The size of maximal systems of square islands

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ABSTRACT
For integers $m > 0$, $n > 0$, and $R = \{(x, y) : 0 \leq x \leq m$ and $0 \leq y \leq n\}$, a set $H$ of closed rectangles that are all subsets of $R$ and the vertices of which have integer coordinates is called a system of rectangular islands if for every pair of rectangles in $H$ one of them contains the other or they do not overlap at all. Let $I_R$ denote the ordered set of systems of rectangular islands on $R$, and let $\max(I_R)$ denote the maximal elements of $I_R$. For $f(m, n) = \max\{|H| : H \in \max(I_R)\}$, G. Czédli [G. Czédli, The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics, in press (doi:10.1016/j.ejc.2008.02.005)] proved $f(m, n) = \left\lfloor \frac{(mn + m + n - 1)}{2} \right\rfloor$. For $g(m, n) = \min\{|H| : H \in \max(I_R)\}$ in [Z. Lengvárszky, The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics 30 (1) 216–219], we proved $g(m, n) = m + n - 1$. Systems of square islands are systems of rectangular islands with $R$ and all members of $H$ being squares. The functions $f(n)$ and $g(n)$ are defined analogously to $f(m, n)$ and $g(m, n)$, and we show $f(n) \leq n(n + 2)/3$ (best polynomial bound), and $g(n) = n$.

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1. Introduction

With primary motivations in coding theory, the notion of systems of rectangular islands was introduced by Czédli [1]. For positive integers $m$ and $n$, consider the $m \times n$ rectangle $R = \{(x, y) : 0 \leq x \leq m$ and $0 \leq y \leq n\}$ in the Cartesian plane. A set of closed rectangles that are all subsets of $R$ and the vertices of which have integer coordinates form a system of rectangular islands $H$ if for every pair of rectangles $R_1, R_2 \in H$ either $R_1 \subseteq R_2$, or $R_2 \subseteq R_1$, or $R_1 \cap R_2 = \emptyset$.

Systems of rectangular islands on a given rectangle $R$ form a partially ordered set $I_R$ with respect to set inclusion. Let $\max(I_R)$ denote the subset of maximal elements of $I_R$. The main results of [1,3] can
be summarized as
\[ f(m, n) = \max\{|H| : H \in \max(I_k)\} = \left\lceil \frac{mn + m + n - 1}{2} \right\rceil, \]
and
\[ g(m, n) = \min\{|H| : H \in \max(I_k)\} = m + n - 1. \]

In this paper we consider the square analogue of systems of rectangular islands. For a positive integer \( n \), let \( S = \{(x, y) : 0 \leq x \leq n \text{ and } 0 \leq y \leq n\} \) be a closed square in the Cartesian plane. A set of closed squares that are subsets of \( S \) and the vertices of which have integer coordinates form a system of square islands \( H \) if \( S_1, S_2 \in H \) implies either \( S_1 \subseteq S_2 \), or \( S_2 \subseteq S_1 \), or \( S_1 \cap S_2 = \emptyset \). Let \( J_S \) be the partially ordered set (with respect to set inclusion) of square island systems on \( S \), and \( \max(J_S) \) be the set of maximal elements of \( J_S \). Note that \( S \in H \) for every \( H \in \max(J_S) \). Define
\[ f(n) = \max\{|H| : H \in \max(J_S)\}, \]
and
\[ g(n) = \min\{|H| : H \in \max(J_S)\}. \]

Our purpose is to show \( g(n) = n \), and \( f(n) \leq \frac{n(n+2)}{2} \), with the right hand side being the best possible polynomial upper bound.

2. Lower bound

**Proposition 1.** \( g(n) = n \).

**Proof.** First note that \( g(n) \leq n \) follows from the fact that there is a sequence of \( n \) squares, each included in the next, that form a maximal system of square islands on a given \( n \times n \) square \( S \). Hence, it is enough to show \( g(n) \geq n \), which is equivalent to saying that for any maximal system of square islands \( H \) on an \( n \times n \) square \( S \), we have \(|H| \geq n\). We will proceed by induction on \( n \) with the case \( n = 1 \) being trivial.

Let \( \max(H) \) denote the set of maximal squares, with respect to set inclusion, in \( H \in \max(J_S) \), and for a given square \( Q \in H \), define \( H|_Q = \{P \in H : P \subseteq Q\} \). Clearly, \( H|_Q \) is a maximal system of square islands on \( Q \).

We will use an argument similar to that applied in [3] by considering squares at the border of \( S \). Let us call a square \( Q \) in \( \max(H) \) south-extreme if its distance from the southern border is at most 1 unit. Similar terminology will be used in relation to the northern, western, and eastern borders of \( S \).

