Lie symmetry reductions and exact solutions of a coupled KdV–Burgers equation

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A R T I C L E   I N F O

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Solitary wave solutions
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A B S T R A C T

In this paper, based on classical Lie group method, with the help of Maple software, we study a coupled KdV–Burgers equation, and get its infinitesimal generator, commutation table of Lie algebra, symmetry group and similarity reductions. In particular, solitary wave solutions and similarity solutions of the coupled KdV–Burgers equation are derived from the reduction equations.

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1. Introduction

KdV–Burgers equation was derived from the liquid flow in bubble and the liquid flow in the elastic pipeline, then some researchers applied it to study turbulence [1,2]. Its form is given as follows

\[ u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} = 0, \]  

where \( \alpha \) is the dissipation coefficient and \( \beta \) is dispersion coefficient. This equation can be regarded as a simplest dissipation model. Therefore, the study of the KdV–Burgers equation has important theoretical and applied significance.

Seeking the exact solutions of nonlinear partial differential equations (NPDEs) has been an interesting and hot topic in mathematics and physics for a long time. Some effective methods to construct exact solutions of NPDEs have been put forward, such as the inverse scattering method, Hirota method, Bäcklund transformation, homogeneous balance method and so on [3–6]. Lie group theory was widely used in many branches of mathematics and physics, it has been developed in differential equations since Bluman and Cole proposed similarity theory for differential equations in 1970s [7]. Now this method becomes one of effective methods to get similarity solutions and solitary wave solutions of NPDEs. The main idea of Lie group method is to use the invariance condition of given NPDEs to get similarity solutions and solitary wave solutions of NPDEs. The main idea of Lie group method is to use the invariance condition of given NPDEs to get similarity solutions and solitary wave solutions of NPDEs by solving the reduction equations [7–14]. Solitary wave solutions and similarity solutions are usually applied to describe physical phenomena and to check on the accuracy and reliability of numerical algorithm, so deriving solitary wave solutions and similarity solutions has a great significance. To the best of our knowledge, related classical Lie group method has not been preformed to the coupled KdV–Burgers equation.

In this paper, we consider the following coupled KdV–Burgers equation

\[
\begin{align*}
    u_t + uu_x + \alpha_1 u_{xx} + \beta_1 u_{xxx} &= 0, \\
    v_t + vv_x + \alpha_2 v_{xx} + \beta_2 v_{xxx} &= 0,
\end{align*}
\]

where \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) are nonzero constant.
In this paper, we first perform Lie symmetry analysis for the coupled KdV–Burgers equation (2). Then, we discuss the Lie symmetry group of Eq. (2). Finally, by using similarity variables to obtain reduction equations, and solving the reduction equation, we obtain solitary wave solutions or similarity solutions of Eq. (2).

2. Lie symmetry analysis of the coupled KdV–Burgers equation

In this section, we perform Lie symmetry analysis for Eq. (2), and obtain its infinitesimal generator, commutation table of Lie algebra.

According to the method of determining the infinitesimal generator of NPDEs, we take the infinitesimal generator of Eq. (2) as follows:

\[
\mathcal{V} = \zeta(x, t, u, \nu) \frac{\partial}{\partial x} + \tau(x, t, u, \nu) \frac{\partial}{\partial t} + \eta_1(x, t, u, \nu) \frac{\partial}{\partial u} + \eta_2(x, t, u, \nu) \frac{\partial}{\partial \nu},
\]

where \(\zeta(x, t, u, \nu), \tau(x, t, u, \nu), \eta_1(x, t, u, \nu)\) and \(\eta_2(x, t, u, \nu)\) are coefficient functions of the infinitesimal generator to be determined.

