Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

J. Math. Anal. Appl. 287 (2003) 430–443

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Recurrence and ergodicity in unital $*$-algebras

Rocco Duvenhage and Anton Ströh *

Department of Mathematics and Applied Mathematics, University of Pretoria, 0002 Pretoria, South Africa

Received 16 May 2002

Submitted by Z.-J. Ruan

Abstract

Results concerning recurrence and ergodicity are proved in an abstract Hilbert space setting based on the proof of Khintchine's recurrence theorem for sets, and on the Hilbert space characterization of ergodicity. These results are carried over to a non-commutative $*$-algebraic setting using the GNS-construction. This generalizes the corresponding measure theoretic results, in particular a variation of Khintchine's theorem for ergodic systems, where the image of one set overlaps with another set, instead of with itself.

1. Introduction

The inspiration for this paper is the following theorem of Khintchine dating from 1934 (see [4] for a proof):

Khintchine's theorem. *Let (X, Σ, μ) be a probability space (that is to say, μ is a measure on a σ-algebra Σ of subsets of a set X, with $\mu(X) = 1$), and consider a mapping $T : X \to X$ such that $T^{-1}(S) \in \Sigma$ and $\mu(T^{-1}(S)) \leqslant \mu(S)$ for all $S \in \Sigma$. Then for any $A \in \Sigma$ and $\varepsilon > 0$, the set*

$$E = \left\{k \in \mathbb{N}\colon\ \mu\left(A \cap T^{-k}(A)\right) > \mu(A)^2 - \varepsilon\right\}$$

is relatively dense in $\mathbb{N} = \{1, 2, 3, \ldots\}$.

* Corresponding author.
E-mail address: astroh@math.up.ac.za (A. Ströh).

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(02)00571-1

We will call (X, Σ, μ, T), as given above, a *measure theoretic dynamical system*. Recall that the relatively denseness of E in $\mathbb{N}$ means that there exists an $n \in \mathbb{N}$ such that $E \cap \{j, j+1, \ldots, j+n-1\}$ is non-empty for every $j \in \mathbb{N}$. Khintchine's theorem is an example of a recurrence result. It tells us that for every $k \in E$, the set A contains a set $A \cap T^{-k}(A)$ of measure larger than $\mu(A)^2 - \varepsilon$ which is mapped back into A by T^k.

A question that arises from Khintchine's theorem is whether, given $A, B \in \Sigma$ and $\varepsilon > 0$, the set

$$F = \left\{k \in \mathbb{N}\colon\ \mu\left(A \cap T^{-k}(B)\right) > \mu(A)\mu(B) - \varepsilon\right\}$$

is relatively dense in $\mathbb{N}$. This is clearly not true in general, for example, if T is the identity and A, B and ε are chosen such that $\mu(A)\mu(B) > \varepsilon$ while $A \cap B$ is empty, then F is empty. T has to "mix" the measure space sufficiently for F to be non-empty. In [5] it is shown for the case where $\mu(T^{-1}(S)) = \mu(S)$ for all $S \in \Sigma$, that if for every pair $A, B \in \Sigma$ of positive measure there exists some $k \in \mathbb{N}$ such that $\mu(A \cap T^{-k}(B)) > 0$, then the dynamical system is ergodic. Ergodicity therefore seems like the natural concept to use when considering the question posed above. This is indeed what we will do.

The notion of ergodicity originally developed as a way to characterize systems in classical statistical mechanics for which the time mean and the phase space mean of any observable are equal. For our purposes it will be most convenient to define ergodicity of a measure theoretic dynamical system (X, Σ, μ, T) as follows (refer to [4], for example): (X, Σ, μ, T) is called *ergodic* if the fixed points of the linear Hilbert space operator $U : L^2(\mu) \to L^2(\mu)$: $f \mapsto f \circ T$ form a one-dimensional subspace of $L^2(\mu)$. (It is easy to verify that U is well-defined on $L^2(\mu)$.)

As we shall see, the ideas we have discussed so far are not really measure theoretic in nature. This is in large part due to the fact that the proof of Khintchine's theorem is essentially a Hilbert space proof using the mean ergodic theorem. This proof can for the most part be written purely in Hilbert space terms, hence giving an abstract Hilbert space result. Along with the Hilbert space characterization of ergodicity given above, this means that a fair amount of ergodic theory can be done purely in an abstract Hilbert space setting. This is the approach taken in Section 3, using the mean ergodic theorem as the basic tool.

Having built up some ergodic theory in abstract Hilbert spaces, nothing is to stop us from applying the results to mathematical structures other than measure theoretic dynamical systems. The mathematical structure we will consider is much more general than measure theoretic dynamical systems and can easily be motivated as follows: From a measure theoretic dynamical system (X, Σ, μ, T) we obtain the unital $*$-algebra $B_\infty(\Sigma)$ of all bounded complex-valued measurable functions defined on X, and two linear mappings

$$\varphi : B_\infty(\Sigma) \to \mathbb{C}\colon \quad f \mapsto \int f \, d\mu$$

and

$$\tau : B_\infty(\Sigma) \to B_\infty(\Sigma)\colon \quad f \mapsto f \circ T \tag{1}$$

with the following properties: $\varphi(1) = 1$, $\varphi(f^* f) \geqslant 0$, $\tau(1) = 1$ and $\varphi(\tau(f)^* \tau(f)) \leqslant \varphi(f^* f)$ for all $f \in B_\infty(\Sigma)$, where $f^* = \overline{f}$ defines the involution on $B_\infty(\Sigma)$, making it a

$*$-algebra. We can view this abstractly by replacing $B_\infty(\Sigma)$ with any unital $*$-algebra and considering linear mappings φ and τ on it with the properties mentioned above. (A *unital* $*$*-algebra* $\mathfrak{A}$ is an algebra with an involution, and a unit element denoted by 1, that is to say $1A = A = A1$ for all $A \in \mathfrak{A}$. We will only work with the case of complex scalars.) The most obvious generalization this brings is that the unital $*$-algebra need not be commutative, for example, the bounded linear operators on a Hilbert space. Also note that τ in (1) is a $*$-homomorphism of $B_\infty(\Sigma)$, but we will not need this property of τ in the abstract $*$-algebraic setting. We describe the $*$-algebraic setting in more detail in Section 2, and in Section 4 the Hilbert space results are applied to this setting using the GNS-construction.

