Annals of Pure and Applied Logic 162 (2011) 667–678

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

The complexity of central series in nilpotent computable groups

Barbara F. Csima [a,*], Reed Solomon [b]

[a] Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1

[b] Department of Mathematics, 196 Auditorium Rd., University of Connecticut, U-3009, Storrs, CT 06269-3009, USA

ARTICLE INFO

Article history:
Received 19 February 2010
Received in revised form 20 December 2010
Accepted 10 January 2011
Available online 22 February 2011
Communicated by A. Nies

MSC:
03D40
20F10
20F18
06F15

Keywords:
Computable groups
Nilpotent groups
Word problems

ABSTRACT

The terms of the upper and lower central series of a nilpotent computable group have computably enumerable Turing degree. We show that the Turing degrees of these terms are independent even when restricted to groups which admit computable orders.

1. Introduction

There are at least two general types of questions that are considered in computable algebra. One set of questions arises from thinking of computable algebra as the study of computable model theory restricted to a particular class of structures. From this point of view, it is natural to consider various computable model theoretic notions such as computable dimension, degree spectra of structures, degree spectra of relations, etc., and to ask how these notions behave within the specified class of structures.

In this sense, the computable algebraic behavior of nilpotent groups is well understood. Hirschfeldt et al. [5] proved that for the commonly considered computable model theoretic notions (such as those mentioned above), any behavior which occurs in some model also occurs in a nilpotent group. To prove this result, they used a coding of integral domains into class 2 nilpotent groups (specifically into Heisenberg groups) originally described by Mal'cev.

A second set of questions arises from thinking of computable algebra as the study of the effectiveness of the basic theorems, constructions and structural properties within the specified class of structures. In the case of nilpotent groups, this perspective leads to the following sorts of questions. How complex is the center or the commutator subgroup of a nilpotent computable group? More generally, how complex are the terms in the upper and lower central series of a nilpotent computable group?

Before discussing these questions further, we give some background on nilpotent groups. Nilpotent groups can be defined in a number of ways and we begin with a definition using the lower central series. Let G be a group written multiplicatively.

* Corresponding author.
E-mail addresses: csima@math.uwaterloo.ca (B.F. Csima), david.solomon@uconn.edu (R. Solomon).
URLs: http://www.math.uwaterloo.ca/~csima (B.F. Csima), http://www.math.uconn.edu/~solomon (R. Solomon).

0168-0072/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2011.01.011

For $x, y \in G$, the *commutator* of x and y is $[x, y] = x^{-1}y^{-1}xy$. If H and K are subgroups of G, then $[H, K]$ is the subgroup generated by the commutators $[h, k]$ with $h \in H$ and $k \in K$.

Definition 1.1. The *lower central series* of a group G is

$$G = \gamma_1 G \trianglerighteq \gamma_2 G \trianglerighteq \gamma_3 G \trianglerighteq \cdots$$

defined inductively by $\gamma_1 G = G$ and $\gamma_{i+1} G = [\gamma_i G, G]$. A group G is *nilpotent* if there is an r such that $\gamma_{r+1} G = 1$. More specifically, G is a *class r nilpotent group* if r is the least such that $\gamma_{r+1} G = 1$.

Nilpotent groups can also be defined by the upper central series. For any normal subgroup H of a group G, there is a natural projection $\pi : G \to G/H$ given by $\pi(g) = gH$. The *center of* G, denoted $C(G)$, is defined by $g \in C(G)$ if and only if $gh = hg$ for all $h \in G$. $C(G)$ is a normal subgroup, so we have the associated projection $\pi : G \to G/C(G)$. Taking the center of $G/C(G)$ and pulling back to G by π^{-1}, one gets another normal subgroup of G. Continuing in this spirit yields the upper central series of G.

Definition 1.2. The *upper central series* of a group G is

$$1 = \zeta_0 G \trianglelefteq \zeta_1 G \trianglelefteq \zeta_2 G \trianglelefteq \cdots$$

defined inductively by $\zeta_0 G = 1$ and $\zeta_{i+1} = \pi^{-1}(C(G/\zeta_i(G))$ for $\pi : G \to G/\zeta_i G$. A group G is *nilpotent* if there is an r such that $\zeta_r G = G$. More specifically, G is a *class r nilpotent group* if r is the least such that $\zeta_r G = G$.

These two definitions are equivalent in the sense that a group G is class r nilpotent under the lower central series definition if and only if it is class r nilpotent under the upper central series definition. The class 1 nilpotent groups are exactly the abelian groups, so the nilpotent class can be thought of as giving a measure of closeness to being abelian.

We will be concerned with the complexity of computing the terms in the upper and lower central series of a nilpotent computable group. A group G is *computable* if its set of elements is a computable subset of $\mathbb{N}$ and the group operation is a partial computable function whose domain includes this set of elements. Because the word problem for finitely generated nilpotent groups is solvable, such groups are computable. (See Miller [9] for a general discussion of the word problem within various classes of groups.) Furthermore, Baumslag et al. [1] proved that the terms in the upper and lower central series can be effectively calculated in such groups. Therefore, we focus our attention on infinitely generated nilpotent computable groups.

If H and K are computably enumerable subgroups of a computable group G, then the commutator subgroup $[H, K]$ is easily seen to be computably enumerable. It follows by induction that the terms of the lower central series of a computable group must be computably enumerable.

It is easy to see that the terms in the upper central series are Π^0_1.

$$g \in \zeta_1 G \Leftrightarrow \forall h\,(gh = hg)$$
$$g \in \zeta_{i+1} G \Leftrightarrow \forall h\,(gh = hg \bmod \zeta_i G) \Leftrightarrow \forall h\,([g, h] \in \zeta_i G)$$

Therefore, the terms in the upper and lower central series of a computable group have c.e. Turing degree. If G is a computable group which is class n nilpotent (for $n \geq 2$), then there are $2n - 2$ many nontrivial terms in these series: $\gamma_i G$ for $2 \leq i \leq n$, and $\zeta_i G$ for $1 \leq i \leq n - 1$. Our main theorem shows that the degrees of these terms are computationally independent in the following sense.

Theorem 1.3. *Fix $n \geq 2$ and c.e. Turing degrees $\mathbf{d_1}, \ldots, \mathbf{d_{n-1}}$ and $\mathbf{e_2}, \ldots, \mathbf{e_n}$. There is a computable group G which is class n nilpotent with $\deg(\zeta_i G) = \mathbf{d_i}$ for $1 \leq i \leq n - 1$ and $\deg(\gamma_i G) = \mathbf{e_i}$ for $2 \leq i \leq n$. Furthermore, G admits a computable order so this computational independence property holds for computable ordered nilpotent groups as well.*

Latkin [7] considered similar questions with respect to the lower central series and proved the following theorem. (See Section 6 for addition results from Latkin [7] and a list of related open questions.)

Theorem 1.4 (*Latkin [7]*). *Fix $n \geq 2$ and c.e. Turing degrees $\mathbf{e_2}, \ldots, \mathbf{e_n}$. There is a torsion-free class n nilpotent group G such that $\deg(\gamma_i G) = \mathbf{e_i}$ for $2 \leq i \leq n$.*

As in [7], we break the proof of Theorem 1.3 into smaller steps using the fact that the terms in the upper and lower central series interact nicely with direct products. (This lemma follows directly from the definitions.)

Lemma 1.5. *For any groups G and H, $\zeta_i(G \times H) = \zeta_i G \times \zeta_i H$ and $\gamma_i(G \times H) = \gamma_i G \times \gamma_i H$.*

If $G = G_1 \times \cdots \times G_k$, with the usual presentation, then the degree of each term in the lower and upper central series is the join of the corresponding terms in the G_i groups. Therefore, to prove Theorem 1.3, it suffices to establish the following two theorems.

Theorem 1.6. *For any $n \geq 2$ and c.e. Turing degree $\mathbf{d}$, there is a class n nilpotent computable group G such that the upper central terms are computable, $\gamma_i G$ is computable for $1 \leq i \leq n - 1$ and $\deg(\gamma_n G) = \mathbf{d}$.*

Theorem 1.7. *For any $n \geq 2$ and c.e. Turing degree $\mathbf{d}$, there is a class n nilpotent computable group G such that the lower central terms are computable, $\zeta_i G$ is computable for $0 \leq i \leq n-2$ and $\deg(\zeta_{n-1}G) = \mathbf{d}$.*

In Section 2, we describe a collection procedure due to Hall Jr. [2] for reducing words in a free nilpotent group to a normal form. In Section 3, we sketch Latkin's proof of Theorem 1.4 and we prove Theorem 1.6 by establishing additional properties of Latkin's construction. In Section 4, we prove Theorem 1.7. In Section 5, we give the basic definitions and properties of ordered groups, and we prove that the groups constructed for Theorem 1.3 are computably orderable. Finally, in Section 6 we list some open questions.

Note that by necessity our constructions differ from those used in Hirschfeldt et al. [5] to code integral domains into Heisenberg groups. The Heisenberg groups G in [5] have the property that $\zeta_1 G = \gamma_2 G$ and hence (since $\zeta_1 G$ is Π^0_1 and $\gamma_2 G$ is Σ^0_1), the center of G is computable in every computable presentation of G.

2. Normal forms and the collection process

In this section, we describe a normal form theorem for free nilpotent groups due to Marshall Hall, Jr. and we sketch the collection process that reduces a given word to its normal form. Additional information, including a proof of the uniqueness of the normal forms, can be found in [2] as well as [3,4,8,10]. Because of our interest in computable groups, we restrict our attention to countable groups throughout this paper.