Using the invariance condition \(pr^3(V(\Delta))\big|_x = 0\), where \(\Delta\) is Eq. (2) and \(pr^3\mathcal{V}\) is the third prolongation of \(\mathcal{V}\), applying the third prolongation of \(\mathcal{V}\) to Eq. (2), and with help of Maple software, we get the following determining equations:

\[
\begin{align*}
\zeta_x &= \zeta_u = \zeta_v = 0, \\
\tau_x &= \tau_t = \tau_u = \tau_v = 0, \\
\eta_{1x} &= \eta_{1t} = \eta_{1u} = \eta_{1v} = 0, \\
\eta_{2x} &= \eta_{2t} = \eta_{2u} = \eta_{2v} = 0, \\
\zeta_t &= \eta_1 = \eta_2.
\end{align*}
\]

Solving above Eqs. (4), we obtain

\[
\begin{align*}
\zeta &= C_1 x + C_2, \\
\tau &= C_3, \\
\eta_1 &= C_1, \\
\eta_2 &= C_2,
\end{align*}
\]

where \(C_1, C_2\) and \(C_3\) are arbitrary constants.

Hence the Lie algebra of infinitesimal symmetries of Eq. (2) is spanned by the following vector fields:

\[
\mathcal{V}_1 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \mathcal{V}_2 = \frac{\partial}{\partial u}, \quad \mathcal{V}_3 = \frac{\partial}{\partial \nu}
\]

Then, all of the infinitesimal generators of Eq. (2) can be expressed as

\[
\mathcal{V} = C_1 \mathcal{V}_1 + C_2 \mathcal{V}_2 + C_3 \mathcal{V}_3
\]

The commutation relations of Lie algebra determined by \(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\), are shown in Table 1. It is obvious that \(\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}\) is closed under the Lie bracket.

3. Symmetry groups of the coupled KdV–Burgers equation

In this section, in order to get some exact solutions from a known solution of Eq. (2), we should find the one-parameter symmetry groups \(g_v : (x, t, u, \nu) \rightarrow (\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu})\) of corresponding infinitesimal generators. To get the Lie symmetry groups, we should solve the following initial problems of ordinary differential equations:

\[
\begin{align*}
\frac{d(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu})}{d\varepsilon} &= (\zeta, \tau, \eta_1, \eta_2), \\
(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu})\big|_{\varepsilon=0} &= (x, t, u, \nu),
\end{align*}
\]

where \(\zeta = \zeta(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu}), \tau = \tau(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu}), \eta_1 = \eta_1(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu}), \eta_2 = \eta_2(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{\nu})\), and \(\varepsilon\) is a group parameter.

<table>
<thead>
<tr>
<th>[\mathcal{V}_1, \mathcal{V}_2]</th>
<th>(\mathcal{V}_1)</th>
<th>(\mathcal{V}_2)</th>
<th>(\mathcal{V}_3)</th>
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<tbody>
<tr>
<td>(\mathcal{V}_1)</td>
<td>0</td>
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<td>(-\mathcal{V}_2)</td>
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<td>(\mathcal{V}_2)</td>
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<tr>
<td>(\mathcal{V}_3)</td>
<td>0</td>
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</table>
For the infinitesimal generator $V = C_1 V_1 + C_2 V_2 + C_3 V_3$, we will take the following different values to obtain the corresponding infinitesimal generators:

**Case 1.** $C_1 = 1, C_2 = C_3 = 0$, the infinitesimal generator is $V_1 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$.

**Case 2.** $C_2 = 1, C_1 = C_3 = 0$, the infinitesimal generator is $V_2 = \frac{\partial}{\partial x}$.

**Case 3.** $C_3 = 1, C_1 = C_2 = 0$, the infinitesimal generator is $V_3 = \frac{\partial}{\partial t}$.

**Case 4.** $C_1 = C_2 = 1, C_3 = 0$, the infinitesimal generator is $V_4 = V_1 + V_2 = (t + 1) \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial x}$.

**Case 5.** $C_1 = C_3 = 1, C_2 = 0$, the infinitesimal generator is $V_5 = V_1 + V_3 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial x}$.

**Case 6.** $C_1 = 0, C_2 = C_3 = k$, the infinitesimal generator is $V_6 = cV_2 + kV_3 = c \frac{\partial}{\partial x} + k \frac{\partial}{\partial t}$.

**Case 7.** $C_1 = C_2 = C_3 = 1$, the infinitesimal generator is $V_7 = V_1 + V_2 + V_3 = (t + 1) \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial x}$.