In Section 5 we obtain the measure theoretic results as a special case, and also briefly discuss another special case, namely von Neumann algebras.

2. $*$-dynamical systems and ergodicity

By a *state* on a unital $*$-algebra $\mathfrak{A}$ we mean a linear functional φ on $\mathfrak{A}$ which is positive (i.e., $\varphi(A^*A) \geqslant 0$ for all $A \in \mathfrak{A}$) with $\varphi(1) = 1$. Motivated by our remarks in Section 1, we give the following definition:

Definition 2.1. Let φ be a state on a unital $*$-algebra $\mathfrak{A}$. Consider any linear function $\tau : \mathfrak{A} \to \mathfrak{A}$ such that

$$\tau(1) = 1$$

and

$$\varphi\big(\tau(A)^*\tau(A)\big) \leqslant \varphi(A^*A)$$

for all $A \in \mathfrak{A}$. Then we call $(\mathfrak{A}, \varphi, \tau)$ a $*$-*dynamical system.*

Let $L(V)$ denote the algebra of all linear operators $V \to V$ on the vector space V.

Definition 2.2. Let φ be a state on a unital $*$-algebra $\mathfrak{A}$. A *cyclic representation* of $(\mathfrak{A}, \varphi)$ is a triple $(\mathfrak{G}, \pi, \Omega)$, where $\mathfrak{G}$ is an inner product space, $\pi : \mathfrak{A} \to L(\mathfrak{G})$ is linear with $\pi(1) = 1$, $\pi(AB) = \pi(A)\pi(B)$, $\Omega \in \mathfrak{G}$, $\pi(\mathfrak{A})\Omega = \mathfrak{G}$, and $\langle \pi(A)\Omega, \pi(B)\Omega \rangle = \varphi(A^*B)$, for all $A, B \in \mathfrak{A}$.

A cyclic representation as in Definition 2.2 exists by the GNS-construction (refer to [1] for example, where the construction is performed for C^*-algebras, but it also works for unital $*$-algebras). We will not need the property $\pi(AB) = \pi(A)\pi(B)$ in this paper however. The term "cyclic" refers to the fact that $\pi(\mathfrak{A})\Omega = \mathfrak{G}$. Note that

$$\iota : \mathfrak{A} \to \mathfrak{G}: \quad A \mapsto \pi(A)\Omega \tag{2}$$

is a linear surjection such that $\iota(1) = \Omega$, and that

$$U_0 : \mathfrak{G} \to \mathfrak{G}: \quad \iota(A) \mapsto \iota\big(\tau(A)\big) \tag{3}$$

is a well-defined linear operator with $\|U_0\| \leqslant 1$ for τ as in Definition 2.1, since $\|\iota(\tau(A))\|^2 = \varphi(\tau(A)^*\tau(A)) \leqslant \varphi(A^*A) = \|\iota(A)\|^2$. We define a seminorm $\|\cdot\|_\varphi$ on $\mathfrak{A}$ by

$$\|A\|_\varphi = \sqrt{\varphi(A^*A)} = \|\iota(A)\|$$

for all $A \in \mathfrak{A}$. We now want to define the concept of ergodicity for a $*$-dynamical system.

Definition 2.3. A $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$ is called *ergodic* if it has the following property: For any sequence (A_n) in $\mathfrak{A}$ such that $\|\tau(A_n) - A_n\|_\varphi \to 0$ and such that for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ for which $\|A_m - A_n\|_\varphi < \varepsilon$ if $m > N$ and $n > N$, it follows that $\|A_n - \alpha\|_\varphi \to 0$ for some $\alpha \in \mathbb{C}$.

In Section 4 we will give a simple example of an ergodic $*$-dynamical system whose $*$-algebra is non-commutative. Recall that for any vectors x and y in a Hilbert space $\mathfrak{H}$, we denote by $x \otimes y$ the bounded linear operator $\mathfrak{H} \to \mathfrak{H}$ defined by $(x \otimes y)z = x\langle y, z\rangle$. The motivation for Definition 2.3 is the following proposition:

Proposition 2.4. *Consider a $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$ and let U_0 be given by* (3) *in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$. Let $U : \mathfrak{H} \to \mathfrak{H}$ be the bounded linear extension of U_0 to the completion $\mathfrak{H}$ of $\mathfrak{G}$, and let P be the projection of $\mathfrak{H}$ onto the subspace of fixed points of U. Then $(\mathfrak{A}, \varphi, \tau)$ is ergodic if and only if $P = \Omega \otimes \Omega$, that is to say, if and only if the fixed points of U form a one-dimensional subspace of $\mathfrak{H}$.*

Proof. Since $\|\Omega\|^2 = \varphi(1^*1) = 1$, we know that $\Omega \otimes \Omega$ is the projection of $\mathfrak{H}$ onto the one-dimensional subspace $\mathbb{C}\Omega$. Also note that $U\Omega = \Omega$, since $\Omega = \iota(1)$, hence $\mathbb{C}\Omega \subset P\mathfrak{H}$.

Suppose $(\mathfrak{A}, \varphi, \tau)$ is ergodic and let x be a fixed point of U. Consider any sequence (x_n) in $\mathfrak{G}$ such that $x_n \to x$, say $x_n = \iota(A_n)$. Then $\|\tau(A_n) - A_n\|_\varphi = \|Ux_n - x_n\| \to 0$, since U is continuous, while for any $\varepsilon > 0$ there exists some N for which $\|A_m - A_n\|_\varphi = \|x_m - x_n\| < \varepsilon$ if $m > N$ and $n > N$. Since $(\mathfrak{A}, \varphi, \tau)$ is ergodic, it follows that $\|x_n - \iota(\alpha)\| = \|A_n - \alpha\|_\varphi \to 0$ for some $\alpha \in \mathbb{C}$, but then $x = \iota(\alpha) = \alpha\Omega$. Therefore $P\mathfrak{H} = \mathbb{C}\Omega$ which means that $P = \Omega \otimes \Omega$.