Note that it is possible for a square \( Q \) in \( \max(H) \) to be extreme in more than one direction. In fact, when \(|\max(H)| = 1\), then \( Q \), the unique member of \( \max(H) \), is extreme in all four directions. In this case, for \( n(Q) \), the side length of \( Q \), we have \( n(Q) = n(S) - 1 = n - 1 \) and the statement \(|H| \geq n\) is immediate by induction.

Let \( P \) and \( Q \) be two south-extreme squares. Then clearly, \( P \) is entirely to the left of \( Q \), or vice versa; thus, there is a natural linear ordering (“left of”) on the set of all south-extreme squares. Let \( \min[|x_1 - x_2| : (x_1, y_1) \in P; (x_2, y_2) \in Q] \) be the distance between \( P \) and \( Q \). If \( P \) is immediately to the left of \( Q \), then their distance is at most 2 since otherwise a square could be added to \( H \), contradicting the assumption that \( H \) is maximal. Similarly, the leftmost south-extreme square is at most 1 unit from the western border of \( S \), and it is also west-extreme, and the rightmost south-extreme square is at most 1 unit form the eastern border of \( S \), and it is also east-extreme. Except for the trivial case when \(|\max(H)| = 1\), a square \( Q \in \max(H) \) that is extreme in two directions cannot be 1 unit away from both borders since then a square with side length \( n(Q) + 1 \) could be added, and \( H \) would not be maximal.

Let \( S_1, S_2, \ldots, S_k \) be an enumeration of the extreme squares in \( \max(H) \) starting, say, at the southwestern corner and going in a counterclockwise fashion along the border of \( S \). We assume \( k > 1 \) since the (trivial) case \( k = 1 \) when \(|\max(H)| = 1 \) has been discussed above. For each \( S_i \), make a projection to the appropriate side(s) of \( S \) depending on whether the square is south-, north-, west-, or east-extreme. The resulting line segment(s) will have a length of \( n_i = n(S_i) \), the side length of square \( S_i \). The following inequality is valid:
\[(2k + 4) + (n_{i_1} + n_{i_2} + n_{i_3} + n_{i_4}) + \sum_{i=1}^{k} n_i \geq 4n.\]

The right side is the length of the border of S. The line segments that are projections of the S_i are represented on the left side by the summation together with \((n_{i_1} + n_{i_2} + n_{i_3} + n_{i_4})\), the latter of which was added because some squares (those at the four corners) are projected in two directions. The term \((2k + 4)\) accounts for the gaps between the projected line segments. We have \(2k\) since there are \(k\) extreme squares and there is a distance of at most 2 between two consecutive squares. The term 4 is added since in the case of the four corners an additional gap of at most 1 unit per corner may occur when the extreme square is projected in two directions. Here we use the fact that unless \(k = 1\), a square that is at the, say, south-western corner, cannot be 1 unit away from both the southern and the western sides of S.

The above inequality implies that one of the following is true:

\[k \geq n - 1 \text{ or } n_{i_1} + n_{i_2} + n_{i_3} + n_{i_4} \geq n - 1 \text{ or } \sum_{i=1}^{k} n_i \geq n - 1.\]

Note that \(k \geq n - 1\) implies \(\sum_{i=1}^{k} n_i \geq n - 1\); thus we will ignore the first possibility. Let us assume \(\sum_{i=1}^{k} n_i \geq n - 1\), and apply induction: \(|H| = 1 + \sum_{i=1}^{k} |H|_{S_i} = 1 + \sum_{i=1}^{k} n_i \geq 1 + (n - 1) = n\).

Consider the remaining case \(n_{i_1} + n_{i_2} + n_{i_3} + n_{i_4} \geq n - 1\). Let \(p = |\{i_1, i_2, i_3, i_4\}|\), and note that \(p\) can be 1, 2, 3, or 4. If \(p = 4\), i.e., the four indices \(i_1, i_2, i_3,\) and \(i_4\) are distinct, then \(\sum_{i=1}^{k} n_i \geq n - 1\) follows, and we are done by the last argument. The case \(p = 1\) is equivalent to \(k = 1\) which we examined above. It is not hard to see that the cases \(p = 2\) or \(p = 3\) can only occur in a very special configuration that can be described as follows: there must be at least one \(S_i\) that is an \((n - 2) \times (n - 2)\) square, and has distances 0, 1, 2, and 1 from, say the southern, eastern, northern, and western borders of S, respectively, and the other \(S_j\)'s, at least \(\lceil n/3 \rceil\) in all, are \(1 \times 1\) squares at the northern border of S. Also, \(p = 2\) implies \(n = 3\), and \(p = 3\) implies \(n \geq 3\). In either case we have \(\sum_{i=1}^{k} n_i \geq (n - 2) + \lceil n/3 \rceil \geq n - 1\), and the argument above can be applied again. \(\square\)