The one-parameter symmetry groups $g_i : (x, t, u, v) \rightarrow (\hat{x}, \hat{t}, \hat{u}, \hat{v})$ of above corresponding the infinitesimal generators are given as follows:

- $g_1 : (x, t, u, v) \rightarrow (x + \hat{e}t, t + \hat{e}, u + \hat{e}, v + \hat{e})$,
- $g_2 : (x, t, u, v) \rightarrow (x + \hat{e}, t, u + \hat{e}, v + \hat{e})$,
- $g_3 : (x, t, u, v) \rightarrow (x, t + \hat{e}, u + \hat{e}, v + \hat{e})$,
- $g_4 : (x, t, u, v) \rightarrow (x + \hat{e}(t + 1), t + \hat{e}, u + \hat{e}, v + \hat{e})$,
- $g_5 : (x, t, u, v) \rightarrow (x + \hat{e}t, t + \hat{e}, u + \hat{e}, v + \hat{e})$,
- $g_6 : (x, t, u, v) \rightarrow (x + \hat{e}c, t + \hat{e}c, u + \hat{e}, v + \hat{e})$,
- $g_7 : (x, t, u, v) \rightarrow (x + \hat{e}(t + 1), t + \hat{e}, u + \hat{e}, v + \hat{e})$,

where $g_1, g_2, g_5, g_7$ are Galilean transformations, $g_2$ is a space translation, $g_3$ is a time translation, and $g_6$ is a space–time translation. $\hat{e}$ is an arbitrary constant.

If $u = f(x, t), v = g(x, t)$ is a known solution of Eq. (2), then by using the above groups $g_i (i = 1, 2, \ldots, 7)$, the corresponding new solutions $u_i, v_i (i = 1, 2, \ldots, 7)$ can be obtained respectively as follows:

- $u_1 = f(x - \hat{e}t, t + \hat{e})$,
- $u_2 = f(x + \hat{e}, t)$,
- $u_3 = f(x, t + \hat{e})$,
- $u_4 = f(x - \hat{e}(t + 1), t + \hat{e})$,
- $u_5 = f(x - \hat{e}t + \hat{e}^2, t - \hat{e})$,
- $u_6 = f(x - \hat{e}c, t - \hat{e}c)$,
- $u_7 = f(x - \hat{e}(t + 1), t + \hat{e})$.

For the known solution $u = f(x, t), v = g(x, t)$, by using one-parameter symmetry groups $g_i (i = 1, 2, \ldots, 7)$ continuously, we can obtain a new solution which can be expressed as the following form:

- $u = f(x - (\hat{e}_1 + \hat{e}_4 + \hat{e}_5 + \hat{e}_7)t - (\hat{e}_2 + \hat{e}_6 + \hat{e}_7) + (\hat{e}_3^2 + \hat{e}_5^2), t - \hat{e}_3 - \hat{e}_5 - k\hat{e}_6 - \hat{e}_7 + (\hat{e}_1 + \hat{e}_4 + \hat{e}_5 + \hat{e}_7)$,
- $v = g(x - (\hat{e}_1 + \hat{e}_4 + \hat{e}_5 + \hat{e}_7)t - (\hat{e}_2 + \hat{e}_6 + \hat{e}_7) + (\hat{e}_3^2 + \hat{e}_5^2), t - \hat{e}_3 - \hat{e}_5 - k\hat{e}_6 - \hat{e}_7 + (\hat{e}_1 + \hat{e}_4 + \hat{e}_5 + \hat{e}_7)$,

where $\hat{e}_i (i = 1, 2, \ldots, 7)$ are arbitrary constants.

### 4. Symmetry reductions and exact solutions of the coupled KdV–Burgers equation

In the previous sections, we obtained the infinitesimal generators $V_i (i = 1, 2, \ldots, 7)$. In this section, we will get similarity variables and its reduction equations, and obtain solitary wave solutions or similarity solutions by solving the reduction equations.

