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Intersections of Solutions of Nonlinear Parabolic Equations

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INTRODUCTION

As an extension of the property of uniqueness of solutions of nonlinear parabolic equations in one space dimension, it can be shown that the number of intersections of any two solutions of such an equation in a certain region cannot increase with time, unless in some way new intersections enter across the boundary of the region. This property enables useful deductions to be made about a solution, given its behavior at the boundaries of a region with respect to a family of known solutions. Such deductions are particularly useful when the known solutions, or some subset of them, act as "cluster points" in a larger space of solutions. The "intermediate asymptotic solutions" described by Zel'dovich and Barenblatt [1] form such cluster points; similarity solutions and traveling wave solutions are frequently occurring examples.

In this paper the general parabolic equation considered is

$$u_t = F(t, x, u, u_x, u_{xx}). \tag{1}$$

The function $F(t, x, p, q, r)$ is assumed to be defined and continuous on some domain E in R^5, and to be an increasing function of r in E.

A UNIQUENESS THEOREM

To consider uniqueness conditions for parabolic equations it is necessary to introduce the idea of a parabolic boundary (cf. [2]). In R^2 a past-neighborhood of a point $(\bar{x}, \bar{t})$ is, for some positive r, the set $\{(x, t): t < \bar{t}, x^2 + t^2 < r^2\}$. Then if G is an open connected domain in R^2 lying between $t = 0$ and $t = t_1$, the parabolic boundary R_p of G is the set of points on the frontier of G which do not have a past-neighborhood in G.

The following uniqueness theorem is a consequence of a theorem of Walter [2]:

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THEOREM 1. *Suppose $u(x, t)$ is a solution of* (1) *existing on $\bar{G}$ and obeying there the growth restriction*

$$e^{hx^2}u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \qquad \text{for all } t\colon 0 \leqslant t < t_1 \quad \text{and} \quad \text{for all } h < 0. \quad (2)$$

Suppose that for all (p, q, r) sufficiently small, for some $K > 0$

$$\begin{aligned} &|F(t, x, u, u_x, u_{xx}) - F(t, x, u + p, u_x + q, u_{xx} + r)| \\ &\quad < K[(1 + x^2)\,|p| + (1 + |x|)\,|q| + |r|]. \end{aligned} \qquad (3)$$

Then if $v(x, t)$ is any other solution of (1) *existing on $\bar{G}$ and taking the same values as $u(x, t)$ on R_p, and if $v(x, t)$ is also subject to the growth condition* (2), *then $u(x, t) = v(x, t)$ on $\bar{G}$.*

The conditions for uniqueness of solutions can be relaxed and varied somewhat; more extensive discussion can be found in [2] or [3].

CONSERVATION OF INTERSECTIONS OF SOLUTIONS

THEOREM 2. *Let G be a connected domain in R^2 lying between $t = 0$ and $t = t_1$, whose boundary consists of an interval I where $t = t_1$ together with a continuous parabolic boundary R_p of G. If solutions $u(x, t)$ and $v(x, t)$ of* (1) *exist on G and obey the growth restriction* (2), *and if in the neighborhood of u the Lipschitz-type condition* (3) *is obeyed, then if the difference $w(x, t) = u(x, t) - v(x, t)$ changes sign n times on R_p, it does not change sign more than n times on I.*

Proof. Suppose $(x_1, t_1) \in I$, and $w(x_1, t_1) > 0$. Let H be the maximal connected open subset of G adjacent to (x_1, t_1) on which $w(x, t) > 0$. Let K_p be the parabolic boundary of H.

By Theorem 1, if w is zero on K_p, then it will be zero at (x_1, t_1). So there must be a point $(\bar{x}, \bar{t})$ on K_p where $w(\bar{x}, \bar{t}) > 0$.

Suppose there is a past-neighborhood of $(\bar{x}, \bar{t})$ lying in G. Then there is a past-neighborhood, possibly smaller, which is in G and for which $w(x, t) > 0$, meaning that it is in H. This contradicts $(\bar{x}, \bar{t}) \in K_p$. Therefore no past-neighborhood of $(\bar{x}, \bar{t})$ lies in G, so $(\bar{x}, \bar{t}) \in R_p$.

So every point on I for which $w > 0$ can be connected with a point on R_p for which $w > 0$. Points where $w < 0$ can also be connected. Since the regions connecting the parts of I and K_p of alternating sign cannot cross, the conclusion of the theorem is established.

A General Application

It is characteristic of diffusive processes that details of the initial conditions have diminishing influence on the shape of solutions as time progresses. Consequently, a subset of solutions may emerge as a focus to which other solutions in some sense tend. Traveling wave solutions for autonomous equations, and similarity solutions for equations with the appropriate invariance properties are examples. As Zel'dovich and Barenblatt [1] indicate in their review of intermediate asymptotic solutions, these two classes are interconvertible, although when converted to traveling waveform, similarity solutions are not solutions of an autonomous equation. For that reason they are not treated here, although the method described can be adapted to deal with them.

