JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 155, 364–370 (1991)

On Uniformly Starlike Functions

A. W. GOODMAN

Department of Mathematics, University of South Florida, Tampa, Florida 33620

Submitted by R. P. Boas

Received December 1, 1988

1. INTRODUCTION

In a recent letter to the author, B. Pinchuk posed the following problem. Let ST denote the usual class of starlike functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

These are normalized functions regular and univalent in E: $|z| < 1$, for which $f(E)$ is starlike with respect to the origin. Let γ be a circle contained in E and let ζ be the center of γ. The Pinchuk question is this: If $f(z)$ is in ST, is it true that $f(\gamma)$ is a closed curve that is starlike with respect to $f(\zeta)$? In Section 5 we will see that the answer is no.

There seems to be no reason to demand that the complete circle γ lies in E, and we replace this condition with the stronger condition that γ is an arc of a circle contained in E, but we still ask that ζ the center of γ is also in E. Thus we have,

DEFINITION 1. A function $f(z)$ is said to be uniformly starlike in E if $f(z)$ is in ST and has the property that for every circular arc γ contained in E, with center ζ also in E, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$. We let UST denote the class of all such functions.

An arc $f(\gamma)$ is starlike with respect to a point $w_0 = f(\zeta)$ if $\arg(f(z) - w_0)$ is nondecreasing as z traces γ in the positive direction. If γ is an arc of a circle, then the positive direction is the usual counterclockwise direction.

In [3, p. 109] we proved that if any arc γ is given by $z(t)$, then $f(\gamma)$ is starlike with respect to w_0 iff

$$\operatorname{Im}\left[\frac{f'(z)}{f(z) - w_0}\frac{dz}{dt}\right] \geqq 0 \tag{1.2}$$

0022-247X/91 $3.00

for z on γ. For a circular arc γ, set $z=\zeta+re^{it}$. Then $z'(t)=i(z-\zeta)$ and a brief computation will give

THEOREM 1. *Let $f(z)$ have the form* (1.1). *Then $f(z)$ is in UST iff*

$$\operatorname{Re}\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\geqslant 0 \tag{1.3}$$

for every pair (z,ζ) in the polydisc $E\times E$.

Thus all the properties of functions in UST are contained implicitly in the relation (1.3). However, obtaining these properties is not easy.

2. FUNCTIONS WITH POSITIVE REAL PART ON THE POLYDISC

Let $P^{(2)}$ denote the set of functions

$$P(z,\zeta)=1+\sum\sum_{m+n>0} b_{mn}z^m\zeta^n \tag{2.1}$$

that are regular in $E\times E$ and satisfy the condition $\operatorname{Re} P\geqq 0$ in that domain. Such functions have been the subject of numerous investigations [6]. Clearly if $f(z)$ is in UST, then

$$Q(z,\zeta)\equiv\frac{fz)-f(\zeta)}{(z-\zeta)f'(z)} \tag{2.2}$$

is in $P^{(2)}$ and $1/Q(z,\zeta)$ is also in $P^{(2)}$, but the set of functions of the form $Q(z,\zeta)$ or $1/Q(z,\zeta)$ does not exhaust the set $P^{(2)}$. We set

$$Q(z,\zeta)\equiv\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}=1+\sum\sum_{m+n>0} b_{mn}z^m\zeta^n \tag{2.3}$$

and we define $p_n(\zeta)$ and $q_n(z)$ by

$$Q(z,\zeta)\equiv\sum_{n=0}^{\infty} p_n(\zeta)\,z^n\equiv\sum_{n=0}^{\infty} q_n(z)\,\zeta^n. \tag{2.4}$$

LEMMA 1. *If $f(z)\in UST$, then*

$$p_0(\zeta)=\frac{f(\zeta)}{\zeta},\qquad \textit{and}\qquad p_1(\zeta)=\frac{f(\zeta)[1-2a_2\zeta]-\zeta}{\zeta^2} \tag{2.5}$$

and

$$q_0(z) = \frac{f(z)}{zf'(z)}, \qquad \textit{and} \qquad q_1(z) = \frac{f(z) - z}{z^2 f'(z)}. \tag{2.6}$$