**Case 1.** For the infinitesimal generator $V_1 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, the similarity variables are $r = t, F(r) = x - tu, G(r) = x - tv$, and the group-invariant solution is $u = \frac{x - F(r)}{t}, v = \frac{x - G(r)}{t}$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation

\[
\begin{align*}
F_t &= 0, \\
G_r &= 0.
\end{align*}
\]

(9)

Obviously, $F = c_1, G = c_2$ is a solution of Eq. (9), where $c_1, c_2$ are arbitrary constants. Therefore, Eq. (2) has a similarity solution as follows:

\[
\begin{align*}
u &= \frac{x - c_1}{t}, \\
v &= \frac{x - c_2}{t}.
\end{align*}
\]

(10)
Case 2. For the infinitesimal generator $V_2 = \frac{\partial}{\partial x}$, the similarity variables are $r = t, F(r) = u, G(r) = v$, and the group-invariant solution is $u = F(r), v = G(r)$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation:

\[
\begin{align*}
F_t &= 0, \\
G_r &= 0.
\end{align*}
\]  

(11)

Therefore, Eq. (2) has a solution $u = c_1, v = c_2$, where $c_1, c_2$ are arbitrary constants. Obviously, the solution is not meaningful.

Case 3. For the infinitesimal generator $V_3 = \frac{\partial}{\partial r}$, the similarity variables are $r = x, F(r) = u, G(r) = v$, and the group-invariant solution is $u = F(r), v = G(r)$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation:

\[
\begin{align*}
FF_t + x \alpha_1 F_r + \beta_1 F_{rr} &= 0, \\
GG_r + x \alpha_2 G_r + \beta_2 F_{rr} &= 0.
\end{align*}
\]  

(12)

Case 4. For the infinitesimal generator $V_4 = (t + 1) \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, the similarity variables are $r = t + (t + 1)u, F(r) = x - (t + 1)v, v = G(r)$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation:

\[
\begin{align*}
F_t &= 0, \\
G_r &= 0.
\end{align*}
\]  

(13)

Obviously, $F = c_1, G = c_2$ is a solution of Eq. (13), where $c_1, c_2$ are arbitrary constants. Therefore, Eq. (2) has a similarity solution as follows:

\[
\begin{align*}
u &= \frac{c_1}{1+t}, \\
u &= \frac{c_2}{1+t}.
\end{align*}
\]  

(14)

Case 5. For the infinitesimal generator $V_5 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, the similarity variables are $r = x - ct, F(r) = u, G(r) = v$, and the group-invariant solution is $u = F(r), v = G(r)$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation:

\[
\begin{align*}
1 + 2FF_t - 4x \alpha_1 G_r - 8 \beta_1 F_{rr} &= 0, \\
1 + 2GG_r - 4x \alpha_2 G_r - 8 \beta_2 F_{rr} &= 0.
\end{align*}
\]  

(15)

Case 6. For the infinitesimal generator $V_6 = c \frac{\partial}{\partial x} + k \frac{\partial}{\partial t}$, the similarity variables are $r = kx - ct, F(r) = u, G(r) = v$, and the group-invariant solution is $u = F(r), v = G(r)$. Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation

\[
\begin{align*}
-cF_t - kFF_t - x \alpha_1^2 G_r - \beta_1 k^2 F_{rr}, \\
cF_r - kGG_r - x \alpha_1^2 G_r - \beta_2 k^2 F_{rr}.
\end{align*}
\]  

(16)

Solving Eq. (16) by using tanh method and then substituting into the group-invariant solution, we can obtain following different exact solitary wave solutions of Eq. (2):

(a) If $\alpha_1 \beta_2 = \alpha_2 \beta_1$ and $\alpha_1 \alpha_2 > 0$, then Eq. (2) has four solitary wave solutions as follows:

\[
\begin{align*}
u(x, t) &= \frac{-250[3c_1^2 + 3c_2^2 + 3x^2]}{25x^2} - \frac{6x^4}{25x^2} \tanh \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right) - \frac{3x^2}{25x^2} \tanh^2 \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right), \\
u(x, t) &= \frac{250[3c_1^2 + 3c_2^2 + 3x^2]}{25x^2} + \frac{6x^4}{25x^2} \tanh \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right) - \frac{3x^2}{25x^2} \tanh^2 \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right), \\
u(x, t) &= \frac{-250[3c_1^2 + 3c_2^2 + 3x^2]}{25x^2} - \frac{6x^4}{25x^2} \tanh \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right) - \frac{3x^2}{25x^2} \tanh^2 \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right), \\
u(x, t) &= \frac{250[3c_1^2 + 3c_2^2 + 3x^2]}{25x^2} + \frac{6x^4}{25x^2} \tanh \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right) - \frac{3x^2}{25x^2} \tanh^2 \left( \frac{2x \sqrt{3/2x}}{105x^2} x - ct \right).
\end{align*}
\]  

