The perturbed Sparre Andersen model with a threshold dividend strategy

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Abstract

In this paper, we consider a Sparre Andersen model perturbed by diffusion with generalized Erlang(\(n\))-distributed inter-claim times and a threshold dividend strategy. Integro-differential equations with certain boundary conditions for the moment-generation function and the \(m\)th moment of the present value of all dividends until ruin are derived. We also derive integro-differential equations with boundary conditions for the Gerber–Shiu functions. The special case where the inter-claim times are Erlang(2) distributed and the claim size distribution is exponential is considered in some details.

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1. Introduction

Dividend strategies for insurance risk model were first proposed by De Finetti [7] to reflect more realistically the surplus cash flows in an insurance portfolio, and he found that the optimal strategy must be a barrier strategy. From then on, barrier strategies have been studied in a number of papers and books, including [2–4,6,8,11,12,17,23]. Some recent papers on dividend barrier strategies are [14–16,18,20,28], as well as the references therein.

The compound Poisson risk model perturbed by a diffusion was first introduced by Gerber [10] and has been further studied by many authors during the last few years, see e.g. [5,9,13,24,25,27] and the references therein.

For a class of compound Poisson process perturbed by diffusion and a threshold dividend strategy, the expected discounted dividend payments prior to ruin and the Gerber–Shiu expected discounted penalty function have been studied by Wan [26]. The analysis of the discounted penalty function in [21] was recently generalized to a Sparre Andersen risk model with generalized Erlang(\(n\))-distributed inter-claim times by [19], and a corresponding generalization of the results on the distribution of dividend payments has been studied by Albrecher et al. [1].

In the recent paper [22], the authors study the expectation of aggregate dividends until ruin for a Sparre Andersen risk process perturbed by diffusion under a threshold strategy, in which claim waiting times have a common generalized Erlang(\(n\)) distribution. They obtain some integro-differential equations satisfied by the expected discounted dividends,

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and further its renewal equations. Furthermore, applying these results to the Erlang(2) risk model perturbed by diffusion, where claims have a common exponential distributions, giving some explicit expressions and numerical analysis.

The purpose of this paper is to present some results on the distribution of dividend payments until ruin and the Gerber–Shiu function under a Sparre Andersen model perturbed by diffusion with generalized Erlang($n$)-distributed inter-claim times and a threshold dividend strategy. In Section 2 we describe the model, then in Sections 3 and 4, integro-differential equations for the moment-generation function and the $m$th moment of the sum of the discounted dividend payments until ruin are derived. We then derive integro-differential equations for the Gerber–Shiu function in Section 5. Finally, in Section 6, we illustrate the results for an Erlang(2)-model with exponential-distributed claim sizes.

2. The model

In the Sparre Andersen model perturbed by a diffusion, the surplus process $U(t)$ of an insurance portfolio is given by

$$U(t) = u + c_1 t - \sum_{i=1}^{N(t)} X_i + \sigma B(t), \quad t \geq 0,$$

where $u \geq 0$ is the initial surplus, $c_1 > 0$ the constant premium income rate. The $X_i$'s is a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $F$ and density function $f$, representing claim sizes. The ordinary renewal process $\{N(t), t \geq 0\}$ denotes the number of claims up to time $t$ and is defined as $N(t) = \sup\{k : T_1 + \cdots + T_k \leq t\}$, where the i.i.d. inter-claim times $\{T_i\}_{i=1}^\infty$ have a common generalized Erlang($n$) distribution. $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\sigma > 0$ is a constant, representing the diffusion volatility parameter. In addition, $\{X_i, i = 1, 2, \ldots\}$, $\{N(t), t \geq 0\}$, and $\{B(t), t \geq 0\}$ are independent. Denote the aggregate claims process by $S(t)$, i.e. $S(t) = \sum_{i=1}^{N(t)} X_i$. The net profit condition is given by $c_1 > E[X_i]/E[T_i]$.

The insurance company will pay dividends to its shareholders. We shall assume that the company pays dividends according to the following strategy governed by parameters $b > 0$ and $\sigma > 0$. Whenever the modified surplus is below the level $b$, no dividends are paid. However, when the modified surplus is above $b$, dividends are paid continuously at a constant rate $\sigma (0 < \sigma \leq c_1)$. Thus, the threshold $b$ plays the role of a breakpoint or a regime-switching boundary. Let denote $c_2 = c_1 - \sigma$ and $\{U_b(t), t \geq 0\}$ to be the modified surplus process under the above assumption, then it can be expressed as

$$U_b(t) = \begin{cases} u + c_1 t + \sigma B(t) - S(t), & U_b(t) < b, \\ u + c_2 t + \sigma B(t) - S(t), & U_b(t) \geq b. \end{cases}$$

Let denote $D(t)$ to be the cumulative amount of dividends paid out up to time $t$ and $\delta > 0$ the force of interest, then

$$D_{u,b} = \int_0^{T_b} e^{-\delta t} dD(t)$$

is the present value of all dividends until $T_b$, where $T_b$ denoted by $T_b = \inf\{t : U_b(t) \leq 0\}$ is the time of ruin. An alternative expression for $D_{u,b}$ is

$$D_{u,b} = \int_0^{T_b} e^{-\delta t} I(U_b(t) > b) \, dt,$$

with $I(\cdot)$ denoting the indicator function.

