\bar{G} = G/U$. Let $x \in G - A$ be of minimal order; then $o(x) \leqslant p^k$. We have $\bar{G} = \langle \bar{x}, \bar{A} \rangle$, $o(\bar{x}) \leqslant p^{k-1}$ since $x^{p^{k-1}} \in \Omega_1(\mho_{k-1}(G)) \leqslant U$, and so $\Omega_{k-1}(\bar{G}) = p^{k-1}$ and $|\bar{G}| = |G/U| = p^{(p-1)(k-1)+1}$. By Theorem 2, $\exp(\bar{G}) = p^{k-1}$ so $\exp(G) \leqslant p^k$, and G is not a counterexample. □

Proof of Theorem 4. If G is of maximal class and exponent $> p^k$, its order is $\geqslant p^{(p-1)k+2}$ and it satisfies the hypothesis [2, Theorem 13.19]; see also Lemma 3(h),(l),(m). Suppose that G is a counterexample of minimal order. Then $\exp(G) \geqslant \exp(\Omega_k(G)) \geqslant p^{k+1}$ so G is irregular, all maximal subgroups of G have exponent $\geqslant p^k$ and G is not of maximal class. By Theorem 2, G has a (proper) subgroup A of order $p^{(p-1)k+1}$ and exponent $\leqslant p^k$. Since A is maximal among subgroups of G of exponent $\leqslant p^k$, we get $\exp(A) = p^k$. In view of Lemma 5, $|G| > p^{(p-1)k+2}$.

Assume that G has a subgroup H of maximal class and index p. Let R, a normal subgroup of G, be of order p^p and exponent p (Lemma 3(g)(i)). Assume, in addition, that $R < H$. Then $|H| = p^{p+1}$ (Lemma 3(l)). By Lemma 3(j), $\exp(G) = \exp(H) = p^2 < p^{k+1}$, a contradiction. Now let $R \nleqslant H$; then $G = RH$, $|R \cap H| = p^{p-1}$ and $G/R \cong (H/(R \cap H)) \times (R/(R \cap H))$. In that case, $\exp(H/(R \cap H)) = \exp(G/R) \geqslant p^k > p$. Then $H/(R \cap H)$ is irregular (otherwise, $\exp(H/(R \cap H)) = p$). In that case, $H/(R \cap H)$ has a subgroup $B/(R \cap H)$ of order $p^{(p-1)(k-1)+1}$ and exponent p^{k-1} [3, Theorems 9.5, 9.6, 13.19]. But $B/(R \cap H) \cong B_0/R$ for some $B_0 < G$. Then $\exp(B_0) \leqslant p^k$ and $|B_0| = p^{(p-1)(k-1)+1+p} = p^{(p-1)k+2} > |A|$, a contradiction. Thus, all subgroups of index p in G are not of maximal class.

The hypothesis is inherited by subgroups of G. Therefore, if M is maximal in G, then, by induction, $\exp(\Omega_k(M)) = p^k$, since M is not of maximal class, by the previous paragraph. If we take, from the start, M so that it contains A, we get $A = \Omega_k(M)$ so A is normal in G. By assumption, there is $x \in G - A$ with $o(x) \leqslant p^k$; then $x^p \in M$, $o(x^p) < p^k$ so $x^p \in \Omega_{k-1}(M) \leqslant A$. Set $B = \langle x, A \rangle$. Then $|B| = p|A| = p^{(p-1)k+2}$, $\exp(B) = p^{k+1}$ and $\Omega_k(B) = B$ so, by Lemma 5 and the choice of A, B must be of maximal class. By the previous paragraph, $|G : B| > p$. Let $B < M < G$, where M is maximal in G. Then, by the above, $\Omega_k(M) = A$, a contradiction since $A < B \leqslant \Omega_k(M)$. The proof is complete. □

For $k = 1$, Theorem 4 is not true. Indeed, let the central product $G = M * C$, where M is a p-group of maximal class and order p^{p+1} with $|\Omega_1(M)| = p^{p-1}$ and C is cyclic of order p^2, $|G| = p^{p+2}$. Then $\Omega_1(G) = G$ has exponent p^2, G has no subgroup of order

$p^{p+1} = p^{(p-1)1+2}$ and exponent p (consider the intersection of that subgroup with M); see in [5, Appendix 31, the paragraph preceding Exercise B].

The following theorem generalizes Theorem 4; its proof is shorter since it is based on other ideas.

Theorem 6. *Let G be a p-group and $k > 1$. Suppose that G has a proper subgroup A of order $p^{(p-1)k+1}$ which is maximal among subgroups of G of exponent $\leqslant p^k$. Then either $\Omega_k(G) = A$ or G is of maximal class* (*in the last case, A is also of maximal class*).