The present application of Theorem 2 is to show how convergence of certain solutions of an autonomous parabolic equation to a wave of prescribed constant shape can be verified.

If (1) is autonomous, solutions of the form $U(x - mt)$ will obey the ordinary equation

$$-mU' = F(U, U', U''), \tag{4}$$

or, letting

$$P = -U', \tag{5}$$

$$mP = F(U, -P, P\, dP/dU). \tag{6}$$

The transformation

$$p(u, t) = -u_x(x, t) \tag{7}$$

is called the hodograph transformation, and the (u, p) plane is called the hodograph plane. To a differentiable function $f(x)$ corresponds a trajectory in the hodograph plane, and conversely a trajectory in the hodograph plane determines a graph of a function in the (x, u) plane. This correspondence will be frequently invoked. Functions of the form $U(x - mt)$ have stationary trajectories in the hodograph plane, and the aim of the analysis to follow is to show that certain solutions of (1) have trajectories converging to a time-invariant curve in the hodograph plane.

Suppose that initial and boundary conditions are specified which determine a solution $u(x, t)$ of (1), and the asymptotic form of this solution is sought. If the asymptotic form is a traveling wave, then it must be consistent, for large times, with those initial and boundary conditions. This requirement is restrictive and may uniquely specify a particular waveshape. Even where it does not (cf. [6]), there are likely to be elementary considerations indicating one of the possible waveforms as a limit. Suppose this wave has the form $U_0(x - m_0 t)$, and its trajectory in the hodograph plane is $P_0(u)$. It can be assumed that in the region of interest U_0 is a decreasing function.

Then some special concepts can be defined. A left comb-function is a function $U(X)$ defined and monotonic on some interval I, for which:

(i) for some $m > m_0$, and all $\alpha \geqslant 0$, $U(\alpha + x - mt)$ has only one intersection with $u(x, t)$ for all t and all x for which $\alpha + x + mt \in I$, and

(ii) $U(x + \alpha - mt)$ is a solution of (1).

The corresponding left comb-region is defined as the set

$$L(t) = \{(x, u)\text{: for some } \alpha \geqslant 0,\, u = U(\alpha + x - mt)\}.$$

A right comb-function and comb-region are defined similarly, with $m < m_0$ and $\alpha \leqslant 0$.

It is easy to verify that at each intersection of a comb-function with u, either $u_x(x, t) \geqslant U'(\alpha + x - mt)$, or $u_x(x, t) \leqslant U'(\alpha + x - mt)$, with the same inequality holding for each α. If the first inequality is true, the comb-function is called steep, and if the second is true, it is called shallow.

The combination of comb-function and region is called a comb. With each left comb there may be associated an overfunction $U^+(X)$, defined on some interval J, and having the following properties:

(i) for some m_+: $m_0 < m_+ < m$, $U^+(x - m_+t)$ is a solution of (1),

(ii) if $[x, U^+(x - m_+t)] \in L(t)$ then $U^+(x - m_+t) > u(x, t)$, and

(iii) the range of U^+ includes that of U.

With each right comb an associated underfunction may be similarly defined. The strategy of proof is then as follows. One seeks a family of steep left combs, with comb-functions arbitrarily close to $P_0(u)$ in the hodograph plane, and each having an associated overfunction. Then, since the comb-function moves to the right with greater speed than the overfunction, eventually the graph of the latter lies within the comb-region. By hypothesis this ensures that $u(x, t)$ also passes through the comb-region. At every point in the comb-region where u intersects a comb-function, it does so in such a way that $u_x \leqslant U'$. Therefore, if $P(m, u)$ is the hodograph trajectory of the comb-function, then for large t, $p(u, t) \leqslant P(m, u)$, where P can be made arbitrarily close to P_0.

Alternatively, one could seek a family of steep right combs, with associated underfunctions, with a similar argument. A third possibility is to find families of right and left steep combs, which will eventually overlap and cover the whole x-interval.

An analogous argument is then used with shallow combs, to show that for large enough times, $p(u, t)$ must exceed functions arbitrarily close to $P_0(u)$, For both arguments the choice of the various methods should be exercised with care, since some may be more difficult than others, or even unavailable.

Theorem 2 will be needed to establish that certain functions are comb-functions, or over- or underfunctions.

Application: The Nonlinear Fokker–Planck Equation

The nonlinear Fokker–Planck equation

$$u_t = [D(u)\, u_x]_x - [K(u)]_x \tag{8}$$

describes the motion under gravity and capillarity of a liquid in an unsaturated porous medium. The diffusivity D and conductivity K are assumed to be positive, monotonic, and convex. With an initial function $u(x, 0)$ prescribed for $x > 0$, and a boundary condition at $x = 0$, $t > 0$, one has the infiltration problem of soil physics. This has been extensively analyzed [4], and a demonstration of a tendency to traveling waveshape for a particular set of conditions has been obtained [5]. The method described here works for various initial and boundary conditions.