Conversely, suppose $P = \Omega \otimes \Omega$ and consider any sequence (A_n) in $\mathfrak{A}$ such that $\|\tau(A_n) - A_n\|_\varphi \to 0$ and such that for any $\varepsilon > 0$ there exists some N for which $\|A_m - A_n\|_\varphi < \varepsilon$ if $m > N$ and $n > N$. Then $x_n = \iota(A_n)$ is a Cauchy sequence and hence convergent in $\mathfrak{H}$, since $\|x_m - x_n\| = \|A_m - A_n\|_\varphi$. Say $x_n \to x$, then $Ux_n \to Ux$ since U is continuous. Since $\|Ux_n - x_n\| = \|\tau(A_n) - A_n\|_\varphi \to 0$, it follows that $Ux_n \to x$, hence $Ux = x$. This means that $x \in P\mathfrak{H}$ which implies that $x = \alpha\Omega$ for some $\alpha \in \mathbb{C}$. Therefore $\|A_n - \alpha\|_\varphi = \|x_n - \alpha\Omega\| \to 0$, and so we conclude that $(\mathfrak{A}, \varphi, \tau)$ is ergodic. □

Proposition 2.4 tells us that Definition 2.3 includes the measure theoretic definition as a special case. This can be seen as follows: From a measure theoretic dynamical system (X, Σ, μ, T) we obtain the $*$-dynamical system $(B_\infty(\Sigma), \varphi, \tau)$, where $\varphi(f) = \int f\, d\mu$ and $\tau(f) = f \circ T$ for all $f \in B_\infty(\Sigma)$. A cyclic representation of $(B_\infty(\Sigma), \varphi, \tau)$ is $(\mathfrak{G}, \pi, \Omega)$ with $\mathfrak{G} = \{[g]\colon\ g \in B_\infty(\Sigma)\}$, $\pi(f)[g] = [fg]$ for all $f, g \in B_\infty(\Sigma)$, and $\Omega = [1]$, where

$[g]$ denotes the equivalence class of all measurable complex-valued functions on the measure space that are almost everywhere equal to g. The completion of $\mathfrak{G}$ is $L^2(\mu)$, and U in Proposition 2.4 is now given by

$$Uf = f \circ T$$

for all $f \in L^2(\mu)$, where here we have dropped the $[\cdot]$ notation, as is standard for L^2-spaces (f and $f \circ T$ now denote equivalence classes of functions). Proposition 2.4 tells us that $(B_\infty(\Sigma), \varphi, \tau)$ is ergodic if and only if the fixed points of U form a one-dimensional subspace of $L^2(\mu)$, in other words if and only if (X, Σ, μ, T) is ergodic, as was mentioned in Section 1.

Finally we remark that we use Definition 2.3 as the definition of ergodicity, since it is formulated purely in terms of the objects $\mathfrak{A}$, φ and τ appearing in the $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$, unlike Proposition 2.4 which involves a cyclic representation of these objects. However, as a characterization of ergodicity, Proposition 2.4 is generally easier to use. Of course, one might wonder if Definition 2.3 could not be simplified by using a single element rather than a sequence. With U as in Proposition 2.4, and $x = \iota(A)$ for some $A \in \mathfrak{A}$, we have $Ux = x$ if and only if $\|Ux - x\| = 0$, which is equivalent to $\|\tau(A) - A\|_\varphi = 0$. For ergodicity we need this to imply that $x = \alpha\Omega$ for some $\alpha \in \mathbb{C}$, which is equivalent to $\|A - \alpha\|_\varphi = \|x - \alpha\Omega\| = 0$. However, we cannot define ergodicity as "$\|\tau(A) - A\|_\varphi = 0$ implies that $\|A - \alpha\|_\varphi = 0$ for some $\alpha \in \mathbb{C}$," since Proposition 2.4 would no longer hold: There would be examples of ergodic $*$-dynamical systems for which the fixed points of U do not form a one-dimensional subspace of $\mathfrak{H}$. (In Appendix A we give such an example.) Our theory would then fall apart, since much of our later work is based on the fact that for ergodic systems the fixed point space of U is one-dimensional. For example, the characterization of ergodicity in terms of the equality of means of the sort mentioned in Section 1 (but extended to $*$-dynamical systems), implies this one-dimensionality. Also, this one-dimensionality is used in our proof of the variation of Khintchine's theorem mentioned in Section 1. (See Sections 3 and 4 for details.) The use of a sequence rather than a single element is therefore necessary in Definition 2.3.

3. Some ergodic theory in Hilbert spaces

Our main tool in this section is the

Mean ergodic theorem. *Consider a linear operator $U : \mathfrak{H} \to \mathfrak{H}$ with $\|U\| \leqslant 1$ on a Hilbert space $\mathfrak{H}$. Let P be the projection of $\mathfrak{H}$ onto the subspace of fixed points of U. For any $x \in \mathfrak{H}$ we then have*

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k x \to Px$$

as $n \to \infty$.

Refer to [4] for a proof. We now state and prove a generalized Hilbert space version of Khintchine's theorem:

Theorem 3.1. *Let $\mathfrak{H}$, U and P be as in the mean ergodic theorem above. Consider any $x, y \in \mathfrak{H}$ and $\varepsilon > 0$. Then the set*

$$E = \left\{k \in \mathbb{N}\colon \left|\langle x, U^k y\rangle\right| > \left|\langle x, Py\rangle\right| - \varepsilon\right\}$$

is relatively dense in $\mathbb{N}$.

Proof. The proof is essentially the same as that of Khintchine's theorem. By the mean ergodic theorem there exists an $n \in \mathbb{N}$ such that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k y - Py\right\| < \frac{\varepsilon}{\|x\|+1}.$$

Since $UPy = Py$ and $\|U\| \leqslant 1$, it follows for any $j \in \mathbb{N}$ that

$$\left\|\frac{1}{n}\sum_{k=j}^{j+n-1} U^k y - Py\right\| \leqslant \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k y - Py\right\| < \frac{\varepsilon}{\|x\|+1}$$

and therefore

$$\left|\left\langle x, \frac{1}{n}\sum_{k=j}^{j+n-1} U^k y - Py\right\rangle\right| \leqslant \|x\|\left\|\frac{1}{n}\sum_{k=j}^{j+n-1} U^k y - Py\right\| < \varepsilon.$$

Hence

$$\left|\langle x, Py\rangle\right| - \varepsilon < \left|\frac{1}{n}\sum_{k=j}^{j+n-1}\langle x, U^k y\rangle\right| \leqslant \frac{1}{n}\sum_{k=j}^{j+n-1}\left|\langle x, U^k y\rangle\right|$$

and so $|\langle x, U^k y\rangle| > |\langle x, Py\rangle| - \varepsilon$ for some $k \in \{j, j+1, \ldots, j+n-1\}$, in other words E is relatively dense in $\mathbb{N}$. □

Khintchine's theorem corresponds to the case where $y = x$. The following two propositions are the Hilbert space building blocks for two characterizations of ergodicity to be considered in the next section.