Proof. Suppose that $\Omega_k(G) > A$; then $\exp(\Omega_k(G)) > p^k$. In that case, G is irregular (Lemma 3(c)(iii)). It follows that $\exp(A) = p^k$.

First suppose that A is normal in G. Let $x \in G - A$ be of minimal order. Then $o(x) \leqslant p^k$, by assumption, and $x^p \in A$ so $B = \langle x, A\rangle$ has order $p^{(p-1)k+2}$ and exponent p^{k+1}, and $\Omega_k(B) = B$. In that case, by Lemma 5, B is of maximal class. It follows from parts (h) and (l) of Lemma 3 that A is also of maximal class. Now let $A < D \leqslant G$ be such that $|D : A| = p$. Since $\exp(D) > p^k = \exp(A)$, it follows from Lemma 3(j) that D must be of maximal class. Thus, all subgroups of G of order $p|A|$, containing A, are of maximal class so G is also of maximal class, by Lemma 3(k).

Now suppose that A is not normal in G. Set $N_G(A) = N$. Since $N < G$, A is not characteristic in N so, by the previous paragraph, N is of maximal class. Then, by [4, Remark 3], G is also of maximal class. The last assertion follows from Lemma 3(h),(l). □

In particular, if a p-group G has only one subgroup, say A, of order $p^{(p-1)k+1}$ and exponent $\leqslant p^k$, then $\Omega_k(G) = A$.[4] This follows from Theorem 6 and Lemma 3(h),(l) if $k > 1$. Now let $k = 1$ and $\Omega_1(G) > A$. First assume that A is normal in G. Let $x \in G - A$ be of order p. Set $B = \langle x, A\rangle$ and let $x \in B_1 < B$ with $|B : B_1| = p$. Then, by the modular law, $B_1 = \langle x\rangle \cdot (B \cap B_1)$ so $\Omega_1(B_1) = B_1$, and we get $B_1 \neq A$, $\exp(B_1) = p$ (Lemma 3(c)(iii)) and $|B_1| = p^p = |A|$, a contradiction. Setting $N_G(A) = N$, we get, by what has just been proved, $\Omega_1(N) = A$ so A is characteristic in N and so $N = G$, i.e., A is normal in G, contrary to the assumption.

Let G be a 2-group of exponent $> 2^k > 2$ and let $A < G$ be of order 2^{k+1} and exponent $\leqslant 2^k$ which is maximal among subgroups of G of exponent $\leqslant 2^k$. We claim that then one of the following holds:

(A) G has a cyclic subgroup of index 2,
(B) $G = \langle x, y \mid x^{2^n} = 1, n > 1, y^4 = x^{2^{n-1}}, x^y = x^{-1}\rangle$.

We assume that G has no cyclic subgroup of index 2; then G is not of maximal class so, by Theorem 6, $A = \Omega_k(G)$. It follows from Lemma 3(m) that A is not of maximal class; then $\mathrm{cl}(A) \leqslant 2$. In that case, $\Omega_2(G) = \Omega_2(A)$ is of order 8. By [2, Lemma 2.1(c)], G is a group from (B).[5]

[4] Compare with Remark 6.

[5] As Janko noticed, two groups in that lemma, corresponding to values $i = 0$ and $i = 1$, are isomorphic.

Proposition 7. *Let G be a group of order $p^{(p-1)k+3}$, $k > 2$. Suppose that $\Omega_k(G) = G$ and $\exp(G) > p^k$. Then one of the following holds*:

(a) *G is of maximal class.*
(b) *G has a subgroup A of index p and exponent p^k, A has a G-invariant subgroup H of order p^p and exponent p such that G/H and A/H are of maximal class.*

Proof. We have $|G| = p^{(p-1)k+3} \leqslant p^{kp}$ since $k \geqslant 3$.

Suppose that G is not of maximal class. Then, by Theorem 4, G has a maximal subgroup A such that $\exp(A) = p^k$; then $\exp(G) = p^{k+1}$ and $|A| = p^{(p-1)k+2}$. By Lemma 3(h), A is neither absolutely regular nor of maximal class. By Lemma 3(i),(c), $\mho_{k-1}(G)$ is absolutely regular since it has an elements of order p^2, and $\mho_{k-1}(A) \leqslant \Omega_1(\mho_{k-1}(G))$ since $\mho_{k-1}(G)$ is generated by elements of order p. By Lemma 3(g)(ii), $\Omega_1(\mho_{k-1}(G)) \leqslant H < G$, where H is a G-invariant subgroup of order p^p and exponent p. Let $x \in G - A$ be such that $o(x)$ is as small as possible; then $o(x) \leqslant p^k$. Set $\bar{G} = G/H$. We have $\exp(\bar{A}) = p^{k-1}$, $|\bar{G}| = p^{(p-1)(k-1)+2}$, $o(\bar{x}) \leqslant p^{k-1}$ since $x^{p^{k-1}} \in \Omega_1(\mho_{k-1}(G)) \leqslant H$, and so $\Omega_{k-1}(\bar{G}) = \bar{G}$. We have $\exp(\bar{G}) > p^{k-1}$ so $H < A$ (otherwise, $G = HA$ and $G/H \cong A/(A \cap H)$ is of exponent p^{k-1}; then $\exp(G) \leqslant p^k$, which is not the case). In that case, $\bar{G}$ is of maximal class, by Lemma 5. It follows from Lemma 3(h),(l) that $\bar{A}$ is also of maximal class. The proof is complete. □

