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The 2-rotational Steiner triple systems of order 25

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Abstract

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In this paper, we enumerate the 2-rotational Steiner triple systems of order 25. There are exactly 140 pairwise non-isomorphic such designs. All these designs have full automorphism groups of order 12. We also investigate the existence of subsystems and quadrilaterals in these designs.

1. Introduction

With the existence problem for STS(v)'s (all terms are defined in Section 2) completely settled (see, e.g., [1]), many researchers have devoted themselves to enumerating such designs. The number of pairwise non-isomorphic STS(v)'s, denoted by $N(v)$, has been determined exactly for all $v \leq 15$; we have $N(3) = N(7) = N(9) = 1$, $N(13) = 2$, and $N(15) = 80$ (cf. [11]). At this point, we encounter a combinatorial explosion effect. It has been shown that $N(19) \geq 2395687$ [15], $N(21) \geq 2160980$, $N(25) \geq 10^{14}$, and $N(27) \geq 10^{11}$ (cf. [6]). These numbers are probably too large to ever be computed exactly. Therefore, as STS(v)'s are so numerous, extra conditions are often imposed in order to enumerate interesting classes of STS(v)'s. Typically, such conditions involve specifying automorphisms or subsystems that the desired designs must possess.

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Table 1

v	Number of pairwise non-isomorphic 2-rotational STS(v)'s	Reference
7	1	[14]
9	1	[14]
15	3	[11, 17]
19	10	[14]
25	140	this paper
27	$\geqslant$1468	this paper

The class of STS(25)'s possessing an automorphism of order 25, i.e. cyclic STS(25)'s, has been enumerated by Bays (cf. Colbourn and Mathon [5]). There are exactly 12 pairwise non-isomorphic such designs. In this paper, we enumerate the class of 2-rotational STS(25)'s and investigate the existence of subsystems and quadrilaterals in these designs. The existence problem for 2-rotational STS(v)'s has been settled by Phelps and Rosa [14] who. proved that the condition $v \equiv 1, 3, 7, 9, 15$ or $19 \pmod{24}$ is both necessary and sufficient for a 2-rotational STS(v) to exist. In particular, they also determined that the number of pairwise non-isomorphic 2-rotational STS(19)'s is 10. In view of the results in this paper, together with that of Phelps and Rosa, and previous results on $N(v)$, the enumeration problem for 2-rotational STS(v)'s is now complete for all $v \leqslant 25$. Table 1 gives the current state of knowledge on the number of pairwise non-isomorphic 2-rotational STS(v)'s for $v \leqslant 27$.

2. Definitions and notations

A *Steiner triple system* is a pair $(X, \mathcal{B})$, where X is a finite set of elements called *points*, and $\mathcal{B}$ is a collection of three-subsets of X called *triples*, such that every two-subset of X is contained in exactly one triple. A Steiner triple system having v points is denoted by STS(v). The number v is called the *order* of the STS(v).

An STS(w), say $(Y, \mathcal{B})$, is a *subsystem* of an STS(v), say $(X, \mathcal{A})$, provided $Y \subseteq X$ and $\mathcal{B} \subseteq \mathcal{A}$. It is easy to see that an STS(w) exists as a subsystem of an STS(v) only if $2w + 1 \leqslant v$. Trivially, any triple of an STS(v) forms a subsystem of order three. We are interested only in *nontrivial* subsystems, i.e. subsystems of order at least seven. An STS(v) with no nontrivial subsystems is commonly called *simple* or *planar*.

A *quadrilateral* in an STS(v) is a subset of four triples whose union contains precisely six points. A quadrilateral must have the following configuration: $\{a, b, c\}$, $\{a, d, e\}$, $\{f, b, d\}$, and $\{f, c, e\}$. STS(v)'s containing no quadrilaterals are said to be *quadrilateral-free*. We denote a quadrilateral-free STS(v) by QFSTS(v).

Two STS(v)'s, say (X_1, $\mathscr{B}_1$) and (X_2, $\mathscr{B}_2$), are said to be *isomorphic* if there exists a bijection $\pi: X_1 \to X_2$ such that $\{x, y, z\} \in \mathscr{B}_1$ implies $\{\pi(x), \pi(y), \pi(z)\} \in \mathscr{B}_2$, Such a bijection is called *an isomorphism.* An *automorphism* of an STS(v), say (X, $\mathscr{B}$), is an isomorphism from (X, $\mathscr{B}$) onto itself. The set of all automorphisms forms a group, called the *full automorphism group,* under functional composition. Any subgroup of the full automorphism group is simply called *an automorphism group.*

An STS(v) is called *k-rotational* if it admits a permutation with one fixed point and $(v-1)/k$ cycles of length k as an automorphism.