Proposition 3.2. *Let $\mathfrak{H}$, U and P be as in the mean ergodic theorem above. Consider an $\Omega \in \mathfrak{H}$ and let $\mathfrak{T}$ be any total set in $\mathfrak{H}$. Then the following hold*:

(i) *If $P = \Omega \otimes \Omega$, then*

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k y - \Omega\langle\Omega, y\rangle\right\| \to 0 \tag{4}$$

as $n \to \infty$, for every $y \in \mathfrak{H}$.

(ii) *If* (4) *holds for every $y \in \mathfrak{T}$, then $P = \Omega \otimes \Omega$.*

Proof. By the mean ergodic theorem we know that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k y - Py \right\| \to 0 \tag{5}$$

for every $y \in \mathfrak{H}$ as $n \to \infty$, but for $P = \Omega \otimes \Omega$ we have $Py = \Omega \langle \Omega, y \rangle$ and this proves (i).

To prove (ii), consider any $y \in \mathfrak{T}$. From (4) and (5) it then follows that $Py = \Omega \langle \Omega, y \rangle = (\Omega \otimes \Omega) y$. Since by definition the linear span of $\mathfrak{T}$ is dense in $\mathfrak{H}$, and since P and $\Omega \otimes \Omega$ are bounded (and hence continuous) linear operators on $\mathfrak{H}$, we conclude that $P = \Omega \otimes \Omega$. □

Proposition 3.3. *Let $\mathfrak{H}$, U and P be as in the mean ergodic theorem above. Consider an $\Omega \in \mathfrak{H}$ and let $\mathfrak{S}$ and $\mathfrak{T}$ be total sets in $\mathfrak{H}$. Then the following hold*:

(i) *If $P = \Omega \otimes \Omega$, then*

$$\frac{1}{n} \sum_{k=0}^{n-1} \langle x, U^k y \rangle \to \langle x, \Omega \rangle \langle \Omega, y \rangle \tag{6}$$

as $n \to \infty$, for all $x, y \in \mathfrak{H}$.

(ii) *If* (6) *holds for all $x \in \mathfrak{S}$ and $y \in \mathfrak{T}$, then $P = \Omega \otimes \Omega$.*

Proof. Statement (i) follows immediately from Proposition 3.2(i) by simply taking the inner product of x with the expression inside the norm in (4).

To prove (ii), consider any $x \in \mathfrak{S}$ and $y \in \mathfrak{T}$. From the mean ergodic theorem it follows that

$$\frac{1}{n} \sum_{k=0}^{n-1} \langle x, U^k y \rangle \to \langle x, Py \rangle$$

as $n \to \infty$. Combining this with (6) we see that $\langle x, Py \rangle = \langle x, \Omega \rangle \langle \Omega, y \rangle = \langle x, (\Omega \otimes \Omega) y \rangle$. Since the linear span of $\mathfrak{S}$ is dense in $\mathfrak{H}$, this implies that $Py = (\Omega \otimes \Omega) y$. Hence $P = \Omega \otimes \Omega$ as in the proof of Proposition 3.2(ii). □

The reason for using total sets will become clear in Sections 4 and 5.

4. Ergodic results for $*$-dynamical systems

In this section we carry the results of Section 3 over to $*$-dynamical systems using cyclic representations. Firstly we give a $*$-dynamical generalization of Khintchine's theorem which follows from Theorem 3.1:

Theorem 4.1. *Let $(\mathfrak{A}, \varphi, \tau)$ be a $*$-dynamical system, and consider any $A \in \mathfrak{A}$ and $\varepsilon > 0$. Then the set*

$$E = \left\{ k \in \mathbb{N} \colon \left| \varphi\left(A^* \tau^k(A)\right) \right| > \left| \varphi(A) \right|^2 - \varepsilon \right\}$$

is relatively dense in $\mathbb{N}$.

Proof. Let U and P be defined as in Proposition 2.4 in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$. Set $x = \iota(A)$. From (3) it is clear that $\Omega = \iota(1)$ is a fixed point of U, so $\langle \Omega, x\rangle = \langle P\Omega, x\rangle = \langle \Omega, Px\rangle$. It follows that $|\varphi(A)| = |\varphi(1^*A)| = |\langle \Omega, x\rangle| \leqslant \|\Omega\|\|Px\| = \|Px\|$. We also have $\varphi(A^*\tau^k(A)) = \langle x, U^k x\rangle$. Hence by Theorem 3.1, with $y = x$, the set E is relatively dense in $\mathbb{N}$. □

A C^*-algebraic version of Theorem 4.1 was previously obtained in [3]. Next we use Theorem 3.1 to prove a variant of Theorem 4.1:

Theorem 4.2. *Let $(\mathfrak{A}, \varphi, \tau)$ be an ergodic $*$-dynamical system, and consider any $A, B \in \mathfrak{A}$ and $\varepsilon > 0$. Then the set*

$$E = \left\{k \in \mathbb{N}\colon \left|\varphi\left(A\tau^k(B)\right)\right| > \left|\varphi(A)\varphi(B)\right| - \varepsilon\right\}$$

is relatively dense in $\mathbb{N}$.