In the sequel we will be interested in the moment generating function

$$M(u, y; b) = E[e^{yD_{u,b}}],$$

(for those values of $y$ where it exits) and the $m$th moment function

$$V_m(u, b) = E[D_{u,b}^m], \quad (m \in \mathbb{N}),$$
and the expected discounted penalty (Gerber–Shiu) function
\[ \phi_b(u) = \mathbb{E}[e^{-\delta T_b} \omega(U_b(T_b^-), |U_b(T_b)|)I(T_b < \infty)|U_b(0) = u], \]
where \( \omega(x, y) \) is a nonnegative function of \( x > 0, y > 0 \).

We will always assume that \( M(u, y; b), V_m(u, b), \) and \( \phi_b(u) \) are sufficiently smooth functions in \( u \) and \( y \), respectively.

### 3. Integro-differential equations for \( M(u, y; b) \)

In this section, we will give the integro-differential equations and boundary conditions satisfied by the moment generating function \( M(u, y; b) \).

Let \( \partial / \partial y \) denote the differentiation operator with respect to \( y \) and correspondingly \( \partial^2 / \partial u^2 \) and \( \partial / \partial u \) the differentiation operators with respect to \( u \). Moreover, define \( \prod_{j=2}^{n} = 1 \).

Clearly, the moment generating function \( M(u, y; b) \) behaves differently, depending on whether its initial surplus \( u \) is below or above the barrier level \( b \). Hence, we write
\[
M(u, y; b) = \begin{cases} 
M_1(u, y; b)(u), & 0 \leq u < b, \\
M_2(u, y; b)(u), & b \leq u < \infty. 
\end{cases}
\]

Let us decompose every inter-occurrence time with generalized Erlang(\( n \)) distribution into the independent sum of \( n \) exponential random variables with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_n \), each causing a sub-claim of size 0 and at the time of the \( n \)th sub-claim an actual claim with distribution function \( F \) occurs. This can be realized by considering \( n \) states of the risk process. Starting at time 0 in state 1, every sub-claim causes a transition to the next state and at the time of the occurrence of the \( n \)th sub-claim, an actual claim with distribution function \( F \) occurs and the risk process jumps into state 1 again. This will allow to use Markovian arguments due to the lack-of-memory property of the exponential distribution.

The following theorem provides integro-differential equations for the function \( M(u, y; b) \).

**Theorem 3.1.** For \( 0 < u < b \), the moment-generating function \( M(u, y; b) \) is the solution of the partial integro-differential equation
\[
\left( \prod_{j=1}^{n} \frac{\partial y \partial - c_1 \partial / \partial u - \frac{\sigma^2 \partial^2}{2 \partial u^2} + \lambda_j}{\lambda_j} \right) M_1(u, y; b) - \int_{0}^{u} M_1(u - v, y; b) dF(v) - F(u) = 0, \tag{3}
\]
and for \( b < u < \infty \), \( M(u, y; b) \) satisfies the partial integro-differential equation
\[
\left( \prod_{j=1}^{n} \frac{\partial y \partial - c_2 \partial / \partial u - \frac{\sigma^2 \partial^2}{2 \partial u^2} + \lambda_j - \alpha y}{\lambda_j} \right) M_2(u, y; b) - \int_{0}^{u-b} M_2(u - v, y; b) dF(v) - \int_{u-b}^{u} M_1(u - v, y; b) dF(v) = 0, \tag{4}
\]
with boundary conditions
\[
\left( \frac{\partial y \partial - c_1 \partial / \partial u - \frac{\sigma^2 \partial^2}{2 \partial u^2}}{\lambda_j} \right)^{j-1} M_1(u, y; b)|_{u=0} = \begin{cases} 
1, & j = 1, \\
0, & j = 2, \ldots, n. 
\end{cases} \tag{5}
\]
\[
\left( \frac{\partial y \partial - c_1 \partial / \partial u - \frac{\sigma^2 \partial^2}{2 \partial u^2}}{\lambda_j} \right)^{j-1} M_1(u, y; b)|_{u=b} = \left( \frac{\partial y \partial - c_2 \partial / \partial u - \frac{\sigma^2 \partial^2}{2 \partial u^2} - \alpha y}{\lambda_j} \right)^{j-1} M_2(u, y; b)|_{u=b}, \quad j = 1, \ldots, n. \tag{6}
\]
and

\[
\left( \delta y \frac{\partial}{\partial y} - c_1 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} \right) \left. \left( j-1 \frac{\partial M_j(u, y; b)}{\partial u} \right) \right|_{u=b-} = \left( \delta y \frac{\partial}{\partial y} - c_2 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} - 2y \right) \left. \left( j-1 \frac{\partial M_j(u, y; b)}{\partial u} \right) \right|_{u=b+}, \quad j = 1, \ldots, n. \tag{7}
\]