To define a free nilpotent group, it is useful to characterize nilpotent groups as varieties in combinatorial group theory. We extend the definition of commutators inductively by $[x_1, x_2, \ldots, x_{n+1}] = [[x_1, x_2, \ldots, x_n], x_{n+1}]$. A group G is nilpotent if and only if there is a $r \geq 1$ such that $[x_1, x_2, \ldots, x_{r+1}] = 1$ for all $x_1, \ldots, x_{r+1} \in G$. For the least such r, G is class r nilpotent. The *free class r nilpotent group on a set X* is the group G/N where G is the free group on X and N is the subgroup generated by $\{[g_1, \ldots, g_{r+1}] \mid g_1, \ldots, g_{r+1} \in G\}$.

Let F be a free class r nilpotent group on the set X. Fix an order $\leq_X$ on X. We generate a set of basic commutators, assign weights to these basic commutators and define an order on them. The weight of a basic commutator c is denoted by $w(c)$.

Definition 2.1. The letters in X are the basic commutators of weight 1 and they are ordered by $\leq_1 = \leq_X$. Assume that the basic commutators of weight $\leq k$ have been defined and we have produced an order $\leq_k$ of them. A commutator $[c, d]$ is a basic commutator of weight $k+1$ if and only if

1. c and d are basic commutators of weight $\leq k$ and $w(c) + w(d) = k + 1$,
2. $d <_k c$, and
3. if the basic commutator c has the form $[u, v]$, then $v \leq_k d$.

We define the order $\leq_{k+1}$ on the basic commutators of weight $\leq k+1$ by $x \leq_{k+1} y$ if and only if

1. $w(x), w(y) \leq k$ and $x \leq_k y$, or
2. $w(x) \leq k$ and $w(y) = k+1$, or
3. $w(x) = w(y) = k+1$, $x = [c, d]$, $y = [u, v]$ and $\langle c, d\rangle \leq_k^{\text{lex}} \langle u, v\rangle$ where $\leq_k^{\text{lex}}$ is the lexicographic order.

Since F has class r, all commutators of weight $> r$ are equivalent to the identity, so we consider only basic commutators of weight $\leq r$ and we use $\leq$ to denote the order $\leq_r$. The normal form theorem is as follows. (In its original form, it was stated for finitely generated free nilpotent groups, but it holds for infinitely generated groups as well.)

Theorem 2.2 (*Hall, Jr. [2]*)**.** *Let F be a free class r nilpotent group on the (possibly finite) set of generators $x_0, x_1, \ldots$ with a fixed order on the basic commutators as above. Each $y \in F$ can be uniquely written as a finite product*

$$c_0^{m_0} c_1^{m_1} \cdots c_l^{m_l}$$

where c_i is a basic commutator, $c_i < c_{i+1}$, and $m_i \in \mathbb{Z} \setminus \{0\}$. Furthermore, each lower central factor $\gamma_i F/\gamma_{i+1} F$ is a free abelian group on the basic commutators of weight i, so $y \in \gamma_i F$ if and only if the normal form contains only basic commutators of weight $\geq i$.

It follows from Theorem 2.2 that a free class r nilpotent group on a computable set of generators has a computable presentation in which the lower central terms are all computable. Furthermore, since $\zeta_i G = \gamma_{r-i+1} G$ in a free class r nilpotent group, it follows that the terms in the upper central series are computable in this presentation as well. Therefore, in the proofs of Theorems 1.6 and 1.7, we can restrict to the case when the c.e. degree is noncomputable.

The details for proving uniqueness in Theorem 2.2 will not play a role in the later sections, but the details of reducing a word over the generators to its normal form will be useful. This process relies on the following definition and lemma. (Whenever a group is described as a free nilpotent group on a set of generators X, the set X comes equipped with an order and this order gives rise to the order on the basic commutators used in the normal forms.)

Definition 2.3. We define the commutator $[x, y^{(n)}]$ by induction on n: $[x, y^{(0)}] = x$ and $[x, y^{(n+1)}] = [[x, y^{(n)}], y]$.

Note that if $[x, y^{(1)}]$ is a basic commutator in a free class r nilpotent group, then $[x, y^{(n)}]$ is a basic commutator for all $0 \leq n < r$ and $[x, y^{(n)}] = 1$ for all $n \geq r$. The following commutator rules can be found in Section 11.1 of Hall [3].

Lemma 2.4. *The following equations hold for any elements x, y of a nilpotent group.*

$$x \cdot y = y \cdot x \cdot [x, y]$$
$$x^{-1} \cdot y = y \cdot [x, y]^{-1} \cdot x^{-1}$$
$$x \cdot y^{-1} = y^{-1} \cdot x \cdot [x, y^{(2)}] \cdot [x, y^{(4)}] \cdots [x, y^{(3)}]^{-1} \cdot [x, y]^{-1}$$
$$x^{-1} \cdot y^{-1} = y^{-1} \cdot [x, y] \cdot [x, y^{(3)}] \cdots [x, y^{(4)}]^{-1} \cdot [x, y^{(2)}]^{-1} \cdot x^{-1}.$$

(The products in the third and fourth equations are finite because $[x, y^{(k)}] = 1$ for all k greater than or equal to the class of the nilpotent group.)

We can now describe the collection process to reduce a word w on the generators X of a free class r nilpotent group to its normal form. We begin by viewing w as a word over X (that is, as a word over the basic commutators of weight 1, allowing each such commutator to occur either positively or negatively). Pick the least generator y (in the fixed order on the basic commutators) such that y or y^{-1} occurs in w and consider the leftmost occurrence of this basic commutator in w. The commutator rules in Lemma 2.4 allow us to pass this basic commutator left across each generator x (that is, across each basic commutator of weight 1) until it reaches the front of w.

Note that since $y < x$, anything of the form $[x, y^{(k)}]$ is a basic commutator. Hence our word has been rewritten in the form $y^{\epsilon_0} w'$ where ϵ_0 is 1 or -1 and w' is a word over our basic commutators. (That is, we keep the basic commutators generated by this process together as single units.) All the basic commutators introduced in this process have weights ≥ 2 and hence come after y in the order on the basic commutators. Furthermore, every new commutator of the form $[u, v]$ generated by this process has $v = y$. We pick the least basic commutator c such that c or c^{-1} occurs in w' and repeat this process to form an equivalent word $y^{\epsilon_0} c^{\epsilon_1} w''$. Notice that if we need to move c or c^{-1} past a basic commutator of the form $[u, v]$ generated in the first step of this process, then $v = y \leq c$, so the commutators generated by our rules are all basic. Continuing in this fashion eventually reduces w to its normal form.

We apply this reduction procedure in the context of free nilpotent groups in Section 3. In Section 4, we apply the same procedure in the context of a slightly different set of reduction rules to give normal forms for elements of a nilpotent group which is not free. The following lemma will be a useful calculation tool in Section 4.

Lemma 2.5. *Let G be a group and let N be a normal subgroup of G. For all $i \geq 1$,*

(1) *if $gN \in \gamma_i(G/N)$, then there is a $g' \in \gamma_i G$ such that $g = g' \bmod N$, and*
(2) *if there is an $h \in N$ such that $gh \in \gamma_i G$, then $gN \in \gamma_i(G/N)$.*

Therefore, for any $g \in G$,

$$gN \in \gamma_i(G/N) \Leftrightarrow gN \cap \gamma_i G \neq \emptyset.$$

Proof. Both statements follow by induction on i using the fact that $[aN, bN] = [a, b]N$. The base cases when $i = 1$ are trivial since $\gamma_1 G = G$ and $\gamma_1(G/N) = G/N$.

For the induction case in (1), assume that $gN \in \gamma_{i+1}(G/N)$. We write gN as a product of commutators $[aN, bN]$ (or their inverses) for which $aN \in \gamma_i(G/N)$ and $bN \in G/N$. By the inductive hypothesis, we can assume that $a \in \gamma_i G$. We obtain (1) since $[aN, bN] = [a, b]N$ and $[a, b] \in \gamma_{i+1} G$. The induction case for (2) is similar. □

3. Latkin's construction

In this section, we prove Theorem 1.6 which is restated here for convenience.

Theorem 3.1. *For any $n \geq 2$ and c.e. Turing degree* **d**, *there is a class n nilpotent computable group G such that the upper central terms are computable, $\gamma_i G$ is computable for $1 \leq i \leq n-1$ and* $\deg(\gamma_n G) = \mathbf{d}$.

The remainder of this section is devoted to proving Theorem 3.1. We use the original construction from Latkin [7] and prove that the upper central terms in the group constructed there are computable. Without loss of generality, we assume that **d** is a noncomputable c.e. degree.

We begin with a description of Latkin's construction. Fix a c.e. set A of degree **d** and let f be a computable 1-to-1 function such that $A = \text{range}(f)$. Let Y denote the ordered set of generators

$$a < y_0 < y_1 < \cdots$$

and let $F_{n-1}(Y)$ be the free class $n-1$ nilpotent group on the ordered generators Y. Let $D_A = \{[y_r, a^{(n-2)}] \mid \exists t\, (f(t) = r)\}$. D_A is c.e. and contained in the center of $F_{n-1}(Y)$.

Let $X = \{b\} \cup \{x_{rt} \mid r, t \in \omega\}$ be a set of generators ordered by $b < x_{rt}$ for all $r, t \in \omega$ and $x_{rt} < x_{uv}$ if and only if $\langle r, t\rangle < \langle u, v\rangle$ in a fixed computable order on ω^2. Let $F_n(X)$ be the free class n nilpotent group on this ordered set of generators and let $E_A = \{[x_{rt}, b^{(n-1)}] \mid f(t) = r\}$. E_A is computable and contained in the center of $F_n(X)$.