(17)
If \( \alpha_1 \beta_2 = -5 \alpha_2 \beta_1 \) and \( \alpha_1 \alpha_2 < 0 \), then Eq. (2) has four solitary wave solutions as follows:

\[
\begin{align*}
\text{(21)} \\
&u(x,t) = \frac{\beta^2 \sqrt{30 \alpha_1 \alpha_2 + 270 \alpha_1 \beta_1}}{15 \alpha_1^2 \beta_1^2} - \frac{\beta_2}{\beta_1^2} \tanh \left( -7 \frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) - \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right), \\
v(x,t) &= -\frac{30 \alpha_1 \alpha_2 \beta_2}{15 \alpha_1^2 \beta_1^2} + \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) + \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right).
\end{align*}
\]

\[
\begin{align*}
\text{(22)} \\
u(x,t) &= \frac{\beta^2 \sqrt{30 \alpha_1 \alpha_2 + 270 \alpha_1 \beta_1}}{15 \alpha_1^2 \beta_1^2} + \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) - \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right), \\
v(x,t) &= -\frac{30 \alpha_1 \alpha_2 \beta_2}{15 \alpha_1^2 \beta_1^2} - \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) + \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right).
\end{align*}
\]

\[
\begin{align*}
\text{(23)} \\
u(x,t) &= \frac{\beta^2 \sqrt{30 \alpha_1 \alpha_2 + 270 \alpha_1 \beta_1}}{15 \alpha_1^2 \beta_1^2} - \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) - \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right), \\
v(x,t) &= \frac{30 \alpha_1 \alpha_2 \beta_2}{15 \alpha_1^2 \beta_1^2} + \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) + \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right).
\end{align*}
\]

\[
\begin{align*}
\text{(24)} \\
u(x,t) &= \frac{\beta^2 \sqrt{30 \alpha_1 \alpha_2 + 270 \alpha_1 \beta_1}}{15 \alpha_1^2 \beta_1^2} + \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) - \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right), \\
v(x,t) &= \frac{30 \alpha_1 \alpha_2 \beta_2}{15 \alpha_1^2 \beta_1^2} - \frac{\beta_2}{\beta_1^2} \tanh \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{\alpha_1 \beta_1} x - ct \right) - \frac{18 \beta_2}{\beta_1^2} \tanh^2 \left( -\frac{\sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right) + \frac{2 \sqrt{30 \alpha_1 \alpha_2}}{2 \alpha_1 \beta_1} x - ct \right).
\end{align*}
\]

**Case 7.** For the infinitesimal generator \( V_1 = V_1 + V_2 + V_3 = (t + 1) \frac{\partial}{\partial t} + \beta_1 \frac{\partial}{\partial \alpha_1} + \alpha_1 \frac{\partial}{\partial \alpha_2} \), the similarity variables are \( r = x - \frac{\alpha_1}{\alpha_2} t - F(r) = t - u, G(r) = t - v \), and the group-invariant solution is \( u = t - F(r), v = t - G(r) \). Substituting the group-invariant solution into Eq. (2), we obtain the following reduction equation:

\[
\begin{align*}
\{ &1 + F_r + FF_r - \alpha_1 G_{rr} - \beta_1 F_{rr} = 0, \\
&1 + G_r + GG_r - \alpha_2 G_{rr} - \beta_2 F_{rr} = 0.
\end{align*}
\]

5. Conclusions

In this paper, we study the symmetry reductions and exact solutions of a coupled KdV–Burgers equation by using the classical Lie group method. First, we perform Lie symmetry analysis for the coupled KdV–Burgers equation, and get its infinitesimal generator, commutation table of Lie algebra. Then, we discuss the Lie symmetry groups of the coupled KdV–Burgers equation. Finally, using similarity variables to obtain reduction equations, and solving the reduction equation, we obtain solitary wave solutions or similarity solutions of Eq. (2).

It can be seen by the results of this paper that the Lie group method is an effective method for studying coupled nonlinear partial differential equations.

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