**Proof.** Let \( M_j^{(j)}(u, y; b) \) denote the moment-generating function of \( D_{u,b} \) when \( 0 \leq u < b \), if the risk process is in state \( j (j = 1, \ldots, n) \). Eventually, we are interested in \( M_1(u, y; b) := M_1^{(1)}(u, y; b) \).

We obtain for \( 0 \leq u < b \) and \( j = 1, \ldots, n-1 \)

\[
M_j^{(j)}(u, y; b) = (1 - \lambda_j dt) E[M_j^{(j)}(u + c_1 dt + \sigma B(dt), ye^{-\delta dr}; b)] \\
+ \lambda_j dt E[M_j^{(j+1)}(u + c_1 dt + \sigma B(dt), ye^{-\delta dr}; b)] + o(dt).
\]

By Taylor expansion we have

\[
E[M_j^{(j)}(u + c_1 dt + \sigma B(dt), ye^{-\delta dr}; b)] \\
= M_j^{(j)}(u, y; b) + c_1 dt \frac{\partial M_j^{(j)}(u, y; b)}{\partial u} + y(e^{-\delta dr} - 1) \frac{\partial M_j^{(j)}(u, y; b)}{\partial y} \\
+ \frac{\sigma^2}{2} dt \frac{\partial^2 M_j^{(j)}(u, y; b)}{\partial u^2} + o(dt).
\]

Subtracting \( M_j^{(j)}(u, y; b) \) from each side of the above equation, dividing by \( dt \) and then letting \( dt \to 0 \), we achieve

\[
-\delta y \frac{\partial M_j^{(j)}(u, y; b)}{\partial y} + c_1 \frac{\partial M_j^{(j)}(u, y; b)}{\partial u} + \frac{\sigma^2}{2} \frac{\partial^2 M_j^{(j)}(u, y; b)}{\partial u^2} - \lambda_j M_j^{(j)}(u, y; b) + \lambda_j M_j^{(j+1)}(u, y; b) = 0,
\]

which can be written as

\[
M_j^{(j+1)}(u, y; b) = \frac{\delta y \frac{\partial}{\partial y} - c_1 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} + \lambda_j}{\lambda_j} M_j^{(j)}(u, y; b). \tag{8}
\]

Similarly, for \( j = n \) we have

\[
M_n^{(n)}(u, y; b) = (1 - \lambda_n dt) E[M_n^{(n)}(u + c_1 dt + \sigma B(dt), ye^{-\delta dr}; b)] \\
+ \lambda_n dt E[M_n^{(1)}(u + c_1 dt + \sigma B(dt) - X, ye^{-\delta dr}; b)] + o(dt) \\
= (1 - \lambda_n dt) E[M_n^{(n)}(u + c_1 dt + \sigma B(dt), ye^{-\delta dr}; b)] \\
+ \lambda_n dt E \left[ \int_0^{u+c_1 dt+\sigma B(dt)} M_n^{(1)}(u + c_1 dt + \sigma B(dt) - v, ye^{-\delta dr}; b) \, dF(v) \right] + o(dt) \\
+ \int_0^{\infty} dF(v) + o(dt),
\]
which leads to
\[ -\delta y \frac{\partial M_1^{(n)}(u, y; b)}{\partial y} + c_1 \frac{\partial M_1^{(n)}(u, y; b)}{\partial u} + \frac{\sigma^2}{2} \frac{\partial^2 M_1^{(n)}(u, y; b)}{\partial u^2} - \lambda M_1^{(n)}(u, y; b) \]
\[ + \lambda_n \int_0^u M_1^{(1)}(u - v, y; b) \, dF(v) + \lambda_n \overline{F}(u) = 0. \]

From (8) it follows that
\[ M_1^{(n)}(u, y; b) = \left( \prod_{j=1}^{n-1} \frac{\delta y \frac{\partial}{\partial y} - c_1 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} + \lambda_j}{\lambda_j} \right) M_1^{(1)}(u, y; b) \]
which together with (9) yields (3).