Let $G = (F_{n-1}(Y) \times F_n(X))/\langle D_A \circ E_A \rangle$ where $D_A \circ E_A$ is the computable set

$$D_A \circ E_A = \{\langle [y_r, a^{(n-2)}]^{-1}, 1\rangle \cdot \langle 1, [x_{rt}, b^{(n-1)}]\rangle \mid f(t) = r\}.$$

Since the elements of $D_A \circ E_A$ are contained in the center of $F_{n-1}(Y) \times F_n(X)$, the subgroup $\langle D_A \circ E_A\rangle$ generated by $D_A \circ E_A$ is normal. Furthermore, it is computable since an element $\langle g, h\rangle$ of $F_{n-1}(Y) \times F_n(X)$ is in this subgroup if and only if when g and h are written in normal form in $F_{n-1}(Y)$ and $F_n(X)$ respectively, we meet the following conditions:

- the normal form of g contains only basic commutators of the form $[y_r, a^{(n-2)}]$ and the normal form of h contains only basic commutators from E_A,
- for every power of a basic commutator $[x_{rt}, b^{(n-1)}]^k$ in the normal form of h, the basic commutator $[y_r, a^{(n-2)}]$ occurs in the normal form of g with power $-k$, and
- for every power of a basic commutator $[y_r, a^{(n-2)}]^k$ in the normal form of g, the basic commutator $[x_{rt}, b^{(n-1)}]$ with $f(t) = r$ occurs in the normal form of h with power $-k$.

Moreover, G is a computable group. Latkin [7] proves (see Lemma 2.1 in [7]) that one can effectively obtain normal forms for the elements of G, which we will call G-normal forms, as follows. Take an element $\langle g, h\rangle \in F_{n-1}(Y) \times F_n(X)$. Write g in $F_{n-1}(Y)$-normal form and write h in $F_n(X)$-normal form. For each power of a basic commutator $[x_{rt}, b^{(n-1)}]^k$ from E_A in the normal form of h, convert $[x_{rt}, b^{(n-1)}]^k$ into $[y_r, a^{(n-2)}]^k$ by removing $[x_{rt}, b^{(n-1)}]^k$ from the normal form of h and placing $[y_r, a^{(n-2)}]^k$ onto the end of the normal form of g. Finally, move the basic commutators $[y_r, a^{(n-2)}]$ into the correct position in the normal form of g. Notice that the converting process does not generate new basic commutators in these normal forms as the basic commutators in D_A and E_A lie in the center of $F_{n-1}(Y)$ and $F_n(X)$, respectively.

We think of the process of converting a pair $\langle g, h\rangle$ in $F_{n-1}(Y) \times F_n(X)$-normal form into a pair $\langle g', h'\rangle$ in G-normal form in terms of group multiplication. That is, from the description above, it is clear that when we view $\langle g', h'\rangle$ as a product of basic commutators in $F_{n-1}(Y) \times F_n(X)$-normal form, the conversion process yields $\langle g', h'\rangle = \langle g, h\rangle \cdot \langle c, d\rangle$ where c is in the center of $F_{n-1}(Y)$ and d is in the center of $F_n(X)$. (That is, d is an appropriate product of basic commutators of the form $[x_{rt}, b^{(n-1)}]^{-k}$ and c is an appropriate product of basic commutators of the form $[y_r, a^{(n-2)}]^k$.) Latkin shows that for each pair $\langle g, h\rangle$ in $F_{n-1}(Y) \times F_n(X)$, there is a *unique* pair $\langle g', h'\rangle$ in G-normal form such that $\langle g, h\rangle = \langle g', h'\rangle \bmod \langle D_A \circ E_A\rangle$. This, together with the fact that the procedure for finding an equivalent G-normal form is effective, shows that G is a computable group. From now on, we represent the elements of G by the unique member of their coset that is in G-normal form and use the fact that this G-normal form is also a $F_{n-1}(Y) \times F_n(X)$-normal form. (That is, we can view any G-normal form as an element of $F_{n-1}(Y) \times F_n(X)$ when convenient.)

Latkin [7] uses these normal forms to prove that the terms $\gamma_1 G, \ldots, \gamma_{n-1} G$ are computable and that $\gamma_n G$ has the same Turing degree as A. It remains to show that the terms in the upper central series of G are computable.

Lemma 3.2. *$\zeta_1 G$ is computable.*

Proof. Let $\langle g, h\rangle \in G$. We will show that $\langle g, h\rangle \in \zeta_1 G \iff \langle g, h\rangle \in \zeta_1(F_{n-1}(Y) \times F_n(X))$. Since $\zeta_1(F_{n-1}(Y) \times F_n(X))$ is computable, this will show that $\zeta_1 G$ is computable.

As G is a quotient of $F_{n-1}(Y) \times F_n(X)$, we certainly have $\langle g, h\rangle \in \zeta_1(F_{n-1}(Y) \times F_n(X)) \Rightarrow \langle g, h\rangle \in \zeta_1 G$. For the other direction, assume for a contradiction that $\langle g, h\rangle \in \zeta_1 G$ but $\langle g, h\rangle \notin \zeta_1(F_{n-1}(Y) \times F_n(X))$.

The condition $\langle g, h\rangle \notin \zeta_1(F_{n-1}(Y) \times F_n(X))$ implies that either $g \notin \zeta_1 F_{n-1}(Y)$ or $h \notin \zeta_1 F_n(X)$. Therefore, there are two cases to consider. First suppose that $g \notin \zeta_1 F_{n-1}(Y)$ and fix an element $z \in F_{n-1}(Y)$ such that $gz \neq zg$. Let w denote the $F_{n-1}(Y)$-normal form of gz and let v denote the $F_{n-1}(Y)$-normal form of zg. Then $v \neq w$ in $F_{n-1}(Y)$. However, $\langle w, h\rangle$ and $\langle v, h\rangle$ are both in G-normal form because h does not contain any basic commutators of the form $[x_{rt}, b^{(n-1)}]$ that need to be converted to obtain the G-normal form. Therefore, we have the following calculations in G between words in G-normal form.

$$\langle g, h\rangle \cdot \langle z, 1\rangle = \langle w, h\rangle$$
$$\langle z, 1\rangle \cdot \langle g, h\rangle = \langle v, h\rangle.$$

Since $\langle w, h\rangle \neq \langle v, h\rangle$ in G, we have a contradiction to the assumption that $\langle g, h\rangle \in \zeta_1 G$.

Second, assume that $h \notin \zeta_1 F_n(X)$. Fix $z \in F_n(X)$ such that $hz \neq zh$. Let w denote the $F_n(X)$-normal form of hz and write w as $w'c$ where c is the product of the elements of E_A occurring in w. (Since the elements of E_A are in the center of $F_n(X)$ we do not generate new basic commutators when we pull these elements to the end of w.) Let c' denote the product of basic commutators in $F_{n-1}(Y)$ formed by converting the basic commutators in c from the form $[x_{rt}, b^{(n-1)}]^k$ (in $F_n(X)$) to the form $[y_r, a^{(n-2)}]^{-k}$ (in $F_{n-1}(Y)$). That is, the G-normal form of $\langle 1, hz\rangle$ and $\langle 1, w'c\rangle$ is $\langle c', w'\rangle$.

Similarly, let v denote the $F_n(X)$-normal form of zh and write v as $v'd$ where d is the product of the elements of E_A occurring in v. Let d' denote the product of basic commutators in $F_{n-1}(Y)$ formed by converting the basic commutators in d from the form $[x_{rt}, b^{(n-1)}]^k$ to the form $[y_r, a^{(n-2)}]^{-k}$. That is, the G-normal form of $\langle 1, zh\rangle$ and $\langle 1, v'd\rangle$ is $\langle d', v'\rangle$.

Since $\langle g, h\rangle \in \zeta_1 G$, we know that $\langle g, h\rangle \cdot \langle 1, z\rangle = \langle 1, z\rangle \cdot \langle g, h\rangle$ in G, so that

$$\langle g^{-1}, 1\rangle \cdot \langle g, h\rangle \cdot \langle 1, z\rangle = \langle g^{-1}, 1\rangle \cdot \langle 1, z\rangle \cdot \langle g, h\rangle \bmod \langle D_A \circ E_A\rangle.$$

Simplifying, this gives $\langle 1, hz\rangle = \langle 1, zh\rangle \bmod \langle D_A \circ E_A\rangle$. By the uniqueness of G-normal form representatives, we have that $\langle c', w'\rangle = \langle d', v'\rangle$ in $F_{n-1}(Y) \times F_n(X)$. So $c' = d'$ and $w' = v'$. Note that from the definitions of c' and d' it follows that we must also have $c = d$. Hence $hz = w = w'c = v'd = v = zh$ in $F_n(X)$, contradicting our assumption that $hz \neq zh$. □

Lemma 3.3. *The upper central series terms in G are computable.*

Proof. This lemma follows immediately from the proof of Lemma 3.2. Since the canonical map $\pi : F_{n-1}(Y) \times F_n(X) \to G$ maps $\zeta_1(F_{n-1}(Y) \times F_n(X))$ onto $\zeta_1 G$ and since $\langle D_A \circ E_A\rangle$ is contained in $\zeta_1(F_{n-1}(Y) \times F_n(X))$, it follows that projections map each of the upper central terms of $F_{n-1}(Y) \times F_n(X)$ onto the corresponding upper central term of G. Because the upper central terms of $F_{n-1}(Y) \times F_n(X)$ are computable, it follows that the upper central terms of G are computable. □

This completes the proof of Theorem 1.7. The next lemma will be used in Section 5 to show that G admits a computable order.

Lemma 3.4. *The groups $\zeta_1 G$ and $\zeta_{i+1}G/\zeta_i G$ (for $1 \leq i \leq n$) are free abelian groups on a computable set of generators.*

Proof. By the proof of Lemma 3.2, $\zeta_1 G$ is a free abelian group with generators $\langle g, 1\rangle$ and $\langle 1, h\rangle$ where g is a basic commutator in $F_{n-1}(Y)$ of weight $n-1$ and h is a basic commutator in $F_n(X)$ of weight n which is not of the form $[x_{rt}, b^{(n-1)}]$ with $f(t) = r$. The quotient groups $\zeta_{i+1}G/\zeta_i(G)$ are free abelian groups on the generators $\langle g, 1\rangle$ and $\langle 1, h\rangle$ where g is a basic commutator in $F_{n-1}(Y)$ of weight $n - i$ and h is a basic commutator in $F_n(X)$ of weight $n - i + 1$. □

4. Upper central series

In this section, we prove Theorem 1.7 which is restated here for convenience.