Proof. Let U and P be defined as in Proposition 2.4 in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$. Set $x = \iota(A^*)$ and $y = \iota(B)$. By Proposition 2.4 we have $Px = \alpha\Omega$ and $Py = \beta\Omega$ where $\overline{\alpha} = \langle x, \Omega\rangle = \varphi(A^{**}1) = \varphi(A)$ and $\beta = \varphi(B)$. Therefore $|\langle x, Py\rangle| = |\langle Px, Py\rangle| = |\overline{\alpha}\beta|\|\Omega\|^2 = |\varphi(A)\varphi(B)|$. Furthermore, $\varphi(A\tau^k(B)) = \langle x, U^k y\rangle$. Hence E is relatively dense in $\mathbb{N}$ by Theorem 3.1. □

We are now going to prove two characterizations of ergodicity using Propositions 3.2 and 3.3, respectively. But first we need to consider a notion of totality of a set in a unital $*$-algebra. (Remember that an abstract unital $*$-algebra has no norm.)

Definition 4.3. Let φ be a state on a unital $*$-algebra $\mathfrak{A}$. A subset $\mathfrak{T}$ of $\mathfrak{A}$ is called *φ-dense* in $\mathfrak{A}$ if it is dense in the seminormed space $(\mathfrak{A}, \|\cdot\|_\varphi)$. A subset $\mathfrak{T}$ of $\mathfrak{A}$ is called *φ-total* in $\mathfrak{A}$ if the linear span of $\mathfrak{T}$ is φ-dense in $\mathfrak{A}$.

Trivially, a unital $*$-algebra is φ-total in itself for any state φ.

Lemma 4.4. *Let φ be a state on a unital $*$-algebra $\mathfrak{A}$, and consider any subset $\mathfrak{T}$ of $\mathfrak{A}$. Let ι be given by* (2) *in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$, and let $\mathfrak{H}$ be the completion of $\mathfrak{G}$. Then $\mathfrak{T}$ is φ-total in $\mathfrak{A}$ if and only if $\iota(\mathfrak{T})$ is total in $\mathfrak{H}$.*

Proof. Suppose $\mathfrak{T}$ is φ-total in $\mathfrak{A}$, that is to say the linear span $\mathfrak{B}$ of $\mathfrak{T}$ is φ-dense in $\mathfrak{A}$. Then $\iota(\mathfrak{B})$ is dense in $\mathfrak{G} = \iota(\mathfrak{A})$, since for any $A \in \mathfrak{A}$ there exists a sequence (A_n) in $\mathfrak{B}$ such that $\|\iota(A_n) - \iota(A)\| = \|A_n - A\|_\varphi \to 0$. But by definition $\mathfrak{G}$ is dense in $\mathfrak{H}$, hence $\iota(\mathfrak{B})$ is dense in $\mathfrak{H}$. Since ι is linear, this means that $\iota(\mathfrak{T})$ is total in $\mathfrak{H}$.

Conversely, suppose $\iota(\mathfrak{T})$ is total in $\mathfrak{H}$, then $\iota(\mathfrak{B})$ is dense in $\mathfrak{H}$. It follows that $\mathfrak{B}$ is φ-dense in $\mathfrak{A}$, since for any $A \in \mathfrak{A}$ there exists a sequence (A_n) in $\mathfrak{B}$ such that $\|A_n - A\|_\varphi = \|\iota(A_n) - \iota(A)\| \to 0$. In other words, $\mathfrak{T}$ is φ-total in $\mathfrak{A}$. □

Proposition 4.5. *Let $(\mathfrak{A}, \varphi, \tau)$ be a $*$-dynamical system, and consider any φ-total set $\mathfrak{T}$ in $\mathfrak{A}$. Then the following hold*:

(i) *If* $(\mathfrak{A}, \varphi, \tau)$ *is ergodic, then*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \tau^k(A) - \varphi(A) \right\|_\varphi \to 0 \tag{7}$$

as $n \to \infty$, *for every* $A \in \mathfrak{A}$.

(ii) *If* (7) *holds for every* $A \in \mathfrak{T}$, *then* $(\mathfrak{A}, \varphi, \tau)$ *is ergodic.*

Proof. Let U and P be defined as in Proposition 2.4 in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$. Suppose $(\mathfrak{A}, \varphi, \tau)$ is ergodic. For any $A \in \mathfrak{A}$ we then have

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \tau^k(A) - \varphi(A) \right\|_\varphi = \left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k \iota(A) - \iota\big(\varphi(A)\big) \right\| \to 0 \tag{8}$$

as $n \to \infty$, by Proposition 3.2(i) and Proposition 2.4, since $\iota(\varphi(A)) = \iota(1)\varphi(A) = \Omega\varphi(1^*A) = \Omega\langle\Omega, \iota(A)\rangle$. This proves (i).

Now suppose (7), and therefore (8), hold for every $A \in \mathfrak{T}$. Since $\iota(\mathfrak{T})$ is total in $\mathfrak{H}$ according to Lemma 4.4, it follows from Proposition 3.2(ii) and the identity $\iota(\varphi(A)) = \Omega\langle\Omega, \iota(A)\rangle$, that $P = \Omega \otimes \Omega$. So $(\mathfrak{A}, \varphi, \tau)$ is ergodic by Proposition 2.4, confirming (ii). □

In the spirit of the original motivation behind the concept of ergodicity, this proposition characterizes ergodic $*$-dynamical systems as those for which the *time mean* of each element A of the $*$-algebra converges in the seminorm $\|\cdot\|_\varphi$ to the "phase space" mean $\varphi(A)$. A better name for the latter would be the *system mean* in this case, since there is no phase space involved. For a measure theoretic dynamical system (X, Σ, τ, μ), the state φ is given by $\varphi(f) = \int f \, d\mu$ which is indeed the phase space mean of $f \in B_\infty(\Sigma)$, where X is the phase space. We will come back to this in Section 5.

For any subset $\mathfrak{S}$ of a $*$-algebra, we write $\mathfrak{S}^* = \{A^*\colon\ A \in \mathfrak{S}\}$.

Proposition 4.6. *Let* $(\mathfrak{A}, \varphi, \tau)$ *be a* $*$*-dynamical system, and consider any* φ*-total sets* $\mathfrak{S}$ *and* $\mathfrak{T}$ *in* $\mathfrak{A}$. *Then the following hold*:

(i) *If* $(\mathfrak{A}, \varphi, \tau)$ *is ergodic, then*

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi\big(A\tau^k(B)\big) \to \varphi(A)\varphi(B) \tag{9}$$

as $n \to \infty$, *for all* $A, B \in \mathfrak{A}$.