Similarly, let \( M_2^{(j)}(u, y; b) \) denote the moment-generating function of \( D_{u, b} \) when \( b \leq u < \infty \), if the risk process is in state \( j \) \( (j = 1, \ldots, n) \).
Then, we obtain for \( b \leq u < \infty \) and \( j = 1, \ldots, n-1 \)
\[ M_2^{(j)}(u, y; b) = \exp\{y z \, d\tau\} \{ (1 - \lambda_j \, d\tau) E[M_2^{(j)}(u + c_2 \, d\tau + \sigma B(d\tau), y \, e^{-\delta d\tau}; b) ] \]
\[ + \lambda_j \, d\tau E[M_2^{(j+1)}(u + c_2 \, d\tau + \sigma B(d\tau), y \, e^{-\delta d\tau}; b) ] \} + o(d\tau). \]

By Taylor expansion and some careful calculations, letting \( d\tau \to 0 \), we have
\[ x y M_2^{(j)}(u, y; b) - \delta y \frac{\partial M_2^{(j)}(u, y; b)}{\partial y} + c_2 \frac{\partial M_2^{(j)}(u, y; b)}{\partial u} + \frac{\sigma^2}{2} \frac{\partial^2 M_2^{(j)}(u, y; b)}{\partial u^2} \]
\[ - \lambda_j M_2^{(j)}(u, y; b) + \lambda_j M_2^{(j+1)}(u, y; b) = 0, \]
which can be written as
\[ M_2^{(j+1)}(u, y; b) = \frac{\delta y \frac{\partial}{\partial y} - c_2 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} + \lambda_j - x y}{\lambda_j} M_2^{(j)}(u, y; b). \]

For \( j = n \) we can get after some careful calculations that
\[ x y M_2^{(n)}(u, y; b) - \delta y \frac{\partial M_2^{(n)}(u, y; b)}{\partial y} + c_2 \frac{\partial M_2^{(n)}(u, y; b)}{\partial u} + \frac{\sigma^2}{2} \frac{\partial^2 M_2^{(n)}(u, y; b)}{\partial u^2} \]
\[ - \lambda_n M_2^{(n)}(u, y; b) + \lambda_n \int_{u-b}^{u} M_2^{(1)}(u - v, y; b) \, dF(v) \]
\[ + \lambda_n \int_{u-b}^{u} M_1^{(1)}(u - v, y; b) \, dF(v) + \lambda_n \overline{F}(u) = 0. \]

From (10) it follows that
\[ M_2^{(n)}(u, y; b) = \left( \prod_{j=1}^{n-1} \frac{\delta y \frac{\partial}{\partial y} - c_1 \frac{\partial}{\partial u} - \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2} + \lambda_j - x y}{\lambda_j} \right) M_2^{(1)}(u, y; b) \]
which together with (11) yields (4).
Condition (5) is obvious: if \( u = 0 \), ruin is immediate and no dividends are paid, then \( M^{(j)}_1(0, y; b) = 1, j = 1, \ldots, n \), which together with (8) yields (5).

According to [26] we have

\[
M^{(j)}(u, y; b)|_{u = b-} = M^{(j)}_2(u, y; b)|_{u = b+}, \quad j = 1, \ldots, n,
\]

and

\[
\frac{\partial M^{(j)}(u, y; b)}{\partial u} \bigg|_{u = b-} = \frac{\partial M^{(j)}_2(u, y; b)}{\partial u} \bigg|_{u = b+}, \quad j = 1, \ldots, n.
\]

For \( j = 1 \), (12) is equivalent to (6), and (13) is equivalent to (7). Now it just remains to express Eqs. (6) and (7) for \( j = 2, \ldots, n \) in terms of \( M^{(1)}_1(u, y; b) = M_1(u, y; b) \) and \( M^{(1)}_2(u, y; b) = M_2(u, y; b) \), which is done by virtue of (8) and (10).

**Remark 3.1.** In particular, for \( n = 1 \) and \( \lambda_1 := \lambda \), we get the classical compound Poisson risk model and indeed (3) and (4) simplify in this case to

\[
\lambda \int_0^u M_1(u - v, y; b) \, dF(v) - \lambda \bar{F}(u) = 0,
\]

for \( 0 < u < b \), and for \( u > b \)

\[
\lambda \int_0^u M_2(u - v, y; b) \, dF(v) - \lambda \bar{F}(u) = 0,
\]

and (5)–(7) simplify to

\[
M(0, y; b) = 1, \quad M_1(u, y; b)|_{u = b-} = M_2(u, y; b)|_{u = b+},
\]

\[
\frac{\partial M_1(u, y; b)}{\partial u} \bigg|_{u = b-} = \frac{\partial M_2(u, y; b)}{\partial u} \bigg|_{u = b+},
\]

which are formulas (6.2)–(6.5) and other formulas in [26].

### 4. Moments of \( D_{u,b} \)

In this section, we will give the integro-differential equations and boundary conditions satisfied by the \( m \)th moment function \( V_m(u, b) \).