Theorem 4.1. *For any $n \geq 2$ and c.e. Turing degree $\mathbf{d}$, there is a class n nilpotent computable group G such that the lower central terms are computable, $\zeta_i G$ is computable for $0 \leq i \leq n-2$ and $\deg(\zeta_{n-1}G) = \mathbf{d}$.*

Before constructing our group G, we build an auxiliary group H. Rather than describe H as the quotient of a free class n nilpotent group, we explicitly describe the elements and the multiplication operation on this group. (Alternately, one can give a description of H in terms of an appropriate quotient of a free class n nilpotent group. However, that approach requires a series of lemmas which are somewhat longer and more technical than those used here.)

H is generated by d and y_i, $i \in \omega$, ordered by $d < y_0 < y_1 < \cdots$. We stipulate that the only nontrivial basic commutators are the generators (each of which have weight 1) and those of the form $[y_i, d^{(l)}]$ (which have weight $l + 1$) for $1 \leq l \leq n-1$. (Recall the convention that $[y_i, d^{(0)}] = y_i$ which we use frequently below and that $w(c)$ denotes the weight of a basic commutator c.) Any commutator of the form $[y_i, d^{(l)}]$ for $l \geq n$ is trivial. The basic commutators are ordered by $c_1 < c_2$ if and only if $w(c_1) < w(c_2)$, or $w(c_1) = w(c_2) = 1$ and $c_1 < c_2$ in the order on the generators, or $w(c_1) = w(c_2) > 1$ and $c_1 = [y_i, d^{(l)}]$, $c_2 = [y_j, d^{(l)}]$ with $i < j$.

A word over the basic commutators is a sequence $c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}$ in which each c_i is a basic commutator, $\alpha_i \in \mathbb{Z} \setminus \{0\}$ and $c_i^{\alpha_i}$ is an abbreviation for c_i (or c_i^{-1}) repeated $|\alpha_i|$ times (depending on whether α_i is positive or negative). Such a word is in H-normal form if $c_1 < c_2 < \cdots < c_k$ in our order on the basic commutators. We typically write an H-normal form word as $d^\alpha X$ where $\alpha \in \mathbb{Z}$ (allowing the possibility of $\alpha = 0$ if d does not appear in the normal form) and X is a word in normal form over the basic commutators $[y_i, d^{(l)}]$ with $0 \leq l \leq n-1$.

The elements of H are the H-normal form words. We multiply two elements $h_1, h_2 \in H$ by concatenating $h_1 h_2$ and reducing the resulting word to H-normal form using the following procedure. To begin, consider the basic commutators in the H-normal forms of h_1 and h_2 as single entities, and let x be the word $h_1 h_2$. If x is not in H-normal form, choose the least basic commutator in x that is out of position, and bring it forward past commutators of greater weight using the following reduction rules. Reset x to be the resulting word viewing all newly generated basic commutators as single entities, and repeat the procedure until x is in H-normal form.

(R1) $[y_i, d^{(k)}]^\alpha\, [y_j, d^{(l)}]^\beta = [y_j, d^{(l)}]^\beta\, [y_i, d^{(k)}]^\alpha$ for all $i, j \in \omega$, $0 \leq k, l \leq n-1$ and $\alpha, \beta \in \mathbb{Z}$
(R2) $[y_i, d^{(l)}]\, d = d\, [y_i, d^{(l)}]\, [y_i, d^{(l+1)}]$
(R3) $[y_i, d^{(l)}]^{-1}\, d = d\, [y_i, d^{(l)}]^{-1}\, [y_i, d^{(l+1)}]^{-1}$
(R4) $[y_i, d^{(l)}]\, d^{-1} = d^{-1}\, [y_i, d^{(l)}]\, [y_i, d^{(l+1)}]^{-1}\, [y_i, d^{(l+2)}]\, [y_i, d^{(l+3)}]^{-1} \ldots$
(R5) $[y_i, d^{(l)}]^{-1}\, d^{-1} = d^{-1}\, [y_i, d^{(l)}]^{-1}\, [y_i, d^{(l+1)}]\, [y_i, d^{(l+2)}]^{-1}\, [y_i, d^{(l+3)}] \ldots$

The products in (R4) and (R5) are finite because the commutators $[y_i, d^{(k)}]$ for $k \geq n$ are trivial. Since new commutators generated by (R2)–(R5) are of strictly greater weight than the basic commutators that generated them, and since there is a maximum weight for the commutators, this procedure must halt. The reduction rules (R2)–(R5) are exactly the reduction rules for basic commutators in a free nilpotent group (as described in Section 2) given that all basic commutators of the form $[y_i, d^{(k)}]$ commute with each other by (R1).

The only nontrivial interaction between basic commutators is between d and $[y_i, d^{(l)}]$. If $h_1 = d^\alpha X$ and $h_2 = d^\beta Y$, then the process of reducing $h_1h_2 = d^\alpha X d^\beta Y$ involves moving d^β leftward across X to form $d^{\alpha+\beta}$ (which will generate new basic commutators according to (R2)–(R5)) and rearranging the remaining basic commutators in order (which will not generate new basic commutators by (R1)). If $c = [y_i, d^{(l)}]$ or $c^{-1} = [y_i, d^{(l)}]^{-1}$ appears in X, then moving d or d^{-1} leftward across this basic commutator will generate basic commutators of the form $[y_i, d^{(l+k)}]$ for various values of $k \geq 1$ (depending on whether d and c occur positively or negatively). Since $[y_i, d^{(l+k)}] = [c, d^{(k)}]$, we typically describe the new basic commutators generated by moving d^β across X as having the form $[c, d^{(k)}]$ where c occurs in X and $k \geq 1$ with the understanding that these new basic commutators may occur positively or negatively.

This describes our group H – the elements are the H-normal form words over the basic commutators, multiplication is given by concatenation followed by reduction and the identity element is the trivial word. We must still show that what we have described is indeed a group – that the group operation is associative and that inverses exist. Once we have shown that the operation is associative, we will know that, given a word over $\{d, y_0, y_1, \ldots\}$, if it is brought into normal form using the rules (R1)–(R5), it will always have the same result, no matter the order in which the rules were applied. Thus once we have proved associativity, it will immediately follow that inverse of $c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k} \in H$ is the normal form of $c_k^{-\alpha_k} \cdots c_2^{-\alpha_2} \cdot c_1^{-\alpha_1}$, as expected. We defer proof of associativity of the group operation to the end of this section, and proceed with the proof of Theorem 1.7.

The next lemma gives an algorithm for calculating the lower central terms of H.

Lemma 4.2. *For $x \in H$ and $1 \leq j \leq n$, $x \in \gamma_{j+1}H$ if and only if x contains only basic commutators from $A_j = \{[y_i, d^{(l)}] \mid i \in \omega$ and $l \geq j\}$.*

Proof. The elements of A_j are clearly in $\gamma_{j+1}H$ and hence any product of them is in $\gamma_{j+1}H$. Therefore, it suffices to show that $\gamma_{j+1}H \subseteq \langle A_j \rangle$ for $1 \leq j \leq n$. Since $j \geq 1$, the elements of A_j commute with each other by (R1) and therefore, for $x \in H$ (i.e. a word in H-normal form), $x \in \langle A_j \rangle$ if and only if each of the basic commutators occurring in x is in A_j.

We show that $\gamma_{j+1}H \subseteq \langle A_j \rangle$ by induction on j. When $j = 1$, an arbitrary generator of $\gamma_2 H$ has the form $[g, h]$ where $g, h \in H$. Write g and h in H-normal form as $g = d^\beta \cdot Y$ and $h = d^\delta \cdot Z$ where Y and Z are products of basic commutators $[y_i, d^{(l)}]$ for $l \geq 0$. Note that the basic commutators in Y and Z commute with each other.

By definition $[g, h] = Y^{-1} \cdot d^{-\beta} \cdot Z^{-1} \cdot d^{-\delta} \cdot d^\beta \cdot Y \cdot d^\delta \cdot Z$. Each basic commutator of the form y_i occurring in $[g, h]$ has the property that the sum of its powers in this product is 0. To put this product into normal form, pass the powers of d left to the front of this word. This process generates new basic commutators of the form $[c, d^{(l)}]$ where c is a basic commutator occurring in Y or Z and $l \geq 1$. The powers of d cancel after being passed to the front of the word and we are left with a word over the basic commutators in Y and Z and new basic commutators of the form $[c, d^{(l)}]$ where c occurred in Y or Z and $l \geq 1$. The remaining basic commutators commute with each other by (R1) and hence the remaining product can be put into H-normal form without generating any new basic commutators. Any basic commutators of the form y_i occurring in Y or Z cancel out because their powers summed to 0 in the original product and we have not generated any new basic commutators of this form. Therefore, the resulting H-normal form contains (possibly a subset of the) basic commutators from Y and Z of the form $[y_i, d^{(l)}]$ with $l \geq 1$ and the newly generated basic commutators. Since the newly generated basic commutators also have the form $[y_i, d^{(l)}]$ with $l \geq 1$, they all lie in A_1 as required. This completes the base case.