### 3. Upper bound

**Proposition 2.** \(f(n) \leq \frac{n(n+2)}{3}\), and this is the best possible polynomial upper bound.

**Proof.** We need to show that if \(H\) is a maximal system of square islands on a given \(n \times n\) square \(S\), then \(|H| \leq n(n + 2)/3\). Let us proceed by induction on \(n\), noting that the case \(n = 1\) is trivial. We examine four cases depending on whether \(|\text{max}(H)| = 1, 2, 3,\) or at least 4. If \(|\text{max}(H)| = 1\), then with max\((H) = \{Q\}\), write

\[|H| = |H|_Q + 1 \leq f(n - 1) + 1 \leq \frac{(n - 1)(n + 1)}{3} + 1 = \frac{n^2 + 2n}{3} = \frac{n(n + 2)}{3}.\]

The case \(|\text{max}(H)| = 2\) can occur in essentially one way: \(n = 3\), and max\((H)\) consists of two \(1 \times 1\) squares at the middle of two opposite sides of \(S\). Since \(|H| = 3 \leq 5 = \frac{n(n+2)}{3}\), we are done.

If \(|\text{max}(H)| = 3\), then max\((H)\) is again of special form: one member of max\((H)\), say \(P\), must be \((n - 2) \times (n - 2)\), and the other two members of max\((H)\), say \(Q\) and \(R\), are \(1 \times 1\) squares, and \(n\) is 3, 4, 5, or 6. Write

\[|H| = 1 + |H|_P + |H|_Q + |H|_R = 1 + |H|_P + 1 + 1 = 3 + |H|_P \leq 3 + f(n - 2) \leq 3 + \frac{(n - 2)n}{3} = \frac{n^2 - 2n + 9}{3} = \frac{n^2 + 2n - 4n + 9}{3} \leq \frac{n^2 + 2n}{3} \leq \frac{n(n + 2)}{3}.\]
Assume $|\text{max}(H)| \geq 4$ now. As in the proof of the main theorem in Czédli [1], we apply an argument that is based on comparing areas. For each $Q \in \text{max}(H) \cup \{S\}$, draw a square $Q'$ with side length $n(Q') = n(Q) + 1$ around $Q$ in such a way that the sides of $Q'$ are parallel to and $1/2$ a unit away from those of $Q$. Then, since the area of $Q'_1 \cap Q'_2$ is $0$ for any two distinct $Q_1, Q_2 \in \text{max}(H)$, the area of $S'$ is larger than or equal to the sum of the areas of the $Q'$. Using this fact first, then the induction hypothesis together with $|\text{max}(H)| \geq 4$, and finally the definition of $f(n)$, we can write

$$\frac{n(n+2)}{3} = -\frac{1}{3} + \frac{1}{3} \cdot (n+1)^2 \geq -\frac{1}{3} + \frac{1}{3} \cdot \sum_{Q \in \text{max}(H)} (n(Q)+1)^2$$

$$= -\frac{1}{3} + \sum_{Q \in \text{max}(H)} \frac{n(Q)(n(Q)+2)}{3} + \frac{1}{3} \geq 1 + \sum_{Q \in \text{max}(H)} f(n(Q))$$

$$\geq 1 + \sum_{Q \in \text{max}(H)} |H| = |H|,$$

which proves the first part of Proposition 2.

To see that our upper bound is best among polynomials, note that for values of the form $n = 2^k - 1$ there is a maximal system of square islands $H_k$ with $|H_k| = \frac{n(n+2)}{3}$. The construction can be described recursively. Assuming $H_{k-1}$ has been defined, use the middle row and column in the $n \times n$ square $S$ to divide it into four squares, and place one copy of $H_{k-1}$ on each of these four squares which together with $S$ will form $H_k$. □

4. Extensions

A higher dimensional generalization of Czédli’s result has been found by Pluhár [5]. Similarly, one may wish to investigate the higher dimensional versions of Propositions 1 and 2. For systems of square islands, another extension is to consider squares on a rectangle, i.e., the members of $H$ would be squares while $S$ would be an arbitrary $m \times n$ rectangle. In addition, systems of triangular islands on triangular grids have been investigated by Horváth, Németh, and Pluhár [2], and by the author in [4].

References