(ii) *If* (9) *holds for all* $A \in \mathfrak{S}^*$ *and* $B \in \mathfrak{T}$, *then* $(\mathfrak{A}, \varphi, \tau)$ *is ergodic.*

Proof. Let U and P be defined as in Proposition 2.4 in terms of any cyclic representation of $(\mathfrak{A}, \varphi)$. Suppose $(\mathfrak{A}, \varphi, \tau)$ is ergodic. Then $P = \Omega \otimes \Omega$ by Proposition 2.4, and so by Proposition 3.3(i) it follows that

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi\big(A\tau^k(B)\big) = \frac{1}{n} \sum_{k=0}^{n-1} \big\langle \iota(A^*), U^k \iota(B) \big\rangle \to \varphi(A)\varphi(B) \tag{10}$$

as $n \to \infty$, since $\langle \iota(A^*), \Omega \rangle = \varphi(A)$ and $\langle \Omega, \iota(B) \rangle = \varphi(B)$, as in the proof of Theorem 4.2. This proves (i). (Alternatively, (i) can be derived from Proposition 4.5(i) using the Cauchy–Schwarz inequality $|\varphi(AC)| \leqslant \|A^*\|_\varphi \|C\|_\varphi$ with $C = (1/n)\sum_{k=0}^{n-1} \tau^k(B) - \varphi(B)$. This is essentially how Proposition 3.3(i) was derived from Proposition 3.2(i).)

Now suppose (9), and therefore (10), hold for all $A \in \mathfrak{S}^*$ and $B \in \mathfrak{T}$. Since $\iota(\mathfrak{S})$ and $\iota(\mathfrak{T})$ are total in $\mathfrak{H}$ according to Lemma 4.4, it follows from Proposition 3.3(ii) and the identities $\langle \iota(A^*), \Omega \rangle = \varphi(A)$ and $\langle \Omega, \iota(B) \rangle = \varphi(B)$, that $P = \Omega \otimes \Omega$. So $(\mathfrak{A}, \varphi, \tau)$ is ergodic by Proposition 2.4, confirming (ii). □

This characterizes ergodicity in terms of *mixing*. We now give a simple example of an ergodic $*$-dynamical system whose $*$-algebra is non-commutative:

Example 4.7. Let $\mathfrak{A}$ be the unital $*$-algebra of (2×2)-matrices with entries in $\mathbb{C}$, the involution being the conjugate transpose. Let φ be the normalized trace on $\mathfrak{A}$, that is to say $\varphi = (1/2)\mathrm{Tr}$. Define $\tau : \mathfrak{A} \to \mathfrak{A}$ by

$$\tau \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & c_1 a_{12} \\ c_2 a_{21} & a_{11} \end{pmatrix}$$

for some fixed $c_1, c_2 \in \mathbb{C}$ with $|c_1| \leqslant 1$, $|c_2| \leqslant 1$, $c_1 \neq 1$ and $c_2 \neq 1$. The conditions $|c_1| \leqslant 1$ and $|c_2| \leqslant 1$ are necessary and sufficient for $(\mathfrak{A}, \varphi, \tau)$ to be a $*$-dynamical system. Note that for any $c \in \mathbb{C}$ with $|c| \leqslant 1$, it follows from the mean ergodic theorem that

$$\frac{1}{n} \sum_{k=0}^{n-1} c^k$$

converges to 0 if $c \neq 1$, and to 1 otherwise. Using this fact and Proposition 4.6(ii) with $\mathfrak{S} = \mathfrak{T} = \mathfrak{A}$ (and some calculations), it can be verified that the conditions $c_1 \neq 1$ and $c_2 \neq 1$ are necessary and sufficient for $(\mathfrak{A}, \varphi, \tau)$ to be ergodic, assuming that $|c_1| \leqslant 1$ and $|c_2| \leqslant 1$.

5. Measure theory and von Neumann algebras

As was mentioned in Section 2, from a measure theoretic dynamical system (X, Σ, μ, T) we obtain the $*$-dynamical system $(B_\infty(\Sigma), \varphi, \tau)$, where $\varphi(f) = \int f \, d\mu$ and $\tau(f) = f \circ T$. This allows us to apply the results of Section 4 to measure theoretic dynamical systems. For example, if (X, Σ, μ, T) is ergodic, then we know from Section 2 that $(B_\infty(\Sigma), \varphi, \tau)$ is ergodic. Hence for this $*$-dynamical system Theorem 4.2 tells us that for any $A, B \in \Sigma$ and $\varepsilon > 0$, the set

$$\left\{ k \in \mathbb{N} \colon \left| \varphi\left(\chi_A \tau^k(\chi_B)\right) \right| > \left| \varphi(\chi_A)\varphi(\chi_B) \right| - \varepsilon \right\}$$

is relatively dense in $\mathbb{N}$, but this set is exactly the set F from Section 1. (Here χ denotes characteristic functions.) So we have answered our original question:

Corollary 5.1. *Let (X, Σ, μ, T) be an ergodic measure theoretic dynamical system. Then for any $A, B \in \Sigma$ and $\varepsilon > 0$, the set*

$$F = \left\{k \in \mathbb{N}\colon\ \mu\left(A \cap T^{-k}(B)\right) > \mu(A)\mu(B) - \varepsilon\right\}$$

is relatively dense in $\mathbb{N}$.

This result says that for every $k \in F$, the set A contains a set $A \cap T^{-k}(B)$ of measure larger than $\mu(A)\mu(B) - \varepsilon$, which is mapped into B by T^k. Using a similar argument, Khintchine's theorem follows from Theorem 4.1.

Likewise, Propositions 4.5 and 4.6 can be applied to the measure theoretic case. For example, Proposition 4.5(i) tells us that if (X, Σ, μ, T) is ergodic, then

$$\int \left| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k - \varphi(f) \right|^2 d\mu \to 0 \tag{11}$$

as $n \to \infty$, for every $f \in B_\infty(\Sigma)$. Note that this result is not pointwise and is therefore not quite as strong as the usual measure theoretic statement of equality of the time mean and the phase space mean. This is of course where Birkhoff's pointwise ergodic theorem comes into play (see, for example, [4]).