The induction case is similar. Assume $j \geq 1$ and $\gamma_{j+1}H \subseteq \langle A_j \rangle$. We show $\gamma_{j+2}H \subseteq \langle A_{j+1} \rangle$. An arbitrary generator of $\gamma_{j+2}H$ has the form $[g, h]$ where $g \in \gamma_{j+1}H$ and $h \in H$. Since $\gamma_{j+1}H = \langle A_j \rangle$, we write g in H-normal form as a product of basic commutators $[y_i, d^{(l)}]$ for $l \geq j \geq 1$. We write h in H-normal form as $d^\delta \cdot Z$ where Z is a product of basic commutators of the form $[y_i, d^{(l)}]$ for $l \geq 0$. Then $[g, h] = g^{-1} \cdot Z^{-1} \cdot d^{-\delta} \cdot g \cdot d^\delta \cdot Z$. Bringing d^δ left across g to cancel with $d^{-\delta}$ yields new basic commutators of the form $[c, d^{(l)}]$ where c is a basic commutator in g and $l \geq 1$. Since $c \in A_j$, these new basic commutators lie in A_{j+1}.

Since the powers of d sum to 0, we are left with a product consisting of the basic commutators in g, the basic commutators in Z and the newly generated basic commutators. These basic commutators commute with each other and hence we can put this product in normal form without generating any new basic commutators. The basic commutators in g and Z cancel (since they occur with opposite powers in g and g^{-1} and in Z and Z^{-1}) leaving us with only the newly generated basic commutators (some of which may cancel as well). However, the newly generated basic commutators are all in A_{j+1} and hence the resulting H-normal form is in $\langle A_{j+1} \rangle$ as required. □

Since $A_n = \{[y_i, d^{(l)}] \mid i \in \omega$ and $l \geq n\}$ and each $[y_i, d^{(n)}]$ is trivial, Lemma 4.2 implies that $\gamma_{n+1}H = 1$ and therefore H is a class n nilpotent group. To prove Theorem 1.7, we construct G out of infinitely many copies of H. For each $k \in \omega$, let H_k be a copy of H. To distinguish the generators of these groups, we denote the generators of H_k by d_k and $y_{i,k}$ for $i \in \omega$. The elements of H_k are words over the basic commutators d_k and $[y_{i,k}, d_k^{(l)}]$ for $0 \leq l \leq n-1$ in H_k-normal form.

Let f be a one-to-one function with infinite and coinfinite range such that range(f) has degree **d**. We use a quotient of H_k to code whether k is in the range of f and then we take a direct sum of the resulting quotient groups to code the entire range of f. Let $T_k \subseteq H_k$ be

$$T_k = \{[y_{i,k}, d_k^{(n-1)}] \mid \neg\exists j \leq i\, (f(j) = k)\}.$$

Since T_k is contained in the center of H_k, the subgroup $\langle T_k \rangle$ is normal. Let $G_k = H_k / \langle T_k \rangle$. We define G_k-normal forms as follows. A word over the basic commutators of H_k is in G_k-normal form if it is in H_k-normal form and does not contain any basic commutators in T_k. We effectively reduce an arbitrary word over the basic commutators to one in G_k-normal form by reducing it to a word in H_k-normal form and removing all basic commutators in T_k. (Because T_k is in the center of H_k, this process picks out a unique representative of each $\langle T_k\rangle$ equivalence class.) The elements of G_k are the G_k-normal form words with multiplication given by concatenation followed by reduction. Thus the G_k groups are computable uniformly in k.

We let $G = \bigoplus_{k\in\omega} G_k$, the direct sum of the groups G_k. That is, members of G are infinite sequences where the kth term is from G_k, cofinitely many terms are the identity, and the group operation is inherited componentwise from the G_k. We view G as a computable group by viewing its members as arbitrarily large finite tuples, where the componentwise multiplication is computable since the G_k are uniformly computable.

We claim that G is the desired group. To show that the lower central terms $\gamma_j G$ for $1 \le j \le n$ and the upper central terms $\zeta_u G$ for $1 \le u < n-1$ are computable, it suffices to show that the corresponding central terms of G_k are computable uniformly in k. We do this below in Lemmas 4.3–4.5.

It remains to show $\deg(\zeta_{n-1}G) = \mathbf{d}$. If $k \notin \text{range}(f)$, then $T_k = \{[y_{i,k}, d_k^{(n-1)}] \mid i \ge 0\}$ and hence G_k is a class $n-1$ nilpotent group. Therefore, $d_k \in \zeta_{n-1}G_k$ because $\zeta_{n-1}G_k = G_k$. However, if $k \in \text{range}(f)$, then fix j such that $f(j) = k$. For any $i \ge j$, $[y_{i,k}, d_k^{(n-1)}] \notin T_k$ and hence G_k is a properly class n nilpotent group. We show in Lemma 4.6 that in this case, $d_k \notin \zeta_{n-1}G_k$. Therefore, $k \in \text{range}(f)$ if and only if $d_k \notin \zeta_{n-1}G_k$, which holds if and only if $\langle 1_{G_0}, \ldots, 1_{G_{k-1}}, d_k\rangle \notin \zeta_{n-1}G$.

Lemma 4.3. *For $1 \le j \le n-1$, an element $g \in G_k$ (written in G_k-normal form) satisfies $g \in \gamma_{j+1}G_k$ if and only if all the basic commutators in g have the form $[y_{i,k}, d_k^{(l)}]$ for $l \ge j$. (The other lower central terms $\gamma_1 G_k = G_k$ and $\gamma_{n+1}G_k = 1_{G_k}$ are trivially computable.)*

Proof. Fix $g \in G_k$ and view it both as an element of G_k and as an element of H_k representing the coset $g\langle T_k\rangle$. (Note that g is in both G_k- and H_k-normal form.) By Lemma 2.5, $g \in \gamma_{j+1}G_k$ if and only if $g\langle T_k\rangle \cap \gamma_{j+1}H_k \neq \emptyset$. By Lemma 4.2, $g \in \gamma_{j+1}H_k$ if and only if g contains only basic commutators from $A_j = \{[y_{i,k}, d_k^{(l)}] \mid i \in \omega \text{ and } l \ge j\}$. Thus if g contains only basic commutators of the form $[y_{i,k}, d_k^{(l)}]$ for $l \ge j$, then $g \in g\langle T_k\rangle \cap \gamma_{j+1}H_k$ and hence $g \in \gamma_{j+1}G_k$.

For the other direction, assume $g \in \gamma_{j+1}G_k$ and fix $h \in H_k$ such that $h \in g\langle T_k\rangle \cap \gamma_{j+1}H_k$. Because $h \in \gamma_{j+1}H_k$, Lemma 4.2 implies that the basic commutators in h have the form $[y_{i,k}, d_k^{(l)}]$ for $l \ge j$. Since $h \in g\langle T_k\rangle$, the normal forms of g and h differ only by basic commutators in T_k. Thus the basic commutators in g must have the form $[y_{i,k}, d_k^{(l)}]$ for $l \ge j$ as required. □

We calculate the terms in the upper central series of G_k. For $0 \le u < n-1$, let

$$C_u = \{[y_{i,k}, d_k^{(l)}] \mid l \ge n-u\} \cup \{[y_{i,k}, d_k^{(l)}] \mid l \ge n-1-u \text{ and } \neg\exists j \le i\,(f(j) = k)\}$$

We show $\langle C_u\rangle = \zeta_u G_k$ for $0 \le u < n-1$. Note that $\langle C_0\rangle = 1_{G_k}$ since the basic commutators in C_0 are either $[y_{i,k}, d^{(n)}]$ or are in T_k. In either case, they are the identity in G_k. Also, note that since the elements of C_u commute with each other, a word in G_k-normal form is in $\langle C_u\rangle$ if and only if each of the basic commutators in the word is in C_u.

Lemma 4.4. *For $0 \le u < n-1$, $\langle C_u\rangle \subseteq \zeta_u G_k$.*

Proof. We proceed by induction on u, using $\langle C_0\rangle = 1_{G_k}$ as the base case. For the induction case, assume that $u < n-2$, and $\langle C_u\rangle \subseteq \zeta_u G_k$. We show that $\langle C_{u+1}\rangle \subseteq \zeta_{u+1}G_k$. Recall that basic commutators in G_k commute with each other, with the exception of d_k. So it suffices to show that for all $c \in C_{u+1}$ and all $\alpha, \beta \in \{+1, -1\}$, we have $c^\alpha \cdot d_k^\beta = d_k^\beta \cdot c^\alpha \bmod \zeta_u G_k$.

The basic commutator c is either of the form $c = [y_{i,k}, d_k^{(l)}]$ for some fixed $l \ge n-(u+1) = n-u-1$, or c is of the form $c = [y_{i,k}, d_k^{(l)}]$ for some fixed $l \ge n-1-(u+1) = n-u-2$ and $\neg\exists j \le i\,(f(j) = k)$. We break into cases depending on the form of c and the values of α and β. We restrict ourselves to two representative cases and leave the remaining cases to the reader.

First, consider the case when $c = [y_{i,k}, d^{(l)}]$ for $l \ge n-u-1$ and $\alpha = \beta = 1$. By (R2),

$$[y_{i,k}, d^{(l)}] \cdot d = d \cdot [y_{i,k}, d^{(l)}] \cdot [y_{i,k}, d^{(l+1)}]$$

Because $l \ge n-u-1$, we have $l+1 \ge n-u$ and hence $[y_{i,k}, d^{(l+1)}] \in C_u \subseteq \zeta_u G_k$ as required.