Further, for z and ζ in E

$$|p_1(\zeta)| \leqq 2 \operatorname{Re} p_0(\zeta), \qquad \textit{and} \qquad |q_1(z)| \leqq 2 \operatorname{Re} q_0(z). \tag{2.7}$$

Proof. Both (2.5) and (2.6) follow immediately from the defining Eq. (2.3) and (2.4). Each of the rearranged series in (2.4) gives a function of one variable with positive real part in E. Then the classical Caratheodory theorem gives (2.7). ∎

3. Two Examples

The first example is given in,

THEOREM 2. *The function*

$$F_1(z) \equiv \frac{z}{1 - Az} = z + \sum_{n=2}^{\infty} A^{n-1} z^n \tag{3.1}$$

is in UST *iff* $|A| \leqq 1/\sqrt{2} \approx 0.7071$.

Proof. We first remark that if α is real, then $f(z)$ is in UST iff $e^{-i\alpha} f(e^{i\alpha} z)$ is in UST. Thus WLOG we may assume $A \geqq 0$ in (3.1). If we apply (2.2) to $F_1(z)$ we find,

$$Q(z, \zeta) = \frac{1 - Az}{1 - A\zeta}. \tag{3.2}$$

Thus the numerator and denominator in (3.2) lie in a disc with center 1 and radius A. Consideration of this disc shows that if $A \leqslant 1/\sqrt{2}$ then $\operatorname{Re} Q(z, \zeta) > 0$ for all (z, ζ) in $E \times E$. Further, if $A > 1/\sqrt{2}$ there is at least one pair in $E \times E$ for which $\operatorname{Re} Q(z, \zeta) < 0$. ∎

COROLLARY 1. *The set UST has infinitely many members.*

COROLLARY 2. *Let $B^{(n)}$ be the least upper bound for $|a_n|$ for all $f(z)$ in UST. Then*

$$\left(\frac{1}{\sqrt{2}}\right)^{n-1} \leqq B^{(n)}. \tag{3.3}$$

It may be that $B^{(n)} = (1/\sqrt{2})^{n-1}$ for small values of n, but the next example shows this cannot be so for $n > 8$.

THEOREM 3. *If* $F_2(z) = z + Az^n$, $n > 1$, *and* $|A| \leqq \sqrt{2}/2n$, *then* $F_2(z)$ *is in* UST.

Proof. For this function

$$Q(z, \zeta) = \frac{1 + A(z^{n-1} + z^{n-2}\zeta + \cdots + \zeta^{n-1})}{1 + nAz^{n-1}}. \tag{3.4}$$

If (z, ζ) is in EXE and $|A| \leqslant \sqrt{2}/2n$, then $\operatorname{Re} Q(z, \zeta) > 0$. ∎

Note that Theorem 3 may still hold with a larger value of $|A|$. In fact a longer analysis will show that $z + Az^2$ is in UST iff $|A| \leqq \sqrt{3}/4$. We omit the details.

If we compare the lower bounds for $B^{(n)}$ obtained in Theorems 2 and 3 we observe that $(1/\sqrt{2})^{n-1} > \sqrt{2}/2n$ for $n = 3, 4, ..., 7$. For $n = 8$ we have equality and for $n > 8$, the inequality sign is reversed. For $n = 2$ we have $1/\sqrt{2} > \sqrt{3}/4$.

It is natural to look for transformations which preserve the set UST. The transformation $e^{-i\alpha} f(e^{i\alpha} z)$ is one such. However, no other transformation seems to be available. Pommerenke [5] introduced the concept of a linear-invariant family M and he showed that numerous theorems about the family M followed immediately if we only know that M is a linear-invariant family. By definition M is a linear-invariant family if

$$\Lambda_\phi[f] \equiv \frac{f(\phi(z)) - f(\phi(0))}{\phi'(0) f'(\phi(0))}, \qquad \phi(z) = \frac{z + c}{1 + \bar{c}z} \tag{3.5}$$

is also in M for every f in M and every c in E. If we apply (3.5) to $f(z) \equiv z/(1 - Az)$, we find that

$$\Lambda_\phi[z/(1 - Az)] = z/(1 - Bz), \qquad B = (A - \bar{c})/(1 - cA). \tag{3.6}$$