What about the converse? Well, in order to effectively apply Propositions 4.5(ii) and 4.6(ii) to the measure theoretic case, we need to know what the measure theoretic significance of a φ-total set in $B_\infty(\Sigma)$ is. The basic fact we will use is the following simple proposition which follows from Lebesgue's dominated convergence theorem:

Proposition 5.2. *Let (X, Σ, μ) be a probability space and set $\varphi(f) = \int f\, d\mu$ for all $f \in B_\infty(\Sigma)$. Then the set $\mathfrak{T} = \{\chi_S\colon\ S \in \Sigma\}$ is φ-total in $B_\infty(\Sigma)$.*

From this we see that if (11) holds for all measurable characteristic functions f, then $(B_\infty(\Sigma), \varphi, \tau)$ is ergodic by Proposition 4.5(ii), hence (X, Σ, μ, T) is ergodic as mentioned in Section 2.

Finally, with reference to Proposition 4.6(ii), we note that $\mathfrak{T}^* = \mathfrak{T}$ for $\mathfrak{T}$ as in Proposition 5.2.

Next we briefly look at von Neumann algebras, as they are well-known examples of unital $*$-algebras. Consider a von Neumann algebra $\mathfrak{M}$ and suppose $(\mathfrak{M}, \varphi, \tau)$ is a $*$-dynamical system. For example, τ might be a $*$-homomorphism leaving φ invariant, that is to say, $\varphi(\tau(A)) = \varphi(A)$ for all $A \in \mathfrak{M}$. Then the results of Section 4 can be applied directly to $(\mathfrak{M}, \varphi, \tau)$. As a more explicit (and ergodic) example, we note that $\mathfrak{A}$ in Example 4.7 is a von Neumann algebra on the Hilbert space $\mathbb{C}^2$. We can also mention that τ in Example 4.7 is not a homomorphism.

We now describe one suitable choice for the φ-total sets appearing in Propositions 4.5 and 4.6. Let $\mathfrak{P}$ be the projections of $\mathfrak{M}$. It is known that $\mathfrak{M}$ is the norm closure of the linear span of $\mathfrak{P}$, as is mentioned, for example, in [2, p. 326]. Since any state φ on $\mathfrak{M}$ is continuous by virtue of being positive, it follows that $\mathfrak{P}$ is φ-total in $\mathfrak{M}$. Note also, regarding Proposition 4.6(ii), that $\mathfrak{P}^* = \mathfrak{P}$. This is all very similar to the measure theoretic case in Proposition 5.2, since the measurable characteristic functions on X are exactly the

projections of $B_\infty(\Sigma)$. This similarity should not be too surprising, since the theory of von Neumann algebras is often described as "non-commutative measure theory" because of the close analogy with measure theory.

Acknowledgment

We would like to thank the National Research Foundation and the Mellon Foundation Mentoring Programme for financial support. We are also grateful to László Zsidó for providing us with the example used in Appendix A.

Appendix A

This appendix is devoted to the construction of a $*$-dynamical system $(\mathfrak{A}, \varphi, \tau)$ with the property that if $\|\tau(A) - A\|_\varphi = 0$, then $\|A - \alpha\|_\varphi = 0$ for some $\alpha \in \mathbb{C}$, but for which the fixed points of the operator U defined in Proposition 2.4 in terms of some cyclic representation, form a vector subspace of $\mathfrak{H}$ with dimension greater than one. This will prove the necessity of a sequence, rather than a single element, in Definition 2.3, in order for Proposition 2.4 to hold.

First some general considerations. Consider a dense vector subspace $\mathfrak{G}$ of a Hilbert space $\mathfrak{H}$, and let $\mathfrak{L}(\mathfrak{H})$ be the bounded linear operators $\mathfrak{H} \to \mathfrak{H}$. Set

$$\mathfrak{A} := \left\{ A|_{\mathfrak{G}} \colon\ A \in \mathfrak{L}(\mathfrak{H}),\ A\mathfrak{G} \subset \mathfrak{G} \text{ and } A^*\mathfrak{G} \subset \mathfrak{G} \right\}$$

where $A|_{\mathfrak{G}}$ denotes the restriction of A to $\mathfrak{G}$. For any $A \in \mathfrak{A}$, denote by $\overline{A}$ the (unique) bounded linear extension of A to $\mathfrak{H}$. Now define

$$A^* := \overline{A}^*|_{\mathfrak{G}}$$

for all $A \in \mathfrak{A}$, then it is easily verified that $\mathfrak{A}$ becomes a unital $*$-algebra. (For example, for $A, B \in \mathfrak{A}$ it is clear that AB is a bounded linear operator $\mathfrak{G} \to \mathfrak{G}$ which therefore has the extension $\overline{A}.\overline{B} \in \mathfrak{L}(\mathfrak{H})$ for which $\overline{A}.\overline{B}\mathfrak{G} \subset \mathfrak{G}$ and $(\overline{A}.\overline{B})^*\mathfrak{G} = \overline{B}^*\overline{A}^*\mathfrak{G} \subset \mathfrak{G}$ by the definition of $\mathfrak{A}$. Hence $AB \in \mathfrak{A}$, and $(AB)^* = (\overline{A}.\overline{B})^*|_{\mathfrak{G}} = (\overline{B}^*\overline{A}^*)|_{\mathfrak{G}} = \overline{B}^*(\overline{A}^*|_{\mathfrak{G}}) = \overline{B}^*A^* = B^*A^*$. Similarly for the other defining properties of a unital $*$-algebra.) Note that for $A \in \mathfrak{A}$ and $x, y \in \mathfrak{G}$ we have

$$\langle x, Ay\rangle = \langle x, \overline{A}y\rangle = \langle \overline{A}^*x, y\rangle = \langle A^*x, y\rangle.$$