Taking $k = 3$ in Proposition 7, we get

Corollary 8. *Let G be a group of order p^{3p}. If $\Omega_3(G) = G$, then one of the following holds*:

(a) $\exp(G) \leqslant p^3$.
(b) *G is of maximal class.*
(c) *G has a subgroup A of index p and exponent p^3, A has a G-invariant subgroup H of order p^p and exponent p such that G/H and A/H are of maximal class.*

Proposition 9. *Let $k > 3$, $p > 2$ and let G be a p-group containing a normal subgroup A of order $p^{(p-1)k+2}$ which is maximal among subgroups of G of exponent $\leqslant p^k$. Then one of the following holds*:

(a) $\Omega_k(G) = A$.
(b) *$|G : A| = p$, there is in A a G-invariant subgroup R of order p^p and exponent p such that G/R and A/R are of maximal class.*

Proof. Assume that $\Omega_k(G) > A$. Then, if $A < U \leqslant G$, where $|U : A| = p$, then $\exp(U) = p^{k+1}$ so $\exp(A) = p^k$. By Lemma 3(h), A is neither absolutely regular nor of maximal class.

By Lemma 3(i),(c), $\mho_{k-1}(A)$ is of order $\leqslant p^{p-1}$ and exponent p since A is generated by elements of order p and $|A| < p^{kp}$ in view of $k > 3$. Then, by Lemma 3(g)(ii), $\mho_{k-1}(A) < R < A$, where R is a G-invariant subgroup of order p^p and exponent p. Set $\bar{G} = G/R$. We have $|\bar{A}| = p^{(p-1)(k-1)+1}$ and $\exp(\bar{A}) = p^{k-1}$. Next, $\bar{A}$ is maximal among subgroups of exponent p^{k-1} in $\bar{G}$, by the choice of A. It follows from Theorem 6 that either $\Omega_{k-1}(\bar{G}) = \bar{A}$ or $\bar{G}$ and $\bar{A}$ are of maximal class. In the second case, $|G : A| = p$ since each normal subgroup of $\bar{G}$ of index $> p$ has center of order $> p$ so is not of maximal class.

It remains to consider the possibility $\Omega_{k-1}(\bar{G}) = \bar{A}$. Then $\Omega_{k-1}(G) \leqslant A$, by Remark 3. By assumption, there exists an element $x \in G - A$ such that $o(x) \leqslant p^k$ and $x^p \in A$. Since $\Omega_{k-1}G) \leqslant A$, we get $o(x) = p^k$. Set $B = \langle x, A\rangle$; then $|B| = p^{(p-1)k+3}$ since A is normal in G, $\Omega_k(B) = B$ and $\exp(B) = p^{k+1}$, by the choice of A. Since the maximal subgroup A of B is neither absolutely regular nor of maximal class and $|B| > p^{p+1}$, B is not of maximal class (Lemma 3(l)). Working by induction on $|G|$, we conclude that there is in A a B-invariant subgroup K of order p^p and exponent p such that A/K and B/K are of maximal class. By Lemma 3(m), $\Omega_2(B/K) = B/K$ so $\Omega_3(B) = B$. In that case, $B \leqslant \Omega_3(G) \leqslant \Omega_{k-1}(G) \leqslant A$, since $k > 3$, and this is a contradiction. Thus, $\Omega_k(G) = A$, and the proof is complete. □

Proposition 10 (Compare with Corollary 8). *Suppose that G is a p-group of order $\leqslant p^{pk}$ such that $\Omega_k(G) = G$ and all irregular sections of G of order p^{p+1} are $\mathcal{P}$-groups.*[6] *Then* $\exp(G) \leqslant p^k$.

Proof. Suppose that G is a counterexample of minimal order; then $k > 1$ (Lemma 3(c)(i)), $\exp(\Omega_k(G)) > p^k$ so G is irregular (Lemma 3(c)(iii)). In that case, by Remark 5, there exists in G a maximal subgroup A such that $\Omega_k(A) = A$. Since A satisfies the hypothesis, we get $\exp(A) \leqslant p^k$, by induction, so $\exp(G) = p^{k+1}$ and $\exp(A) = p^k$.