Now set $A = \frac{1}{2} < 1/\sqrt{2}$ and $c = -\frac{1}{2}$. Then $B = 4/5 > 1/\sqrt{2}$. So f is in UST but $\Lambda_\phi[f]$ is not. In fact with a little more labor we can prove that for each $f(z)$ in UST, there is a $\phi(z)$ such that $\Lambda_\phi[f]$ is not in UST.

4. COEFFICIENT BOUNDS FOR UST

Sakaguchi [7; 4; 3, pp. 164–165] introduced the concept of functions starlike with respect to symmetrical points. Sakaguchi gave a purely geometric definition, but it is equivalent to the following.

DEFINITION 2. A function $f(z)$ of the form (1.1) is said to be starlike with respect to symmetrical points if

$$\operatorname{Re} \frac{2zf'(z)}{f(z)-f(-z)} \geqslant 0 \tag{4.1}$$

for all z in E. We let STS denote the set of all such functions.

Clearly, the left side of (4.1) is identical with $\operatorname{Re}[1/Q(z, -z)]$, see (2.2) and hence STS $\supset$ UST. Sakaguchi proved that if $f(z)$ is in STS and has the form (1.1), then $|a_n| \leqq 1$ for $n = 2, 3, \ldots$. Consequently $|a_n| \leqq 1$ also holds for the set UST. However, as Charles Horowitz showed (in a letter to the author) we can do much better.

THEOREM 4 (C. Horowitz). *If $f(z)$ is in UST then $f'(z)$ lies in a halfplane bounded by a line through the origin, for all z in E. Further*

$$|a_n| \leqq 2/n, \qquad n = 2, 3, 4, \ldots. \tag{4.2}$$

Proof. Set

$$P(z, \zeta) = f'(z)\, Q(z, \zeta) = \frac{f(z)-f(\zeta)}{z-\zeta}. \tag{4.3}$$

Suppose that $f(z)$ and $f(\zeta)$ are diametrically opposite for some pair z, ζ in E. In other words suppose that $\arg f(z) = \arg f(\zeta) + \pi$. Now by symmetry $P(z, \zeta) = P(\zeta, z)$ and hence $Q(z, \zeta) = P(z, \zeta)/f'(z)$ and $Q(\zeta, z) = P(\zeta, z)/f'(\zeta)$ are diametrically opposite to each other. Since z and ζ are interior points of E, it follows that for some neighboring points in E either $\operatorname{Re} Q(z, \zeta) < 0$ or $\operatorname{Re} Q(\zeta, z) < 0$. This contradicts (1.3). Hence for some real α, $\operatorname{Re} e^{i\alpha} f'(z) > 0$ for all z in E. Consequently for the derivative $n\,|a_n| \leqq 2\,|\cos \alpha| \leqq 2$. ∎

THEOREM 5. *If $f(z)$ is in UST and $|z| = r < 1$, then*

$$\frac{r}{1+2r} \leqq |f(z)| \leqslant -r + 2 \ln \frac{1}{1-r}. \tag{4.4}$$

Proof. The coefficient bounds (4.2) and the theory of dominant power series give the right side of (4.4). For the left side of (4.4) we return to (2.4) and (2.6). Since $\operatorname{Re} Q > 0$, we have

$$|q_1(z)| \leqq 2 \operatorname{Re} q_0(z) \leqq 2\,|q_0(z)| \tag{4.5}$$

or, on multiplying by $|z^2 f'(z)|$,

$$|f(z) - z| \leqslant 2\,|zf(z)|. \tag{4.6}$$

This, and the triangle inequality will give the left side of (4.4). ∎

COROLLARY 3. *Let $K(UST)$ be the Koebe constant for the family UST. Thus $K(UST) = \sup R$ such that $f(E)$ contains the disc $|w| < R$ for every $f(z)$ in UST. Then*

$$1/3 \leqq K(UST) \leqq 1 - \sqrt{3}/4 \approx 0.56699. \tag{4.7}$$

Proof. The lower bound follows from (4.4) as $r \to 1^-$. The upper bound is provided by the function $F(z) = z + Az^2$, where $A = \sqrt{3}/4$. ∎