Let \( d^2 / du^2 \) and \( d / du \) denote the differentiation operators with respect to \( u \). We write

\[
V_m(u, b) = \begin{cases} 
V_{m,1}(u, b), & 0 \leq u < b, \\
V_{m,2}(u, b), & b \leq u < \infty.
\end{cases}
\]

Then we have the following theorem.
Theorem 4.1. For $0 < u < b$, $V_m(u, b)(m = 1, 2, \ldots)$ is the solution of the following ordinary integro-differential equation

$$
\left( \prod_{j=1}^{n} \frac{\delta \Pi_1 - c_1 \frac{d}{du} - \frac{\sigma^2}{2} \frac{d^2}{du^2} + \lambda_j}{\lambda_j} \right) V_{m,1}(u, b) - \int_{0}^{u} V_{m,1}(u - v, b) dF(v) = 0,
$$

and for $0 < u < \infty$, $V_m(u, b)$ satisfies the partial integro-differential equation

$$
\left( \prod_{j=1}^{n} \frac{\delta \Pi_1 - c_2 \frac{d}{du} - \frac{\sigma^2}{2} \frac{d^2}{du^2} + \lambda_j - \alpha \Pi_2}{\lambda_j} \right) V_{m,2}(u, b) - \int_{0}^{u-b} V_{m,2}(u - v, b) dF(v) = 0,
$$

with the operators $\Pi_1 V_m := m V_m$, $\Pi_2 V_m := m V_{m-1}$. Moreover, $V_0 = 1$, $V_{-i} = 0 (i \in \mathbb{N})$. Note that the product $\Pi_2 \Pi_1$ of operators is not commutative and is given by

$$(\Pi_1 \Pi_2) V_m = \Pi_2 (\Pi_1 V_m) = m^2 V_{m-1}, \quad (\Pi_2 \Pi_1) V_m = \Pi_1 (\Pi_2 V_m) = m(m-1) V_{m-1}.$$

The boundary conditions are

$$
\left( \frac{d}{du} + \frac{\sigma^2}{2} \frac{d^2}{du^2} \right)^{j-1} V_{m,1}(u, b)|_{u=0} = 0, \quad j = 1, \ldots, n
$$

and

$$
\left( \frac{d}{du} + \frac{\sigma^2}{2} \frac{d^2}{du^2} \right)^{j-1} V_{m,1}(u, b)|_{u=b-} = \left( \frac{d}{du} + \frac{\sigma^2}{2} \frac{d^2}{du^2} - \alpha \Pi_2 \right)^{j-1} V_{m,2}(u, b)|_{u=b+}, \quad j = 1, \ldots, n.
$$

and

$$
\left( \frac{d}{du} + \frac{\sigma^2}{2} \frac{d^2}{du^2} \right)^{j-1} \frac{dV_{m,1}(u, b)}{du}|_{u=b-} = \left( \frac{d}{du} + \frac{\sigma^2}{2} \frac{d^2}{du^2} - \alpha \Pi_2 \right)^{j-1} \frac{dV_{m,2}(u, b)}{du}|_{u=b+}, \quad j = 1, \ldots, n.
$$

Proof. Recall that $V_m(u, b) = E[D_{u,b}^m]$ and $M(u, y; b) = E[e^{yD_{u,b}}]$. Using the representation

$$
M(u, y; b) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} V_m(u, b),
$$

and equating the coefficients of $y^m (m \in \mathbb{N})$ in (3)–(7) lead to the ordinary integro-differential equations (14)–(18).

Remark 4.1. When $m = 1$, with some careful calculations we can get the integro-differential equations satisfied by first moment of the aggregate dividends. For $0 < u < b$,

$$
\left( \prod_{j=1}^{n} \frac{\delta + \lambda_j - c_1 \frac{d}{du} - \frac{\sigma^2}{2} \frac{d^2}{du^2}}{\lambda_j} \right) V_1(u, b) - \int_{0}^{u} V_1(u - v, b) dF(v) = 0,
$$
and for $b < u < \infty$,

$$
\left( \prod_{j=1}^{n} \frac{\delta + \lambda_j - c_1 \frac{d}{du} - \frac{\sigma^2}{2} \frac{d^2}{du^2}}{\lambda_j} \right) V_1(u, b) - \int_{0}^{u} V_1(u - v, b) \, dF(v)
$$

$$
- \left\{ \sum_{k=1}^{n-1} \left[ \prod_{j=k}^{n-1} \left( 1 + \frac{\delta}{\lambda_{j+1}} \right) \right] \frac{\alpha}{\lambda_k} + \frac{\alpha}{\lambda_n} \right\} = 0.
$$

Two such equations were obtained by Meng et al. in [22].