The *block intersection graph* of an STS(v) is a graph H such that the vertices of H are the triples of the STS(v), and two vertices are adjacent if and only if the corresponding triples intersect. It is easy to see that non-isomorphic STS(v)'s have non-isomorphic block intersection graphs for $v \geq 19$ (and it is still true, though less easy to show, for all v).

3. Constructing designs with a given group

Let G be a group acting on a set X. Then there is a natural action of G on the two-subsets and three-subsets of X. Let $A(G)$ be a matrix with rows and columns indexed by G-orbits of two-subsets and three-subsets of X respectively, such that the (i, j)th entry of $A(G)$, a_{ij}, is the number of three-subsets in the G-orbit indexing column j containing a fixed two-subset in the G-orbit indexing row i. A result of Kramer and Mesner [9] shows that an STS(v) exists with G as an automorphism group if and only if there is a (0, 1)-vector $\boldsymbol{u}$ satisfying the matrix equation $A(G)\boldsymbol{u} = \boldsymbol{j}$, where $\boldsymbol{j}$ is the vector of all 1's. The vector $\boldsymbol{u}$ determines which orbits of triples are to be present in the STS(v) in a natural way.

Kreher and Radziszowski [10] proposed several efficient heuristics for computing (0, 1)-vectors $\boldsymbol{u}$ that satisfy $A(G)\boldsymbol{u} = \boldsymbol{j}$. They observed that if $\boldsymbol{u}$ is any integer vector satisfying $A(G)\boldsymbol{u} = \boldsymbol{j}$, then $\binom{\boldsymbol{u}}{\boldsymbol{0}}$ is a vector in the *lattice* $\mathscr{L}$ spanned by the columns of the matrix

$$B = \begin{pmatrix} I & \boldsymbol{0} \\ A(G) & -\boldsymbol{j} \end{pmatrix},$$

i.e. $\mathscr{L}$ is the set of all integer linear combinations of the columns of B. They also made the observation that a (0, 1)-vector $\boldsymbol{u}$ satisfying $A(G)\boldsymbol{u} = \boldsymbol{j}$ is often a short vector in $\mathscr{L}$. A modified basis reduction algorithm is then used to obtain a reduced basis B' for a new lattice $\mathscr{L}'$ that contains all the integer vectors $\boldsymbol{u}$ satisfying $A(G)\boldsymbol{u} = \boldsymbol{j}$. In addition, the reduced basis B' contains relatively short vectors of $\mathscr{L}'$ and often a (0, 1)-vector $\boldsymbol{u}$ satisfying $A(G)\boldsymbol{u} = \boldsymbol{j}$. These heuristics of Kreher and Radziszowski have been used with much success in the construction of designs with specified automorphism groups [2].

4. Admissible configurations

Let $G = \langle \alpha \rangle$, where $\alpha = (0)(1\,2\ldots12)(13\,14\ldots24)$. It follows from the discussion in the previous section that if we want to construct a 2-rotational STS(25) on the set of points $X = \{0, 1, \ldots, 24\}$, we need only look for a $(0, 1)$-solution to $A(G)\boldsymbol{u} = \boldsymbol{j}$. We can compute the number of orbits of G on two-subsets and three-subsets of X, denoted $\rho_2(G)$ and $\rho_3(G)$ respectively, from Burnside's lemma.

Lemma 1 (Burnside). *The number of orbits of G on t-subsets of X is*

$$\left(\sum_{\pi \in G} |\mathrm{fix}(\pi)|\right) \Big/ |G|,$$

where fix(π) *is the set of t-subsets of X fixed by* π.

This gives $\rho_2(G) = 26$ and $\rho_3(G) = 194$. Table 2 provides a detailed break-down for the 194 orbits of three-subsets of X.