It is well known that if $\sum_{n=2}^{\infty} n\,|a_n| \leqslant 1$, then $f(E)$ is starlike with respect to $w = 0$. See [2], but note there are quite a few unimportant misprints. For UST we have

THEOREM 6. *If $\sum_{n=2}^{\infty} n\,|a_n| \leqslant \sqrt{2}/2$, then $f(z)$ given by* (1.1) *is in UST.*

The proof is left for the reader. However, it is possible that in this theorem $\sqrt{2}/2$ can be replaced by the larger constant $\sqrt{3}/2$.

5. THE PINCHUK QUESTION

We return to the question raised in Section 1. Here we are concerned with $f(\gamma)$ when the circle γ lies completely in E. We can assume that the circle γ is internally tangent to ∂E, and then obtain our conclusion by a continuity argument. Thus in $z = \zeta + re^{it}$ we must impose the condition $|\zeta| + r \leqslant 1$. Let $F(z) = z/(1+z)^2$ and $\zeta = Ce^{i\alpha}$. Then a brief computation will give,

$$Q = \frac{1 - z\zeta}{(1+\zeta)^2} \cdot \frac{1+z}{1-z}. \tag{5.1}$$

There is a domain D of points (C, r, α, t) in $R^{(4)}$ for which $C + r < 1$ and $\operatorname{Re} Q < 0$. For our purpose we only need one such point on the boundary of D. Using degree measure, the point $C = 0.18$, $r = 0.82$, $\alpha = 175°$, and $t = 130°$ will give

$$\operatorname{Re}(1 - z\zeta)(1+\zeta)^2\,(1+z)(1-\bar{z}) \approx -0.0287 < 0. \tag{5.2}$$

Thus, the answer to the question raised by Pinchuk is no.

After this paper was completed I learned that Johnny E. Brown [1] had found the same negative answer to Pinchuk's question. In his work Brown considers starlikeness on circles γ: $|z - z_0| < \rho$ that lie in E, and for each r in $(0, 1)$ he finds $\sup \rho$ such that if $f(z)$ is in S and $|z_0| = r$ with $r + \rho < 1$, then $f(z)$ maps γ onto a closed curve starlike with respect to $f(z_0)$. Thus Brown's work, which was done independently and probably earlier, goes much deeper into Pinchuk's question, but is more complicated than our Section 5. Except for this section, there is no duplication of results in the two papers.

6. Uniformly Convex Functions

The idea contained in Definition 1 can be extended in a large number of ways. Here we mention only one.

Definition 3. A function of the form (1.1), regular and univalent in E is said to be uniformly convex in E if for every circular arc γ contained in E whose center ζ is also in E, the arc $f(\gamma)$ is a convex curve. We let UCV denote the set of all such functions.

It is easy to prove that if $g(z)$ has the form (1.1), then $g(z)$ is in UCV iff

$$\operatorname{Re}\left[1 + \frac{g''(z)}{g'(z)}(z - \zeta)\right] > 0, \tag{6.1}$$

in $E \times E$. The tools for the proof can be found in [3, p. 110].

References

1. J. E. Brown, Images of disks under convex and starlike functions, *Math. Z.* **202** (1989), 457–462.
2. A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.* **8** (1957), 598–601.
3. A. W. Goodman, "Univalent Functions," Polygonal, Washington, NJ, 1983.
4. M. Obradovic, Some theorems on subordination by univalent functions, *Mat. Vesnik* **37** (1985), 211–214.
5. C. Pommerenke, Linear-invariante Familien analytischer Funktionen, I, *Math. Ann.* **155** (1964), 108–154.
6. W. Rudin, "Function Theory in Polydiscs," Benjamin, New York, 1969.
7. K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan* **11** (1959), 72–75.