**Remark 4.2.** For $n = 1$ and $\lambda_1 := \lambda$, (12)–(18) simplify to

$$(\delta m + \lambda) V_{m,1}(u, b) - c_1 V'_{m,1}(u, b) - \frac{\sigma^2}{2} V''_{m,1}(u, b) - \lambda \int_{0}^{u} V_{m,1}(u - v, b) \, dF(v) = 0,$$

$$(\delta m + \lambda) V_{m,2}(u, b) - c_2 V'_{m,2}(u, b) - \frac{\sigma^2}{2} V''_{m,2}(u, b) - \alpha m V_{m-1,2}(u, b) - \lambda \int_{0}^{u} V_{m,2}(u - v, b) \, dF(v) = 0,$$

$$V_{m,1}(0, b) = 0, \quad V_{m,1}(u, b)|_{u=b} = V_{m,2}(u, b)|_{u=b}, \quad V'_{m,1}(u, b)|_{u=b} = V''_{m,2}(u, b)|_{u=b},$$

which coincide with (6.8)–(6.10) and other equations in [26]. If in addition we assume that $m = 1$, we found the above equations coincide with equations (2.1)–(2.4) and (2.6) in [26].

5. The Gerber–Shiu functions

In the following we will discuss the famous Gerber–Shiu expected discounted penalty function. For $\delta > 0$ we define

$$\phi_{b,d}(u) = E\{e^{-\delta T_b} I(T_b < \infty, U_b(T_b) = 0)|U_b(0) = u\},$$

with $\phi_{b,d}(0) = 1$, to be the Laplace transform of the ruin time $T_b$ with respect to $\delta$ when the ruin is due to the oscillations, and define

$$\phi_{b,s}(u) = E\{e^{-\delta T_b} I(U_b(T_b-), |U_b(T_b)|)|T_b < \infty, U_b(T_b) < 0)|U_b(0) = u\},$$

with $\phi_{b,s}(0) = 0$, to be the expected discounted penalty function if the ruin is caused by a claim. Then

$$\phi_b(u) = \phi_{b,d}(u) + \phi_{b,s}(u),$$

is the expected discounted penalty function.

We also write

$$\phi_{b,d}(u) = \begin{cases} \phi_{b,d,1}(u), & 0 \leq u < b, \\ \phi_{b,d,2}(u), & b \leq u < \infty, \end{cases}, \quad \phi_{b,s}(u) = \begin{cases} \phi_{b,s,1}(u), & 0 \leq u < b, \\ \phi_{b,s,2}(u), & b \leq u < \infty. \end{cases}$$

Then, we have

$$\phi_b(u) = \begin{cases} \phi_{b,1}(u) = \phi_{b,d,1}(u) + \phi_{b,s,1}(u), & 0 \leq u < b, \\ \phi_{b,2}(u) = \phi_{b,d,2}(u) + \phi_{b,s,2}(u), & b \leq u < \infty. \end{cases}$$

By similar derivation to (3)–(7), we get the following theorems:

**Theorem 5.1.** For $0 < u < b$, $\phi_{b,d}(u)$ satisfies integro-differential equation

$$
\left( \prod_{j=1}^{n} \frac{\sigma^2}{2} \frac{d^2}{du^2} - c_1 \frac{d}{du} + \frac{\lambda_j + \delta}{\lambda_j} \right) \phi_{b,d,1}(u) - \int_{0}^{u} \phi_{b,d,1}(u - v) \, dF(v) = 0,
$$

(20)
and for \( b < u < \infty \), \( \phi_{b,d}(u) \) satisfies integro-differential equation

\[
\left( \prod_{j=1}^{n} \left( -\frac{\sigma^2}{2} \frac{d^2}{du^2} - c_2 \frac{d}{du} + \lambda_j + \delta \right) \right) \phi_{b,d,2}(u) - \int_{u-b}^{u} \phi_{b,d,2}(u - v) \, dF(v) - \int_{u-b}^{u} \phi_{b,d,1}(u - v) \, dF(v) = 0,
\]

with boundary conditions

\[
\left( -\frac{\sigma^2}{2} \frac{d^2}{du^2} - c_2 \frac{d}{du} + \delta \right)^{j-1} \phi_{b,d,1}(u)|_{u=0} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, \ldots, n \\ \end{cases}
\]

and for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{d}{du} \right)^{j-1} \phi_{b,d,1}(u)|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{d}{du} \right)^{j-1} \phi_{b,d,2}(u)|_{u=b+},
\]

moreover, for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{d}{du} \right)^{j-1} \frac{d\phi_{b,d,1}(u)}{du} \bigg|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{d}{du} \right)^{j-1} \frac{d\phi_{b,d,2}(u)}{du} \bigg|_{u=b+}.
\]