Suppose that $0 \leqslant u(x, 0) < 1$ for $x \geqslant 0$, and that for some $a > 0$, $u(x, 0) = 0$ for $x \geqslant a$. Suppose too that at $x = 0$ the constant flux condition,

$$D(u)\, u_x(0, t) - K(u) = -K(1), \tag{9}$$

applies. Equation (6) becomes

$$m = d/du(DP + K) \tag{10}$$

with the explicit solution

$$P(u) = [-K(u) + mu + A]/D(u). \tag{11}$$

Here the choice of m and A is arbitrary. Since D is assumed positive, the solutions are quite regular when $0 \leqslant u \leqslant 1$. There is just one choice of m and A for which $P(0) = P(1) = 0$; the corresponding wave is the only one which could be of the right form to indicate the eventual shape of the time-varying solution. Its parameter m is denoted m_0; then $A = K(1) - m_0$ and the corresponding hodograph trajectory is denoted $P_0(u)$. By decreasing m slightly, and increasing or decreasing A, functions $P_+(u)$ and $P_-(u)$ are produced which are respectively greater and less than $P_0(u)$, but arbitrarily close, in [0, 1]. The functions $U_+(x - mt - \alpha)$ corresponding to P_+ have negative slope for all u in [0, 1] and so, for large α, have only one intersection with $u(x, 0)$ on the line $u = 0$. Also for large α, $U_+(-mt - \alpha) > 1$, so unless $u(0, t) > 1$ for some t, U_+ is a steep right comb-function.

Since $u =$ constant is a solution of (8) and since from (9) $u_x(0, t) > 0$ if $u(0, t) > 1$, it is not possible that $u(0, t) > 1$. For if at $t = t_2$, $u(0, t)$ reaches

$1 + \epsilon > 1$ for the first time, then $u(x, t_2) \leqslant 1 + \epsilon$ for $x > 0$, from Theorem 2, since $u = 1 + \epsilon$ is a solution of (8). This contradicts $u_x(0, t) > 0$.

When $A < A_0$ and $m < m_0$, the zeros of $P_-(u)$ lie between 0 and 1, and so do the limits of the corresponding traveling wave U_-. The upper limit can be made arbitrarily close to 1, and since $u(x, 0)$ is continuous and less than 1 in $[0, a]$ it has an upper bound $b < 1$. So there exist right shallow combs which have initially no intersections at all with $u(x, 0)$, and subsequently can have no more than one, since at $x = 0$

$$\begin{aligned} \partial U_-/\partial x &= [K(U) - mU - A]/D(U), \\ &> [K(U) - K(1)]/D(U), \qquad \text{since } K(1) = A_0 + m_0 > A + mU, \\ &= \partial u/\partial x \qquad\qquad\qquad\quad \text{if} \quad u(0, t) = U. \end{aligned}$$

An underfunction U^- can be constructed with $A = A_0$ and $m < m_0$, with m arbitrarily close to m_0. The range of U^- includes 0, so for some α, $U^-(\alpha) = 0$. Then $U^-(x + \alpha)$ has no intersection with $u(x, 0)$ when $x \geqslant 0$, and when $t > 0$,

$$\begin{aligned} \partial/\partial x\, U^-(\alpha + x - mt) &= [K(U^-) - mU^- - A_0]/D(U^-), \\ &> [K(U^-) - m_0 U^- - A_0]/D(U^-), \\ &= u_x(0, t) \qquad \text{if} \qquad u(0, t) = U^-. \end{aligned}$$

Then by Theorem 2 it is impossible that U^- and u should intersect for $x > 0$ unless there is first an intersection at $x = 0$; this is impossible by the inequality just proved. So U^- is an underfunction.

So by the argument outlined above, with steep and shallow right combs each with an associated underfunction, it can be shown that after a sufficient period of time the shape of $u(x, t)$ is arbitrarily close to that of U_0, the traveling wave solution.

It is clear from the method that it is not necessary that the flux at $x = 0$ be constant; it is enough that it tends to a constant value. The flux condition prevents $u(x, t)$ from forming new intersections with comb- or underfunctions at $x = 0$; a restriction on $u(0, t)$ could do this as well, and the method can be used with moving boundary conditions. The condition that $u(x, 0)$ be zero for large x can be relaxed.

Remarks

Use of Theorem 2 yields methods which are insensitive to the details of the initial and boundary conditions; this is to be expected, because in diffusive processes these details become progressively less important.

The method described has been applied [6] to solutions of a diffusion equation with regeneration $u_t = u_{xx} + f(u)$. Here $f(0) = f(1) = 0$, and $\int_0^1 f(u)\, du > 0$.

The initial function $u(x, 0)$ is prescribed for all x. The problem is complicated by singularities in the first-order hodograph equation (6), so that there are various possible waves, and the choice is determined by the way in which u tends to zero as x tends to infinity.

The application described here uses functions having one intersection; functions having two intersections can be used to show that the velocity of the profile is locally well behaved, and converges to that of the traveling wave.

The fundamental theorem, Theorem 2, appears to apply only to equations with one space dimension, since its proof uses the property that in two dimensions one continuous curve cannot appear on both sides of another curve without crossing it.

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