Second, consider the case when $\alpha = \beta = -1$ and $c = [y_{i,k}, d_k^{(l)}]$ for $l \ge n-u-2$ and $\neg\exists j \le i\,(f(j) = k)$. By (R5),

$$[y_{i,k}, d_k^{(l)}]^{-1} \cdot d_k^{-1} = d_k^{-1} \cdot [y_{i,k}, d_k^{(l)}]^{-1} \cdot [y_{i,k}, d_k^{(l+1)}] \cdot [y_{i,k}, d_k^{(l+2)}]^{-1} \cdot [y_{i,k}, d_k^{(l+3)}] \cdots$$

Since $l \ge n-u-2$, we have $l+p \ge n-1-u$ for each $p \ge 1$. Since $\neg\exists j \le i\,(f(j) = k)$, each of the basic commutators $[y_{i,k}, d_k^{(l+p)}]$ for $p \ge 1$ is in $C_u \subseteq \zeta_u G_k$. Therefore, the product $[y_{i,k}, d_k^{(l+1)}] \cdot [y_{i,k}, d_k^{(l+2)}]^{-1} \cdot [y_{i,k}, d_k^{(l+3)}] \cdots$ is in $\langle C_u\rangle \subseteq \zeta_u G_k$ as required. □

Lemma 4.5. *For $0 \le u < n-1$, $\langle C_u\rangle = \zeta_u G_k$.*

Proof. By the previous lemma, it suffices to show $\zeta_u G_k \subseteq \langle C_u\rangle$. Since we have already noted this fact for $u = 0$, we proceed by induction using $u = 0$ as the base case. For the induction case, assume $u < n-2$ and $\langle C_u\rangle = \zeta_u G_k$. We show $\zeta_{u+1}G_k \subseteq \langle C_{u+1}\rangle$. Consider an arbitrary element $g \in \zeta_{u+1}G_k$ written in G_k-normal form as $d_k^{\delta} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m}$. We show that $g \in \langle C_{u+1}\rangle$.

First, we show that $\delta = 0$. Suppose for a contradiction that $\delta \neq 0$. Because $g \in \zeta_{u+1}G_k$ and $\langle C_u\rangle = \zeta_u G_k$, we have

$$y_{0,k} \cdot d_k^{\delta} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} = d_k^{\delta} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} \cdot y_{0,k} \bmod \langle C_u\rangle$$

(That is, by the definition of the upper central series, elements of $\zeta_{u+1}G_k$ commute with all elements of G_k, in particular with $y_{0,k}$, modulo $\zeta_u G$.) Putting the element of the right side of this equation in G_k-normal form yields

$$d_k^{\delta} \cdot y_{0,k} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m}$$

because $y_{0,k}$ commutes with all basic commutators except d_k. To put the element on the left side of this equation into G_k-normal form, we move d_k^{δ} across $y_{0,k}$. If $\delta > 0$, then by (R2) and induction on δ we have

$$y_{0,k} \cdot d_k^{\delta} = d_k^{\delta} \cdot y_{0,k} \cdot [y_{0,k}, d_k]^{\delta} \cdot X$$

where X is a G_k-normal form word over the basic commutators of the form $[y_{0,k}, d_k^{(l)}]$ for $l \geq 2$. Similarly, if $\delta < 0$, then by (R4) and induction on $|\delta|$, we have

$$y_{0,k} \cdot d_k^{\delta} = d_k^{\delta} \cdot y_{0,k} \cdot [y_{0,k}, d_k]^{\delta} \cdot X$$

where X is a G_k-normal form word over the basic commutators of the form $[y_{0,k}, d_k^{(l)}]$ with $l \geq 2$. In either case, the newly generated commutators (which have the form $[y_{0,k}, d^{(l)}]$ for $l \geq 1$) commute with the c_i elements. It follows from $y_{0,k} \cdot g = g \cdot y_{0,k} \bmod \langle C_u\rangle$ that $[y_{0,k}, d_k]^{\delta} \cdot X \in \langle C_u\rangle$. However, since a normal form word is in $\langle C_u\rangle$ if and only if each basic commutator in the normal form is in C_u, we conclude that $[y_{0,k}, d_k] \in C_u$.

Recall that $u < n-2$ and hence $3 \leq n-u$ and $2 \leq n-1-u$. By definition, the elements of C_u have the form $[y_{i,k}, d^{(l)}]$ for $l \geq n-u \geq 3$ or the form $[y_{i,k}, d^{(l)}]$ for $l \geq n-1-u \geq 2$ and $\neg\exists j \leq i(f(j) = k)$. Therefore, $[y_{0,k}, d_k] \notin C_u$ for the desired contradiction.

We now know that g must have the form $c_1^{\alpha_1} \cdots c_m^{\alpha_m}$ where each $c_i > d$. Since $g \in \zeta_{u+1}G_k$ and $\langle C_u\rangle = \zeta_u G_k$, we have

$$d_k \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} = c_1^{\alpha_1} \cdots c_m^{\alpha_m} \cdot d_k \bmod \langle C_u\rangle$$

By (R2), (R3) and induction on $|\alpha|$, it follows that $c_i^{\alpha_i} \cdot d_k = d_k \cdot c_i^{\alpha_i} \cdot [c_i, d_k]^{\alpha_i}$. Since the $[c_i, d_k]$ basic commutators commute with each other as well as with the c_j commutators, this equation is equivalent to

$$d_k \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} = d_k \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} \cdot [c_1, d_k]^{\alpha_1} \cdots [c_m, d_k]^{\alpha_m} \bmod \langle C_u\rangle$$

which is true if and only if $[c_i, d_k] \in C_u$ for $1 \leq i \leq m$. However, by the definitions of C_u and C_{u+1}, we have that if $[c_i, d_k] \in C_u$, then $c_i \in C_{u+1}$. Therefore, $g \in \zeta_{u+1}G_k$ implies that the G_k-normal form of g is a product of basic commutators from C_{u+1}. In other words, $\zeta_{u+1}G_k \subseteq \langle C_{u+1}\rangle$ as required. □

Lemma 4.6. *For any $k \in range(f)$ and $w \in G_k$ (in G_k-normal form), $w \in \zeta_{n-1}G_k$ if and only if the basic commutators in w have the form $[y_{i,k}, d_k^{(l)}]$ for $l \geq 1$ or the form $y_{i,k}$ and $\neg\exists j \leq i(f(j) = k)$. In particular, $d_k \notin \zeta_{n-1}G_k$.*

Proof. Fix $k \in \mathrm{range}(f)$ and let

$$C_{n-1} = \{[y_{i,k}, d_k^{(l)}] \mid l \geq 1\} \cup \{y_{i,k} \mid \neg\exists j \leq i\,(f(j) = k)\}.$$

We have $\langle C_{n-1}\rangle \subseteq \zeta_{n-1}G_k$ exactly as in the proof of Lemma 4.4. To show that $\zeta_{n-1}G_k \subseteq \langle C_{n-1}\rangle$, we need to modify the proof of Lemma 4.5 to use the hypothesis that $k \in \mathrm{range}(f)$ when showing that $\delta = 0$ for the potential d^{δ} term. To do this, rather than looking at a calculation involving $y_{0,k}$, we let i be such that $\exists j \leq i\,(f(j) = k)$. Now, the fact that

$$y_{i,k} \cdot d_k^{\delta} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} = d_k^{\delta} \cdot c_1^{\alpha_1} \cdots c_m^{\alpha_m} \cdot y_{i,k} \bmod \langle C_{n-2}\rangle$$

implies that $[y_{i,k}, d_k] \in C_{n-2}$. Since we know that $\exists j \leq i\,(f(j) = k)$, this means that $[y_{i,k}, d_k]$ has the form $[y_{i,k}, d_k^{(l)}]$ for some $l \geq n-(n-2) = 2$, which gives the desired contradiction. The remainder of the proof is the same. □

This completes the proof of Theorem 1.7 with the exception of showing that the multiplication in the original definition of the group H is associative, which we now establish.

Recall that the elements of H are the H-normal form words. We will write elements of H as $d^{\alpha}X$, where X is an H-normal form word that does not contain (any non-zero power of) the basic commutator d. To show this multiplication is associative, we need to show that, for $d^{\alpha}X$, $d^{\beta}Y$, and $d^{\gamma}Z$ in H,

$$(d^{\alpha}X \cdot d^{\beta}Y) \cdot d^{\gamma}Z = d^{\alpha}X \cdot (d^{\beta}Y \cdot d^{\gamma}Z).$$

For X, an H-normal form word not containing the basic commutator d, and for any $\alpha \in \mathbb{Z}$, it is easy to see that $X \cdot d^{\alpha}$ has H-normal form $d^{\alpha}Y$, where Y does not contain the basic commutator d. We introduce the notation $X_{\alpha} = Y$.

The proof of associativity relies on two lemmas: for all X and Y not containing the basic commutator d, and all $\alpha, \beta \in \mathbb{Z}$, we have $(X \cdot Y)_{\alpha} = X_{\alpha} \cdot Y_{\alpha}$ and $(X_{\alpha})_{\beta} = X_{\alpha+\beta}$.

By (R1), it is easy to see that the members of H not containing the basic commutator d form an abelian subgroup of H. Now, given these facts and our rules for multiplication, for arbitrary $d^\alpha X$, $d^\beta Y$ and $d^\gamma Z \in H$, we have

$$\begin{aligned}(d^\alpha X \cdot d^\beta Y) \cdot d^\gamma Z &= (d^{\alpha+\beta}(X_\beta \cdot Y)) \cdot d^\gamma Z\\ &= d^{\alpha+\beta+\gamma}((X_\beta \cdot Y)_\gamma \cdot Z)\\ &= d^{\alpha+\beta+\gamma}(((X_\beta)_\gamma \cdot Y_\gamma) \cdot Z)\\ &= d^{\alpha+\beta+\gamma}((X_{\beta+\gamma} \cdot Y_\gamma) \cdot Z)\\ &= d^{\alpha+\beta+\gamma}(X_{\beta+\gamma} \cdot (Y_\gamma \cdot Z))\\ &= d^\alpha X \cdot d^{\beta+\gamma}(Y_\gamma \cdot Z)\\ &= d^\alpha X \cdot (d^\beta Y \cdot d^\gamma Z)\end{aligned}$$

as desired. It remains to prove the lemmas.

