MATHEMATICS

A UNIVERSAL ISOMORPHISM FOR P-TYPICAL FORMAL GROUPS AND OPERATIONS IN BROWN-PETERSON COHOMOLOGY

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ABSTRACT

We construct an abstract isomorphism of p-typical formal groups which is universal for isomorphisms of p-typical formal groups over $\mathbf{Z}_{(p)}$-algebras or characteristic zero rings. Associated to this universal isomorphism is a homomorphism of rings $\mathbf{Z}[V_1, V_2, \ldots] \to \mathbf{Z}[V_1, V_2, \ldots; T_1, T_2, \ldots]$ which (after localization at p) can be identified with the right unit homomorphism $\eta_R: BP_*(pt) \to BP_*(BP)$ of the Hopf-algebra $BP_*(BP)$ of Brown-Peterson (co)homology. We calculate η_R modulo the ideal $(T_1, T_2, \ldots)^2$. These results are then used to obtain information on some of the operations of Brown-Peterson cohomology.

1. INTRODUCTION

Choose a prime number p and let $\mathbf{Q}$ denote the rational numbers. Let $a_i(V)$, $a_i(V, T)$ in

$$\mathbf{Q}[V]=\mathbf{Q}[V_1, V_2, \ldots] \text{ and } \mathbf{Q}[V; T]=\mathbf{Q}[V_1, V_2, \ldots; T_1, T_2, \ldots]$$

be the polynomials defined by the equations

$$(1.1) \qquad pa_i(V)=\sum_{k=1}^{i} a_{i-k}(V)V_k^{p^{i-k}}, \quad a_0(V)=1$$

$$(1.2) \qquad a_i(V, T)=\sum_{k=0}^{i} a_k(V)T_{i-k}^{p^k}, \quad a_0(V, T)=1.$$

Now define the power series

$$(1.3) \qquad f_V(X)=\sum_{n=0}^{\infty} a_n(V)X^{p^n}, \quad f_{V,T}(X)=\sum_{n=0}^{\infty} a_n(V, T)X^{p^n}$$

$$(1.4) \qquad F_V(X, Y)=f_V^{-1}(f_V(X)+f_V(Y)), \quad F_{V,T}(X, Y)=f_{V,T}^{-1}(f_{V,T}(X)+f_{V,T}(Y))$$

$$(1.5) \qquad \alpha_{V,T}(X)=f_{V,T}^{-1}(f_V(X))$$

where $f_V^{-1}(X)$ and $f_{V,T}^{-1}(X)$ are the inverse power series to $f_V(X)$ and $f_{V,T}(X)$ respectively, i.e. $f_V^{-1}(f_V(X))=X$ and $f_{V,T}^{-1}(f_{V,T}(X))=X$. One then has (cf. [3], [4] and [5] part I).

1.6. THEOREM.

The power series $F_V(X, Y)$, $F_{V,T}(X, Y)$, $\alpha_{V,T}(X)$ have their coefficients in $\mathbf{Z}[V]$, $\mathbf{Z}[V; T]$, $\mathbf{Z}[V; T]$.

The power series $F_V(X, Y)$ and $F_{V,T}(X, Y)$ therefore define p-typical (one dimensional commutative) formal groups over $\mathbf{Z}[V]$ and $\mathbf{Z}[V; T]$ respectively, which are strictly isomorphic via $\alpha_{V,T}(X)$. In addition one has (cf. [4] and [5] part I).

1.7. THEOREM.

The triple $(F_V(X, Y), \alpha_{V,T}(X), F_{V,T}(X, Y))$ over $\mathbf{Z}[V; T]$ is universal for triples $(F(X, Y), \alpha(X), F(X, Y))$ consisting of two p-typical formal groups and a strict isomorphism between them defined over a ring A which is a $\mathbf{Z}_{(p)}$-algebra or a characteristic zero ring.