If G is of maximal class, it is irregular; then G is of order p^{p+1}, by the last sentence of Remark 7; then $\exp(G) = p^2 < p^{k+1}$, which is a contradiction.

Assume that A is of maximal class; then A is irregular, $|A| = p^{p+1}$ (Remark 7) so $\exp(A) = p^2$ (it is easy to see that then $p > 2$ but we do not use this fact). In that case, by Lemma 3(j), $\exp(G) = \exp(A) = p^2 < p^{k+1}$, and G is not a counterexample.

Now assume that A is absolutely regular. Then, by [2, Theorem 7.5], $\Omega_1(G)$ is of order p^p and exponent p. In that case, obviously, $\exp(\Omega_k(G)) = p^k$, contrary to the assumption.

Next we let A be neither absolutely regular nor of maximal class.

By Lemma 3(i), $\mho_{k-1}(G)$ is absolutely regular since it contains an element of order p^2. Then $\mho_{k-1}(A) \leqslant \Omega_1(\mho_{k-1}(G))$ since $\Omega_{k-1}(A)$ is generated by elements of order p, and so $\mho_{k-1}(A)$ is of order $\leqslant p^{p-1}$ and exponent p. By Lemma 3(g)(ii), $\Omega_1(\mho_{k-1}(G)) < H < G$, where H is a G-invariant subgroup of order p^p and exponent p. Set $\bar{G} = G/H$. Let $y \in G - A$ be of minimal order. Then $o(y) \leqslant p^k$, $\bar{G} = \langle \bar{y}, \bar{A}\rangle$, where $o(\bar{y}) \leqslant p^{k-1}$ since $y^{p^{k-1}} \in \Omega_1(\mho_{k-1}(G)) \leqslant H$, $\exp(\bar{A}) = p^{k-1}$ so $\Omega_{k-1}(\bar{G}) = \bar{G}$, and $|\bar{G}| \leqslant p^{p(k-1)}$. Obviously, $\bar{G}$ satisfies the hypothesis with $k-1$ instead of k. Then, by induction, $\exp(\bar{G}) \leqslant p^{k-1}$ so $\exp(G) \leqslant p^k$ and G is not a counterexample. The proof is complete. □

[6] For definition of $\mathcal{P}$-groups, see the paragraph preceding Remark 7.

Corollary 11. *Let $k \in \mathbb{N}$ and let A be a proper subgroup of a p-group G which is maximal among subgroups of G of exponent $\leqslant p^k$. Suppose that all irregular sections of the subgroup $\Omega_k(G)$, having order p^{p+1}, are $\mathcal{P}$-groups. Then, if $|A| < p^{kp}$, then $\Omega_k(G) = A$.*

Proof. We are working by induction on $|G|$. Set $\mathrm{N}_G(A) = N$. If $N < G$, then N satisfies the hypothesis so, by induction, $\Omega_k(N) = A$. In that case, A is characteristic in N so $N = G$, contrary to the assumption. Thus, A is normal in G. Let $x \in G - A$ be such that $o(x)$ is as small as possible. Then $o(x) \leqslant p^k$, $x^p \in A$ so $B = \langle x, A\rangle$ is of order $\leqslant p^{pk}$ since A is normal in G, and $B = \Omega_k(B)$. By Proposition 10, $\exp(B) \leqslant p^k$, contrary to the choice of A. $\square$

Question 1. Study the structure of a p-group G, $p > 2$, provided there exists only one $(p-1)$-admissible Hall chain in G.

Question 2. Let $A < G$ be p-groups with $|A| = p^{(p-1)k+2}$, $\exp(A) = p^k$, where $k > 1$. Suppose that A is maximal among subgroups of G of exponent p^k. Study the structure of G provided A is not normal in G. (See Proposition 9.)

Question 2 is nontrivial even in the case $p = 2 = k$.

Question 3. Let G be a p-group and let $E < G$ be extraspecial of exponent p^2. Suppose that, whenever $E < E_1 \leqslant G$, then $\exp(E_1) > p^2$. Study the embedding of E in G. The case where E is the unique subgroup of G of order $|E|$ and exponent p^2, is of special interest.

Question 3 is surprisingly complicated. Only in the case $|E| = p^3$ the answer is known: G is a 2-group of maximal class.[7] Indeed, take $E_1 > E$ such that $|E_1 : E| = p$; then E_1 has a cyclic subgroup of index p, by hypothesis. Since E_1 is not minimal nonabelian, we have $p = 2$. Then E_1 is of maximal class. Since E_1 is arbitrary, G is of maximal class, by Lemma 3(k).

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[7] Janko proved that a p-group G is extraspecial if and only if $\Omega_2(G)$ is extraspecial. Moreover, he proved that G is extraspecial or semidihedral of order 16 if $\langle x \in G \mid o(x) = p^2\rangle$ is extraspecial; see [5, §83].