**Theorem 5.2.** For \( 0 < u < b \), \( \phi_{b,s}(u) \) satisfies the following integro-differential equation:

\[
\left( \prod_{j=1}^{n} \left( -\frac{\sigma^2}{2} \frac{d^2}{du^2} - c_1 \frac{d}{du} + \lambda_j + \delta \right) \right) \phi_{b,s,1}(u) - \int_{0}^{u} \phi_{b,s,1}(u - v) \, dF(v) - \int_{u}^{\infty} \omega(u, v - u) \, dF(v) = 0,
\]

and for \( b < u < \infty \), \( \phi_{b,s}(u) \) satisfies the following integro-differential equation

\[
\left( \prod_{j=1}^{n} \left( -\frac{\sigma^2}{2} \frac{d^2}{du^2} - c_2 \frac{d}{du} + \lambda_j + \delta \right) \right) \phi_{b,s,2}(u) - \int_{0}^{u-b} \phi_{b,s,2}(u - v) \, dF(v) - \int_{u-b}^{\infty} \omega(u, v - u) \, dF(v) = 0,
\]

with boundary conditions

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{d}{du} \right)^{j-1} \phi_{b,s,1}(u)|_{u=0} = 0, \quad j = 1, \ldots, n,
\]

and for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{d}{du} \right)^{j-1} \phi_{b,s,1}(u)|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{d}{du} \right)^{j-1} \phi_{b,s,2}(u)|_{u=b+},
\]

moreover, for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{d}{du} \right)^{j-1} \frac{d\phi_{b,s,1}(u)}{du} \bigg|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{d}{du} \right)^{j-1} \frac{d\phi_{b,s,2}(u)}{du} \bigg|_{u=b+}.
\]
Combining Theorem 5.1 with Theorem 5.2 and note that \( \phi_b(u) = \phi_{b,x}(u) + \phi_{b,d}(u) \), we get \( \phi_b(u) \) satisfies the following integro-differential equations:

**Theorem 5.3.** For \( 0 < u < b \),

\[
\left( \sum_{j=1}^{n} \frac{\sigma^2}{2} \frac{d^2}{du^2} - \frac{1}{\lambda_j} \frac{d}{du} + \lambda_j + \delta \right) \phi_{b,1}(u) - \int_0^u \phi_{b,1}(u-v) \, dF(v) - \int_u^\infty \omega(u, v-u) \, dF(v) = 0, \tag{30}
\]

and for \( b < u < \infty \),

\[
\left( \sum_{j=1}^{n} \frac{\sigma^2}{2} \frac{d^2}{du^2} - \frac{1}{\lambda_j} \frac{d}{du} + \lambda_j + \delta \right) \phi_{b,2}(u) - \int_0^{u-b} \phi_{b,2}(u-v) \, dF(v) - \int_{u-b}^u \phi_{b,1}(u-v) \, dF(v) - \int_u^\infty \omega(u, v-u) \, dF(v) = 0, \tag{31}
\]

with boundary conditions

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} - \lambda_1 + \delta \right)^{j-1} \phi_{b,1}(u)|_{u=0} = \left\{ \begin{array}{ll} 1, & j = 1, \\ 0, & j = 2, \ldots, n \end{array} \right. \tag{32}
\]

and for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{1}{du} \right)^{j-1} \phi_{b,1}(u)|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{1}{du} \right)^{j-1} \phi_{b,2}(u)|_{u=b+}, \tag{33}
\]

moveover, for \( j = 1, \ldots, n \),

\[
\left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_1 \frac{1}{du} \right)^{j-1} \frac{d\phi_{b,1}(u)}{du} \bigg|_{u=b-} = \left( \frac{\sigma^2}{2} \frac{d^2}{du^2} + c_2 \frac{1}{du} \right)^{j-1} \frac{d\phi_{b,2}(u)}{du} \bigg|_{u=b+}. \tag{34}
\]

For \( n = 1 \) and \( \lambda_1 := \lambda \), (30)–(34) simplify to

\[
\frac{1}{2} \sigma^2 \phi_{b,1}''(u) + c_1 \phi_{b,1}'(u) + \lambda \int_0^u \phi_{b,1}(u-z) \, p(z) \, dz + \lambda \int_u^\infty \omega(u, v-u) \, dF(v) = (\lambda + \delta) \phi_{b,1}(u), \quad 0 < u < b,
\]

\[
\frac{1}{2} \sigma^2 \phi_{b,2}''(u) + c_2 \phi_{b,2}'(u) + \lambda \int_0^{u-b} \phi_{b,2}(u-z) \, p(z) \, dz + \lambda \int_{u-b}^u \phi_{b,1}(u-z) \, p(z) \, dz
\]

\[+ \lambda \int_u^\infty \omega(u, v-u) \, dF(v) = (\lambda + \delta) \phi_{b,2}(u), \quad u > b.
\]

\[\phi_{b,1}(0) = 1, \, \phi_{b,1}(b-) = \phi_{b,2}(b+), \, \phi_{b,1}'(b-) = \phi_{b,2}'(b+),\]

which are formulas (3.2)–(3.4), (3.6) and (3.7) in [26].