I.e. for every such triple $(F(X, Y), \alpha(X), G(X, Y))$ there is a unique homomorphism $\phi: \mathbf{Z}[V; T] \to A$ such that $F(X, Y)=F_V^\phi(X, Y)$, $\alpha(X)= =\alpha_{V,T}^\phi(X)$, $G(X, Y)=F_{V,T}^\phi(X, Y)$.

If we restrict attention to $\mathbf{Z}_{(p)}$-algebras A theorem 1.7 implies that $\mathbf{Z}_{(p)}[V; T]$ represents the functor $\mathscr{I}$: $A \mapsto$ set of all triples $(F(X, Y)$, $\alpha(X)$, $G(X, Y))$. Now $BP_*(BP)=\mathbf{Z}_{(p)}[V; T]$, cf. [1] part II, theorem 16.1, or [2], so that $\mathscr{I}$ is also represented by $BP_*(BP)$ where BP is the Brown-Peterson spectrum. This fact has been used to derive all the structure maps of the Hopf algebra $BP_*(BP)$, cf. [7]. $F_V(X, Y)$ is a p-typically universal p-typical formal group and $F_{V,T}(X, Y)$ is a p-typical formal group. It follows that there are unique polynomials $\bar{V}_n \in \mathbf{Z}[V; T]$ such that $F_{\bar{V}}(X, Y)=F_{V,T}(X, Y)$. It follows that we have for the polynomials $\bar{V}_n$

$$pa_n(V, T)=\sum_{k=1}^{n} a_{n-k}(V, T)\bar{V}_k^{p^{n-k}}. \tag{1.8}$$

The assignment $V_n \mapsto \bar{V}_n$ defines a homomorphism $\mathbf{Z}[V] \to \mathbf{Z}[V, T]$. Now $BP_*(pt)=\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ where the v_i are defined by

$$pl_n=l_{n-1}v_1^{p^{n-1}}+\ldots+l_1v_{n-1}^{p}+v_n \tag{1.9}$$

where $l_n=p^{-n}[\mathbf{CP}^{p^n-1}] \in BP_*(pt) \otimes \mathbf{Q} \subset MU_*(pt) \otimes \mathbf{Q}$. Now identify $\mathbf{Z}_{(p)}[V]$ with $BP_*(pt)$ by means of $V_i \mapsto v_i$ and $\mathbf{Z}_{(p)}[V; T]$ with $BP_*(BP)$ by means of $V_i \mapsto v_i$, $T_i \mapsto t_i$ where the t_i are the elements of $BP_*(BP)$ described in theorem 16.1 of part II of [1]. The homomorphism $\mathbf{Z}[V] \to \to \mathbf{Z}[V; T]$ (when localized at p) then becomes the right unit map $\eta_R: BP_*(pt) \to BP_*(BP)$.

Below we give a recursion formula for $\overline{V}_n$. On the one hand this formula can be used to give a noncohomological proof of the Lubin-Tate formal moduli theorem and a new proof of Lazard's classification theorem for one dimensional formal groups over an algebraically closed field. On the other hand the formula gives information about η_R, and thus gives information about the BP-cohomology operations. Cf. section 3 below.

2. SOME FORMULAS CONCERNING $\overline{V}_n$

Let $B_n = p^n a_n(V)$ where $a_n(V)$ is as in (1.1) above. Let J denote the ideal $(T_1, T_2, \ldots)^2$ in $\mathbf{Z}[V; T]$ and let I be the ideal generated by the elements pT_i, $i=1, 2, \ldots$ and the elements T_iT_j, $i, j=1, 2, \ldots$. Then we have

2.1. THEOREM.

$$(2.1.1)\quad \begin{cases} \overline{V}_n = \sum_{k=1}^{n-1} a_{n-k}\{(V_k^{p^{n-k}} - \overline{V}_k^{p^{n-k}}) + \sum_{\substack{i+j=k \\ i,j\geq 1}} (V_i^{p^{n-k}} T_j^{p^{n-j}} - T_j^{p^{n-k}} \overline{V}_i^{p^{n-j}})\} \\ \qquad + \sum_{\substack{i+j=n \\ i,j\geq 1}} (V_i T_j^{p^i} - T_j \overline{V}_i^{p^j}) + V_n + pT_n. \end{cases}$$