A careful examination of the 26 by 194 matrix $A(G)$ that arises reveals that 38 of the 194 columns contain some entry $a_{ij} > 1$. Hence, none of these 38 corresponding orbits of three-subsets can be part of a 2-rotational STS(25). These 38 orbits all have length 12. Let $\tilde{A}(G)$ be the matrix $A(G)$ with these 38 columns deleted, then a 2-rotational STS(25) exists if and only if there is a $(0, 1)$-vector $\boldsymbol{u}$ satisfying $\tilde{A}(G)\boldsymbol{u} = \boldsymbol{j}$.

Let n_i be the number of orbits of three-subsets of X of length i that are present in a 2-rotational STS(25). We have the following integer linear program:

$$4n_4 + 6n_6 + 12n_{12} = 100,$$
$$0 \leqslant n_4 \leqslant 2, \quad 0 \leqslant n_6 \leqslant 2, \quad 0 \leqslant n_{12} \leqslant 156.$$

The only feasible solutions to the above integer linear program are

$$n_4 = 1,\ n_6 = 0,\ n_{12} = 8 \quad \text{and} \quad n_4 = 1,\ n_6 = 2,\ n_{12} = 7.$$

A 2-rotational STS(25) with the structure $n_4 = 1$, $n_6 = 0$, and $n_{12} = 8$ will be referred to as Type I; a 2-rotational STS(25) with the structure $n_4 = 1$, $n_6 = 2$, and $n_{12} = 7$ will be referred to as Type II.

Table 2

G-orbits of three-subsets of *X*

Length of *G*-orbit	Number of *G*-orbits
4	2
6	2
12	190
	194

We used the algorithm of Kreher and Radziszowski to obtain a reduced basis B for the lattice $\mathscr{L}$ that contains all the integer vectors $\boldsymbol{u}$ satisfying $\tilde{A}(G)\boldsymbol{u}=\boldsymbol{j}$. We observed that for all the vectors $\boldsymbol{b} \in B$, $\|\boldsymbol{b}\|^2$ is even. Thus, if $\boldsymbol{b}'$ is any vector in $\mathscr{L}$, then $\|\boldsymbol{b}'\|^2$ is also even. Therefore, as the existence of a Type I 2-rotational STS(25) implies that there is a $(0,1)$-vector $\boldsymbol{u}$ satisfying $\tilde{A}(G)\boldsymbol{u}=\boldsymbol{j}$ with the property that $\|\boldsymbol{u}\|^2=9$, we have the following result.

Lemma 2. *All 2-rotational* STS(25)'*s are of Type* II.

5. Computational details

Without loss of generality, we may assume that in a 2-rotational STS(25), the starter triples $\{1,5,9\}$, $\{0,1,7\}$, and $\{0,13,19\}$ are the orbit representatives of the one orbit of length four and the two orbits of length six, respectively. This leaves us with the task of selecting seven orbit representatives for the orbits of length 12 to complete the STS(25).

To restrict the search further, we observe that the two-subset $\{1,2\}$ is not contained in any triples of the partial STS(25) that arises from the three starter triples given above. Thus, we may assume that one of the seven orbit representatives for G-orbits of length 12 have the form $\{1,2,*\}$. It is easy to verify that without loss of generality, the only candidates for $*$ are $*=4$ or 13. The remaining six orbit representatives in each case are then completed by a backtracking algorithm using depth-first search. At this stage, it is possible to reduce the search still further. However, working on the combined principles that it is better to let the machine do the work, and that more complex programs are more likely to contains mistakes, we decided to leave in this amount of redundancy. We also executed two computer programs that were written independently for the enumeration to be certain that no mistakes have crept in. In each case, the final results agreed with the other.

Isomorphism testing of the designs was done by using **nauty**, the isomorphism testing algorithm of McKay [12, 13], on the block intersection graphs. Graphs arising from designs are notoriously difficult for isomorphism checking, and it was necessary to first use a fairly sophisticated routine to partition the vertices before using **nauty**. The result is that there are exactly 140 pairwise non-isomorphic 2-rotational STS(25)'s. The isomorphism testing also yielded the result that all 140 designs have full automorphism groups of order 12.

6. Subsystems of order 7 and 9

The only orders for which a nontrivial subsystem can exist in an STS(25) are seven and nine. A direct backtracking algorithm is applied to the 140 2-rotational

STS(25)'s in a search for designs containing subsystems of order seven. The result is astonishing; none of the designs contains a subsystem of order seven. This, however, renders the search for subsystems of order 9 more efficient.