We begin by giving two extensions of the commutator rules for H. Applying (R2) and (R3) with induction on $|\alpha|$ we obtain (R6), and applying (R4) and (R5) with induction on $|\alpha|$ we obtain (R7).

(R6) For all $\alpha \in \mathbb{Z}$ and $c = [y_i, d^{(l)}]$ for $l \geq 0$,

$$c^\alpha\, d = d\,(c^{\frac{\alpha}{|\alpha|}}\,[c, d]^{\frac{\alpha}{|\alpha|}})^{|\alpha|}$$

(R7) For all $\alpha \in \mathbb{Z}$ and $c = [y_i, d^{(l)}]$ for $l \geq 0$,

$$c^\alpha\, d^{-1} = d^{-1}\,(c^{\frac{\alpha}{|\alpha|}}\,[c, d]^{-\frac{\alpha}{|\alpha|}}\,[c, d^{(2)}]^{\frac{\alpha}{|\alpha|}}\,[c, d^{(3)}]^{-\frac{\alpha}{|\alpha|}}\cdots)^{|\alpha|}$$

Lemma 4.7. *For all H-normal forms X and Y not containing the basic commutator d, $(X \cdot Y) \cdot d = X \cdot (Y \cdot d)$ and $(X \cdot Y) \cdot d^{-1} = X \cdot (Y \cdot d^{-1})$. In other words, $(X \cdot Y)_1 = X_1 \cdot Y_1$ and $(X \cdot Y)_{-1} = X_{-1} \cdot Y_{-1}$.*

Proof. Let $c_1, \ldots, c_n$ denote the basic commutators occurring in both X and Y. Let X be some arrangement of $a_1^{\alpha_1}, \ldots, a_k^{\alpha_k}, c_1^{\delta_1}, \ldots, c_n^{\delta_n}$, and let Y be some arrangement of $b_1^{\beta_1} \cdots b_l^{\beta_l} c_1^{\gamma_1} \cdots c_n^{\gamma_n}$. Here all the a_i, b_j and c_m are distinct basic commutators, and only the c_m occur in both X and Y. Since all basic commutators occurring in X and Y commute with one another, and since all basic commutators generated by (R2)–(R5) commute with one another, the particular order in which the basic commutators occur in X and Y is not important for this discussion.

To compute $(X \cdot Y) \cdot d$, we first use (R1) to bring $X \cdot Y$ into H-normal form. So $X \cdot Y$ is some arrangement of $a_1^{\alpha_1}, \ldots, a_k^{\alpha_k}, b_1^{\beta_1}, \ldots, b_l^{\beta_l}, c_1^{\delta_1+\gamma_1}, \ldots, c_n^{\delta_n+\gamma_n}$. We then move d to the front of the word. By (R6), the resulting word is dw, where w is some arrangement of the words $(a_1^{\frac{\alpha_1}{|\alpha_1|}}[a_1, d]^{\frac{\alpha_1}{|\alpha_1|}})^{|\alpha_1|}, \ldots, (a_k^{\frac{\alpha_k}{|\alpha_k|}}[a_k, d]^{\frac{\alpha_k}{|\alpha_k|}})^{|\alpha_k|}$, $(b_1^{\frac{\beta_1}{|\beta_1|}}[b_1, d]^{\frac{\beta_1}{|\beta_1|}})^{|\beta_1|}, \ldots, (b_l^{\frac{\beta_l}{|\beta_l|}}[b_l, d]^{\frac{\beta_l}{|\beta_l|}})^{|\beta_l|}$, $(c_1^{\frac{\delta_1+\gamma_1}{|\delta_1+\gamma_1|}}[c_1, d]^{\frac{\delta_1+\gamma_1}{|\delta_1+\gamma_1|}})^{|\delta_1+\gamma_1|}, \ldots, (c_n^{\frac{\delta_n+\gamma_n}{|\delta_n+\gamma_n|}}[c_n, d]^{\frac{\delta_n+\gamma_n}{|\delta_n+\gamma_n|}})^{|\delta_n+\gamma_n|}$. The terms in w are then commuted using (R1) into H-normal form. Because all of the terms in w commute without generating new basic commutators, we end up with an appropriate rearrangement (with possible cancellation) of $a_i^{\alpha_i}, [a_i, d]^{\alpha_i}, b_i^{\beta_i}, [b_i, d]^{\beta_i}, c_i^{\delta_i+\gamma_i}$ and $[c_i, d]^{\delta_i+\gamma_i}$.

To compute $X \cdot (Y \cdot d)$, we first bring d to the front of Y. By (R6), the resulting word is dv, where v is some arrangement of the words $(b_1^{\frac{\beta_1}{|\beta_1|}}[b_1, d]^{\frac{\beta_1}{|\beta_1|}})^{|\beta_1|}, \ldots, (b_l^{\frac{\beta_l}{|\beta_l|}}[b_l, d]^{\frac{\beta_l}{|\beta_l|}})^{|\beta_l|}$, $(c_1^{\frac{\delta_1}{|\delta_1|}}[c_1, d]^{\frac{\delta_1}{|\delta_1|}})^{|\delta_1|}, \ldots, (c_n^{\frac{\delta_n}{|\delta_n|}}[c_n, d]^{\frac{\delta_n}{|\delta_n|}})^{|\delta_n|}$. We then commute the terms in v, using (R1), to get its normal form v'. Next, we bring d to the front of X. The resulting word is duv', where u is some arrangement of the words $(a_1^{\frac{\alpha_1}{|\alpha_1|}}[a_1, d]^{\frac{\alpha_1}{|\alpha_1|}})^{|\alpha_1|}, \ldots, (a_k^{\frac{\alpha_k}{|\alpha_k|}}[a_k, d]^{\frac{\alpha_k}{|\alpha_k|}})^{|\alpha_k|}$, $(c_1^{\frac{\gamma_1}{|\gamma_1|}}[c_1, d]^{\frac{\gamma_1}{|\gamma_1|}})^{|\gamma_1|}, \ldots, (c_n^{\frac{\gamma_n}{|\gamma_n|}}[c_n, d]^{\frac{\gamma_n}{|\gamma_n|}})^{|\gamma_n|}$. Finally we commute all terms in uv' using (R1). As the terms in v rearrange to those in v' without generating new basic commutators, and the terms in uv' rearrange without generating new basic commutators, the result of these rearrangements yields an appropriate rearrangement (with possible cancellation) of $a_i^{\alpha_i}, [a_i, d]^{\alpha_i}, b_i^{\beta_i}, [b_i, d]^{\beta_i}, c_i^{\delta_i}, [c_i, d]^{\delta_i}, c_i^{\gamma_i}$ and $[c_i, d]^{\gamma_i}$. Thus the two processes give the same H-normal form.

The key point in the above argument is the symmetry in rules (R2) and (R3) as expressed in (R6). Since we have the same symmetry in (R4) and (R5), as expressed in (R7), we obtain $(X \cdot Y) \cdot d^{-1} = X \cdot (Y \cdot d^{-1})$ in a similar fashion. □

Corollary 4.8. *For all $X_1, X_2, \ldots, X_k$ not containing the basic commutator d, and for $\alpha \in \{-1, 1\}$, we have $(X_1 \cdot X_2 \cdots X_k)_\alpha = (X_1)_\alpha \cdot (X_2)_\alpha \cdots (X_k)_\alpha$.*

Proof. We note that since, $X_1, X_2, \ldots, X_k$, and $(X_1)_\alpha, (X_2)_\alpha, \ldots, (X_k)_\alpha$ are all part of the same abelian subgroup of H (by (R1)), there is no harm in omitting brackets in the products — they can be reinserted in any way. The corollary follows immediately from Lemma 4.7 by induction on k. □

We use Corollary 4.8 in the following manner. Let w be a word over the basic commutators other than d and let Y be the H-normal form of w. By Corollary 4.8, reducing the string $w\, d^\alpha$ (i.e. passing d^α across w to obtain $d^\alpha\, v$ and then reducing v using (R1)) gives the same H-normal form as reducing $Y\, d^\alpha$. In other words, for a string w over the basic commutators other than d, we obtain the same H-normal form either by first reducing w to Y with (R1), then passing d^α across Y and reducing again with (R1), or by first passing d^α across w and then reducing the resulting string with (R1).

Lemma 4.9. *For all H-normal forms X not containing the basic commutator d and for all $n \in \mathbb{N}$, $(X \cdot d^n) \cdot d = X \cdot d^{n+1}$ and $(X \cdot d^{-n}) \cdot d^{-1} = X \cdot d^{-n-1}$. In other words, $(X_n)_1 = X_{n+1}$ and $(X_{-n})_{-1} = X_{-n-1}$.*

Proof. To calculate the H-normal form of $(X \cdot d^n) \cdot d$, we pass n copies of d across X to obtain $(d^n\, w) \cdot d$ where w is not necessarily in H-normal form but does not contain the basic commutator d. Then we reduce w to H-normal form Y using (R1). Finally, we reduce the string $d^n\, Y\, d$ by passing d across Y and reducing the result to H-normal form using (R1).

To calculate the H-normal form of $X \cdot d^{n+1}$, we pass n copies of d across X to obtain the string $d^n\, w\, d$. Before combining terms, we pass the final copy of d across w and then reduce using (R1). However, by Corollary 4.8, the H-normal forms of $Y\, d$ and $w\, d$ are the same and hence we obtain the same H-normal form in each case. The case for d^{-n} is similar. □

We now prove the two lemmas required for the associativity of multiplication in H.