Modulo the ideal J we have (in $\mathbf{Z}[V; T]$).

$$(2.1.2)\quad \begin{cases} \overline{V}_n \equiv \sum (-1)^t (B_{s_1} V_{n-s_1}^{p^{s_1}-1})(B_{s_2} V_{n-s_1-s_2}^{p^{s_2}-1}) \ldots (B_{s_t} V_{n-s_1-\ldots-s_t}^{p^{s_t}-1})(-T_i V_j^{p^i}) \\ + \sum (-1)^t (B_{s_1} V_{n-s_1}^{p^{s_1}-1})(B_{s_2} V_{n-s_1-s_2}^{p^{s_2}-1}) \ldots (B_{s_t} V_{n-s_1-\ldots-s_t}^{p^{s_t}-1})(pT_i) + V_n \end{cases}$$

where the first sum is over all sequences $(s_1, \ldots, s_t, i, j)$ such that $s_k, i, j \in \mathbf{N}$, $s_1 + \ldots + s_t + i + j = n$, $t \in \mathbf{N} \cup \{0\}$, and the second sum is over all sequences $(s_1, \ldots, s_t, i)$ such that $s_k, i \in \mathbf{N}$, $s_1 + \ldots + s_t + i = n$, $t \in \mathbf{N} \cup \{0\}$.

And, finally modulo the ideal I we have in $\mathbf{Z}[V; T]$

$$(2.1.3)\quad \begin{cases} \overline{V}_n \equiv \sum (-1)^t \, V_1^{(p-1)^{-1}(p^{s_1} + \ldots + p^{s_t} - t)} V_{n-s_1}^{p^{s_1}-1} \ldots V_{n-s_1-\ldots-s_t}^{p^{s_t}-1}(-T_i V_j^{p^i}) \\ + V_n - T_1 V_{n-1}^p - T_2 V_{n-2}^p - \ldots - T_{n-1} V_1^{p^{n-1}} \end{cases}$$

where the sum is over all sequences $(s_1, \ldots, s_t, i, j)$ such that $s_k, i, j, t \in \mathbf{N}$ and $s_1 + \ldots + s_t + i + j = n$.

2.2. These formula's can be used to give a noncohomological proof of the Lubin-Tate formal moduli theorem ([9]) and Lazard's classification theorem for one dimensional formal groups over an algebraically closed field, ([8]). Cf. [5] part V. Warning: formula (2.2.1) in [5] part V is not correct and should be replaced with (2.1.3) above. The proofs in [5] part V remain mutatis mutandi the same.

3. APPLICATIONS TO BROWN-PETERSON COHOMOLOGY OPERATIONS

A stable BP cohomology operation can be described as a $BP_*(pt)$-linear homomorphism $BP_*(BP) \to BP_*(pt)$, where $BP_*(BP)$ is seen as a left $BP_*(pt)$ module. To find out what such an operation does to elements of $BP_*(pt)$ one composes with $\eta_R\colon BP_*(pt) \to BP_*(BP)$. Cf. [1] part II, section 16 for all this. Let $E=(e_1, e_2, \dots)$ be a sequence of integers $\geqslant 0$ which are almost all zero.

Write $BP_*(BP)=BP_*(pt)\ [t_1, t_2, \dots]$ where the t_i are as in [1] part II, section 16. The cohomology operation r_E is defined as: = coefficient of t^E. One assigns to the exponent sequence E the weight

$$\|E\|=(p-1)e_1+(p^2-1)e_2+\dots.$$

Let Δ_i denote the exponent sequence $\Delta_i=(0, \dots, 0, 1, 0, \dots)$ with the 1 in the i-th place, let $\Delta_0=(0, 0, \dots)$. Scalar multiplication with an element of $\mathbf{N}$ and addition of exponent sequences are defined component wise.

A first application of (2.1.1) is then the following slight generalization of lemma 1.9 of [6] (sometimes known as the Budweiser lemma).