For a given norm one $\Omega \in \mathfrak{G}$ we define a state φ on $\mathfrak{A}$ by

$$\varphi(A) = \langle \Omega, A\Omega\rangle.$$

Next we construct a cyclic representation of $(\mathfrak{A}, \varphi)$. Let

$$\pi : \mathfrak{A} \to L(\mathfrak{G}): \quad A \mapsto A;$$

then clearly π is linear with $\pi(1) = 1$ and $\pi(AB) = \pi(A)\pi(B)$. Note that for any $x, y \in \mathfrak{G}$ we have $(x \otimes y)^* = y \otimes x$, hence $(x \otimes y)\mathfrak{G} \subset \mathfrak{G}$ and $(x \otimes y)^*\mathfrak{G} \subset \mathfrak{G}$, so $(x \otimes y)|_{\mathfrak{G}} \in \mathfrak{A}$. Now, $\pi((x \otimes \Omega)|_{\mathfrak{G}})\Omega = x\langle \Omega, \Omega\rangle = x$, hence $\pi(\mathfrak{A})\Omega = \mathfrak{G}$. Furthermore, $\langle \pi(A)\Omega, \pi(B)\Omega\rangle = \langle A\Omega, B\Omega\rangle = \langle \Omega, A^*B\Omega\rangle = \varphi(A^*B)$. Thus $(\mathfrak{G}, \pi, \Omega)$ is a cyclic representation of $(\mathfrak{A}, \varphi)$.

Suppose we have a unitary operator $U:\mathfrak{H}\to\mathfrak{H}$ such that $U\mathfrak{G}=\mathfrak{G}$ and $U\Omega=\Omega$. Then $U^*\mathfrak{G}=U^{-1}\mathfrak{G}=\mathfrak{G}$, so $V:=U|_{\mathfrak{G}}\in\mathfrak{A}$, and $V^*=U^*|_{\mathfrak{G}}$. It follows that $VAV^*\in\mathfrak{A}$ for all $A\in\mathfrak{A}$, hence we can define a linear function $\tau:\mathfrak{A}\to\mathfrak{A}$ by

$$\tau(A)=VAV^*.$$

Clearly $V^*V=1=VV^*$, so $\tau(1)=1$ and $\varphi(\tau(A)^*\tau(A))=\varphi(VA^*AV^*)=\langle U^*\Omega, A^*AU^*\Omega\rangle=\varphi(A^*A)$, since $U^*\Omega=U^{-1}\Omega=\Omega$. Therefore $(\mathfrak{A},\varphi,\tau)$ is a $*$-dynamical system. Note that $U|_{\mathfrak{G}}$ satisfies (3), namely $U\pi(A)\Omega=UA\Omega=UAU^*\Omega=\tau(A)\Omega=\pi(\tau(A))\Omega$, hence U is the operator which appears in Proposition 2.4.

Assume $\{x\in\mathfrak{G}\colon Ux=x\}=\mathbb{C}\Omega$. If $\|\tau(A)-A\|_\varphi=0$, it then follows for $x=\iota(A)$, with ι given by (2), that $\|Ux-x\|=\|\iota(\tau(A)-A)\|=\|\tau(A)-A\|_\varphi=0$, so $x=\alpha\Omega$ for some $\alpha\in\mathbb{C}$. Therefore $\|A-\alpha\|_\varphi=\|\iota(A-\alpha)\|=\|x-\alpha\Omega\|=0$.

In other words, assuming that the fixed points of U in $\mathfrak{G}$ form the one-dimensional subspace $\mathbb{C}\Omega$, it follows that $\|\tau(A)-A\|_\varphi=0$ implies that $\|A-\alpha\|_\varphi=0$ for some $\alpha\in\mathbb{C}$.

It remains to construct an example of a U with all the properties mentioned above, whose fixed point space in $\mathfrak{H}$ has dimension greater than one. The following example was constructed by Zsidó:

Let $\mathfrak{H}$ be a separable Hilbert space with an orthonormal basis of the form

$$\{\Omega, y\}\cup\{u_k\colon k\in\mathbb{Z}\}$$

(that is to say, this is a total orthonormal set in $\mathfrak{H}$) and define the linear operator $U:\mathfrak{H}\to\mathfrak{H}$ by

$$U\Omega=\Omega,$$
$$Uy=y,$$
$$Uu_k=u_{k+1},\quad k\in\mathbb{Z}.$$

Since U is a surjective isometry, it is unitary. Let $\mathfrak{G}$ be the linear span of

$$\{\Omega\}\cup\{y+u_k\colon k\in\mathbb{Z}\}.$$

Then $U\mathfrak{G}=\mathfrak{G}$. Furthermore, $\mathfrak{G}$ is dense in $\mathfrak{H}$. Indeed,

$$\left\|y-\frac{1}{n}\sum_{k=1}^{n}(y+u_k)\right\|=\frac{1}{n}\left\|\sum_{k=1}^{n}u_k\right\|=\frac{1}{\sqrt{n}}\to 0$$

implies that $y\in\overline{\mathfrak{G}}$, the closure of $\mathfrak{G}$, hence also

$$u_k=(y+u_k)-y\in\overline{\mathfrak{G}}$$

for $k\in\mathbb{Z}$.

Next we show that

$$\{x\in\mathfrak{G}\colon Ux=x\}=\mathbb{C}\Omega.$$

If $\alpha\Omega+\sum_{k=-n}^{n}\beta_k(y+u_k)\in\mathfrak{G}$ is left fixed by U, then

$$\alpha\Omega+\sum_{k=-n}^{n}\beta_k y+\sum_{k=-n}^{n}\beta_k u_{k+1}=\alpha\Omega+\sum_{k=-n}^{n}\beta_k y+\sum_{k=-n}^{n}\beta_k u_k$$

and it follows that $\beta_{-n} = 0$, and that $\beta_{k+1} = \beta_k$ for $k = -n, \ldots, n-1$. Thus

$$\alpha\Omega + \sum_{k=-n}^{n} \beta_k(y + u_k) = \alpha\Omega.$$

On the other hand,

$$\{x \in \mathfrak{H}\colon\ Ux = x\}$$

clearly contains the two-dimensional vector space spanned by Ω and y.

References

[1] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics 1, 2nd ed., Springer, New York, 1987.
[2] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, Vol. 1, American Mathematical Society, Providence, 1997.
[3] C.P. Niculescu, A. Ströh, L. Zsidó, Noncommutative extensions of classical and multiple recurrence theorems, J. Operator Theory, in press.
[4] K. Petersen, Ergodic Theory, Cambridge University Press, Cambridge, 1983.
[5] P. Walters, An Introduction to Ergodic Theory, Springer, New York, 1982.