Lemma 4.10. *For all H-normal forms X and Y not containing the basic commutator d and all $\alpha \in \mathbb{Z}$, $(X \cdot Y) \cdot d^{\alpha} = X \cdot (Y \cdot d^{\alpha})$. In other words, $(X \cdot Y)_{\alpha} = X_{\alpha} \cdot Y_{\alpha}$.*

Proof. We proceed by induction on $|\alpha|$. The base case when $|\alpha| = 1$ is given by Lemma 4.7. For the induction case, consider when $\alpha = n + 1$ is positive.

$$(X \cdot Y)_{n+1} = ((X \cdot Y)_n)_1 = (X_n \cdot Y_n)_1 = (X_n)_1 \cdot (Y_n)_1 = X_{n+1} \cdot Y_{n+1}.$$

The first equality is from Lemma 4.9, the second equality is from the inductive hypothesis, the third equality is from Lemma 4.7 and the last equality is from Lemma 4.9. The calculation when $\alpha = -n - 1$ is similar. □

Lemma 4.11. *For all X not containing the basic commutator d, and all $\alpha, \beta \in \mathbb{Z}$, we have $(X_{\alpha})_{\beta} = X_{\alpha+\beta}$. That is, $(X \cdot d^{\alpha}) \cdot d^{\beta} = X \cdot d^{\alpha+\beta}$.*

Proof. If α and β are both positive or both negative, this holds by Lemma 4.9 and induction.

Consider the case when α is positive and β is negative. We first show by induction on $n \geq 1$ that for all X, $(X_n)_{-1} = X_{n-1}$, i.e., that $(X \cdot d^n) \cdot d^{-1} = X \cdot d^{n-1}$. When $n = 1$, it suffices to show (by repeated applications of Lemma 4.7) that for all basic commutators $c = [y_i, d^{(l)}]$ and all $k \in \mathbb{Z}$ we have $(c^k \cdot d) \cdot d^{-1} = c^k$. Now

$$\begin{aligned}(c^k \cdot d) \cdot d^{-1} &= (d\,(c^{\frac{k}{|k|}}\,[c,d]^{\frac{k}{|k|}})^{|k|}) \cdot d^{-1} \\ &= (d\,c^k\,[c,d]^k) \cdot d^{-1} \\ &= d\,c^k\,d^{-1}\,([c,d]^{\frac{k}{|k|}}\,[c,d^{(2)}]^{-\frac{k}{|k|}}\,[c,d^{(3)}]^{\frac{k}{|k|}}\cdots)^{|k|} \\ &= d\,d^{-1}\,(c^{\frac{k}{|k|}}\,[c,d]^{-\frac{k}{|k|}}\,[c,d^{(2)}]^{\frac{k}{|k|}}\,[c,d^{(3)}]^{-\frac{k}{|k|}}\cdots)^{|k|} \\ &\qquad ([c,d]^{\frac{k}{|k|}}\,[c,d^{(2)}]^{-\frac{k}{|k|}}\,[c,d^{(3)}]^{\frac{k}{|k|}}\cdots)^{|k|}\end{aligned}$$

Since the d's cancel and the remaining terms commute by (R1), this product reduces to c^k as required. For the induction case, we have $(X_{n+1})_{-1} = ((X_n)_1)_{-1} = X_n$, where the first equality is by Lemma 4.9 and the second follows from the base case applied to X_n.

Next we show by induction on $m \geq 1$ that for all X, $(X_n)_{-m} = X_{n-m}$. The base case is the result for the previous paragraph. For the induction case,

$$(X_n)_{-m-1} = ((X_n)_{-m})_{-1} = (X_{n-m})_{-1} = X_{n-m-1}.$$

The first and third equalities follow from Lemma 4.9 and the second equality is the induction hypothesis. This completes the proof of the lemma in the case when α is positive and β is negative. The remaining case (when α is negative and β is positive) is similar. □

5. Computable orders

In this section, we show that the groups constructed in this paper admit computable orders. It follows that our independence result on the complexity of terms in the upper and lower central series holds within the class of computable ordered nilpotent groups. We begin by reviewing some basic definitions and facts about ordered groups.

Definition 5.1. An *ordered group* consists of a group G and a linear order $\leq_G$ on G such that for all $g, h, k \in G$, if $g \leq_G h$, then $gk \leq_G hk$ and $kg \leq_G kh$.

Lemma 5.2. *If G is a computable group which is free abelian on a computable set of generators, then G admits a computable order.*

Proof. Fix any computable order on the set of generators and extend this order lexicographically to the group. □

Lemma 5.3. *Let $(G_i, \leq_i)$ be a (possibly infinite) uniform sequence of computable ordered groups. The direct sum $\bigoplus_i G_i$ admits a computable order.*

Proof. Order G lexicographically using the computable order $\leq_i$ on component G_i. □

To show that the groups in Theorem 1.3 admit computable orders, it suffices by Lemma 5.3 to show that the groups constructed for Theorems 1.6 and 1.7 admit computable orders. Proofs of the classical versions of Lemmas 5.4 and 5.5 can be found in standard texts on ordered groups such as Kokorin and Kopytov [6] and are easily seen to be effective.

Lemma 5.4. *Let G be a class r nilpotent computable group for which the terms in the lower central series are computable, $\gamma_r G$ admits a computable order and each factor $\gamma_i G/\gamma_{i+1} G$ for $1 \leq i < r$ admits a computable order. Then G admits a computable order.*

Lemma 5.5. *Let G be a class r nilpotent computable group for which the terms in the upper central series are computable, $\zeta_1 G$ admits a computable order and each factor $\zeta_{i+1} G/\zeta_i G$ for $1 \leq i < r$ admits a computable order. Then G admits a computable order.*

Theorem 5.6. *The group G constructed in Section 3 to satisfy Theorem 1.6 admits a computable order.*

Proof. By Lemmas 3.4 and 5.2, $\zeta_1 G$ admits a computable order and each upper central factor $\zeta_{i+1} G/\zeta_i G$ admits a computable order. Therefore, by Lemma 5.5, G admits a computable order. □

Theorem 5.7. *The group G constructed in Section 4 to satisfy Theorem 1.7 admits a computable order.*

Proof. By Lemma 5.3, it suffices to show that the groups G_k admit a uniformly computable sequence of orders. Each G_k is a nilpotent computable group with computable lower central series. The subgroup $\gamma_n G_k$ is free abelian on the basic commutators $[y_{i,k}, d_k^{(n-1)}]$ for which $\exists j \leq i\,(f(j) = k)$. (Note that we do not need to know whether there are any such commutators to produce a computable order for $\gamma_n G_k$. Given any pair of elements in $\gamma_n G_k$ we can order them lexicographically by declaring that $[y_{a,k}, d_k^{(n-1)}] < [y_{b,k}, d_k^{(n-1)}]$ whenever both of these commutators are in G_k and $a < b$.) The factor $\gamma_1 G_k/\gamma_2 G_k$ is a free abelian group on the generators d_k and $y_{i,k}$. The remaining factors $\gamma_j G_k/\gamma_{j+1} G_k$ are free abelian groups on the generators $[y_{i,k}, d_k^{(j)}]$. Therefore, by Lemmas 5.2 and 5.4, G_k admits a computable order, uniformly in k. □

6. Open questions

In addition to proving Theorem 1.4, Latkin [7] proves the following theorem showing that one can fix the c.e. Turing degrees of the terms in the lower central series in every computable copy of a class n nilpotent group in any desired way.

Theorem 6.1 (*Latkin [7]*). *For each $n \geq 2$ and c.e. degrees $\mathbf{e_2}, \ldots, \mathbf{e_n}$, there is a class n nilpotent computable group G such that in every computable copy $H \cong G$, $\deg(\gamma_i H) = \mathbf{e_i}$ for $2 \leq i \leq n$.*

The construction of G in Theorem 6.1 uses torsion elements and hence G does not admit an order (computable or otherwise). This observation raises the question of whether one can obtain a similar result using a torsion-free nilpotent group, and if so, whether such a group could admit a computable order (in some or possibly all computable copies).

Theorem 6.1 also raises the natural question of whether one can obtain a similar result for the terms in the upper central series, and if so, whether one can do it with a torsion-free (or possibly computably orderable) nilpotent computable group.

Acknowledgements

B. Csima was partially supported by Canadian NSERC Discovery Grant 312501, and visits to the Institut des Hautes Études Scientifiques 32 and the Max Planck Institute for Mathematics.

References

[1] G. Baumslag, F.B. Cannonito, D.J.S. Robinson, D. Segal, The algorithmic properties of polycyclic-by-finite groups, J. Algebra 142 (1991) 118–149.
[2] M. Hall Jr., A basis for free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc. 6 (1950) 575–581.
[3] M. Hall Jr., The Theory of Groups, Macmillan, New York, 1959.
[4] P. Hall, Some word problems, J. Lond. Math. Soc. 33 (1958) 482–496.
[5] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, A.M. Slinko, Degree spectra and computable dimensions in algebraic structures, Ann. Pure Appl. Logic 115 (2002) 71–113.
[6] A.I. Kokorin, V.M. Kopytov, Fully Ordered Groups, J. Wiley, New York, 1974.
[7] I.V. Latkin, Algorithmic complexity of the problem of occurrence in commutants and the members of the lower central series, Sib. J. Math. (1987) 772–779.
[8] Magnus, Karrass, Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
[9] C.F. Miller III, Decision problems for groups – survey and reflections, in: G. Baumslag, C.F. Miller III (Eds.), Algorithms and Classification in Combinatorial Group Theory, Springer-Verlag, Heidelberg, 1992.
[10] H. Neumann, Varieties of Groups, Springer-Verlag, Heidelberg, 1967.
[11] D.J.S. Robinson, A Course in the Theory of Groups, Springer-Verlag, Heidelberg, 1996.
[12] R. Solomon, Reverse mathematics and fully ordered groups, Notre Dame J. Form. Log. 39 (1998) 157–189.
[13] R.I. Soare, Recursively Enumerable Sets and Degrees, Springer-Verlag, Heidelberg, 1987.