3.1. Lemma.

(i) For $n \geqslant 3$ and $2 \leqslant l \leqslant n-1$ we have that

$$r_E(v_n) \equiv 0 \bmod (p^{p+1}, v_1, \dots, v_{l-1})$$

if $p^n-p^{l-1} > \|E\| \geqslant p^n-p^l$ except in the cases

$$E=p^l\Delta_{n-l},\ E=\Delta_1+(p-1)\Delta_{n-1}+p^l\Delta_{n-l-1}.$$

In these two cases $r_E(v_n)$ is respectively congruent to v_l and $-p^pv_l \bmod (p^{p+1}, v_1, \dots, v_{l-1})$.

(ii) For $n \geqslant 3$ (and $l=1$) we have that $r_E(v_n) \equiv 0 \bmod (p^{p+1})$ if $p^n-1 > \|E\| \geqslant p^n-p$ except in the cases

$$E=p\Delta_{n-1},\ E=\Delta_1+(p-1)\Delta_{n-1}+p\Delta_{n-2}.$$

In these two cases $r_E(v_n)$ is respectively congruent to $v_1(1-p^{p-1})$ and $-p^pv_1 \bmod (p^{p+1})$.

(iii) For $n \geqslant 3$ (and $l=0$) we have that $r_E(v_n) \equiv 0 \bmod (p^{p+2})$ if $\|E\| \geqslant p^n-1$ except in the cases $E=\Delta_n$, $E=\Delta_1+p\Delta_{n-1}$. In these two cases $r_E(v_n)$ is respectively congruent to p and $-p^p \bmod (p^{p+2})$.

(There are slightly different formulae for the cases $n=1, 2$).

A second application is the calculation of the $r_{\Delta_i}(v_n)$. Let $b_n \in BP_*(pt)$ stand for the element p^nl_n. Then we have immediately from (2.1.2).

3.2. THEOREM.

For $0<i<n$ we have

$$(3.2.1)\quad \begin{cases} r_{\Delta_i}(v_n) = \sum (-1)^t (b_{s_1} v_{n-s_1}^{p^{s_1}-1}) \cdot \ldots \cdot (b_{s_t} v_{n-s_1-\ldots-s_t}^{p^{s_t}-1})(-v_{n-s_1-\ldots-s_t-i}^{p^i}) \\ +p \sum (-1)^t (b_{s_1} v_{n-s_1}^{p^{s}-1}) \cdot \ldots \cdot (b_{s_t} v_{n-s_1-\ldots-s_t}^{p^{s}-1}) - v_{n-i}^{p^i} \end{cases}$$

where the first sum is over all sequences $(s_1, \ldots, s_t)$ with $s_1+\ldots+s_t<n-i$, $s_k, t \in \mathbf{N}$ and the second sum is over all sequences $(s_1, \ldots, s_t)$ with $s_1+\ldots+s_t=n-i$, $s_k, t \in \mathbf{N}$. Modulo p we have for $0<i<n$.

$$(3.2.2)\quad \begin{cases} r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} + \sum (-1)^t v_1^{(p-1)^{-1}(p^{s_1}+\ldots+p^{s_t}-t)} \\ v_{n-s_1}^{p^{s_1}-1} \cdot \ldots \cdot v_{n-s_1-\ldots-s_t}^{p^{s_t}-1}(-v_j^{p^i}) \end{cases}$$

where the sum is over all sequences $(s_1, \ldots, s_t, j)$ such that $s_k, t, j \in \mathbf{N}$, $s_1+\ldots+s_t+j=n-i$.

3.3. COROLLARY.

For $0<i<n$ we have $r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} \bmod (p, v_1)$. More generally: let $r = \min(n-i-1, p)$, then we have $\bmod (p, v_1^{p+1})$

$$(3.3.1)\quad r_{\Delta_i}(v_n) \equiv -v_{n-i}^{p^i} + \sum_{t=1}^{r} (-1)^{t+1} v_1^t (v_{n-1} \ldots v_{n-t})^{p-1} v_{n-i-t}^{p^i}.$$

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