It was proven by Colbourn, Colbourn and Stinson [4] that the existence of subsystems in STS(v)'s can be decided in polynomial time. Their algorithm is based on the observation that a subsystem has the property that every triple intersects the subsystem in 0, 1, or 3 points. Therefore, given a subset Y of points, we can *close* Y by repeatedly introducing all points from triples that intersect Y in more than one point. Therefore, taking any three points that is not a triple, and closing it, yields either a proper nontrivial subsystem or the design itself. Consequently, this algorithm when applied to the 2-rotational STS(25)'s, either finds a subsystem of order nine or else proves that no nontrivial subsystems exist.

Of all the 140 2-rotational STS(25)'s tested with this algorithm, only four contain subsystems of order nine. The other 136 designs are planar.

7. Quadrilateral-free systems

The existence of QFSTS(v)'s have been previously investigated. Doyen [7] proved that if $v \equiv 3 \pmod 6$, and 7 does not divide v, then there exists a QFSTS(v). It was also shown by Grannell, Griggs, and Phelan [8] that there exists a QFSTS(v) whenever the order of $-2 \pmod p$ is congruent to 2 (mod 4) for every prime divisor p of $v-2$. A QFSTS(25) can be constructed by this

Table 3

2-rotational STS(25)'s

Number of quadrilaterals	Number of designs
0	4
4	16
12	16
16	29
24	17
28	17
36	15
40	11
48	6
52	5
60	1
64	2
72	1
	140

Table 4

2-Rotational STS(25)'s containing subsystems of order 9

Common starter triples: {1, 5, 9}, {0, 1, 7}, {0, 13, 19}, {1, 2, 13}

Design #	Starter triples					
123	{1, 3, 18}	{1, 4, 14}	{1, 6, 15}	{1, 17, 20}	{1, 19, 21}	{13, 14, 18}
124	{1, 3, 18}	{1, 4, 14}	{1, 6, 15}	{1, 17, 20}	{1, 19, 21}	{13, 14, 21}
133	{1, 3, 18}	{1, 4, 17}	{1, 6, 15}	{1, 19, 21}	{1, 20, 23}	{13, 14, 18}
134	{1, 3, 18}	{1, 4, 17}	{1, 6, 15}	{1, 19, 21}	{1, 20, 23}	{13, 14, 21}

method and is the only QFSTS(25) previously known. Other constructions of QFSTS(v)'s were recently given by Stinson and Wei [16].

We carried out an enumeration of quadrilaterals in the 140 2-rotational STS(25)'s and found exactly four quadrilateral-free designs among them. A summary of the results obtained are given in Table 3.

8. Concluding remarks

The starter triples for the four 2-rotational STS(25)'s with subsystems of order nine are given in Table 4.

A complete enumeration actually proves that each of the four designs given in Table 4 has exactly three distinct subsystems of order nine.

The starter triples for the four quadrilateral-free 2-rotational STS(25)'s are listed in Table 5.

Using similar approaches as those given in this paper, we have been able to prove that there are at least 1468 pairwise non-isomorphic 2-rotational STS(27)'s. This improves greatly the best previous known bound of Phelps and Rosa [14], who showed that there are exactly 35 pairwise non-isomorphic 1-rotational STS(27)'s (and thus at least 35 pairwise non-isomorphic 2-rotational STS(27)'s).

A complete catalogue for all the pairwise non-isomorphic 2-rotational STS(25)'s that we have found is too long to be included here. However, interested readers can obtain a catalogue of these designs [3] from either one of

Table 5

Quadrilateral-free 2-rotational STS(25)'s

Common starter triples: {1, 5, 9}, {0, 1, 7}, {0, 13, 19}, {1, 2, 13}

Design #	Starter triples					
84	{1, 3, 17}	{1, 4, 19}	{1, 6, 14}	{1, 18, 20}	{1, 22, 23}	{13, 16, 21}
88	{1, 3, 17}	{1, 4, 19}	{1, 6, 23}	{1, 14, 21}	{1, 20, 22}	{13, 14, 22}
104	{1, 3, 17}	{1, 4, 22}	{1, 6, 14}	{1, 16, 23}	{1, 18, 20}	{13, 14, 22}
132	{1, 3, 18}	{1, 4, 17}	{1, 6, 15}	{1, 19, 20}	{1, 21, 23}	{13, 16, 21}

the authors. The numbers used to denote the designs in Table 4 and Table 5 correspond to those used in the catalogue.

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