6. Example

Consider the case of a perturbed renewal risk process with Erlang\( (2, \lambda) \) inter-claim times, i.e.

\[P(T_i \leq t) = 1 + (\lambda t + 1)e^{-\lambda t}, \quad t \geq 0.\]
This corresponds to the case $n = 2$ and $\lambda_1 = \lambda_2 := \lambda$. In addition we assume that the claim amounts are exponentially distributed, namely $F(v) = 1 - e^{-\mu v}$ ($v > 0$). Then, the Laplace transform of $f(v)$ is
\[
\hat{f}(s) = \int_0^\infty e^{-sv} f(v) \, dv = \frac{\mu}{\mu + s}.
\]

We just consider the case $m = 1$ for $V_m(u, b)$. From Eqs. (14) and (15) we achieve
\[
\frac{1}{4} \sigma^4 V_{1,1}^{(4)}(u, b) + c_1 \sigma^2 V_{1,1}^{(3)}(u, b) + \left[ c_1^2 - \sigma^2 (\lambda + \delta) \right] V_{1,1}''(u, b) - 2c_1(\lambda + \delta) V_{1,1}'(u, b)
+ (\lambda + \delta)^2 V_{1,1}(u, b) + \lambda^2 \int_0^u V_{1,1}(u - v, b) \, dv = 0, \quad 0 < u < b,
\]
and
\[
\frac{1}{4} \sigma^4 V_{1,2}^{(4)}(u, b) + c_2 \sigma^2 V_{1,2}^{(3)}(u, b) + \left[ c_2^2 - \sigma^2 (\lambda + \delta) \right] V_{1,2}''(u, b) - 2c_2(\lambda + \delta) V_{1,2}'(u, b)
+ (\lambda + \delta)^2 V_{1,2}(u, b) + \lambda^2 \int_0^{u-b} V_{1,2}(u - v, b) \, dv + \lambda^2 \int_{u-b}^u V_{1,2}(u - v, b) \, dv
- \lambda(2\lambda + \delta) = 0, \quad b < u < \infty.
\]

Applying the operator $d/du + \mu$ to (35) and (36) produce
\[
\frac{1}{4} \sigma^4 V_{1,1}^{(5)}(u, b) + \sigma^2 \left( c_1 + \frac{\mu}{4} \sigma^2 \right) V_{1,1}^{(4)}(u, b) + \left[ c_1^2 + \sigma^2(c_1\mu - \lambda - \delta) \right] V_{1,1}^{(3)}(u, b)
+ \left[ c_2^2 + \sigma^2(c_2\mu - \lambda - \delta) \right] V_{1,1}'(u, b)
+ (\lambda + \delta)^2 V_{1,1}(u, b) + \lambda^2 \int_0^u V_{1,1}(u - v, b) \, dv
- \mu\delta(2\lambda + \delta) V_{1,1}(u, b) = 0, \quad 0 < u < b,
\]
and
\[
\frac{1}{4} \sigma^4 V_{1,2}^{(5)}(u, b) + \sigma^2 \left( c_2 + \frac{\mu}{4} \sigma^2 \right) V_{1,2}^{(4)}(u, b) + \left[ c_2^2 + \sigma^2(c_2\mu - \lambda - \delta) \right] V_{1,2}^{(3)}(u, b)
+ (\lambda + \delta)^2 V_{1,2}(u, b) + \lambda^2 \int_0^{u-b} V_{1,2}(u - v, b) \, dv + \lambda^2 \int_{u-b}^u V_{1,2}(u - v, b) \, dv
- \mu\delta(2\lambda + \delta) V_{1,2}(u, b) - \mu\lambda(2\lambda + \delta) = 0, \quad b < u < \infty.
\]

The solution of (37) is of the form
\[
V_{1,1}(u, b) = \sum_{i=1}^5 a_i e^{R_i u},
\]
\[
\text{where } \Re(R_1), \Re(R_2) > 0 \text{ and } \Re(R_3), \Re(R_4), \Re(R_5) \leq 0 \text{ are the roots of}
\]
\[
\frac{\sigma^4}{4} R^5 + \sigma^2 \left( c_1 + \frac{\mu}{4} \sigma^2 \right) R^4 + \left[ c_1^2 + \sigma^2(c_1\mu - \lambda - \delta) \right] R^3 + \left[ c_2^2 + \sigma^2(c_2\mu - \lambda - \delta) \right] R^2
+ (\lambda + \delta)(\lambda + \delta - 2c_1\mu) R - \mu\delta(2\lambda + \delta) = 0, \quad u > 0,
\]
and $a_i (i = 1, \ldots, 5)$ are constants to be determined in the following.

**Remark 6.1.** Note that the above equation can be written as
\[
\left[ \frac{(\lambda + \delta) - c_1 s - \frac{\sigma^2}{2} s^2}{\lambda} \right]^2 = \frac{\mu}{\mu + s},
\]
which coincides with the Generalized Lundberg Fundamental equation in [22].
Substituting (39) into (35), we get
\[ \sum_{i=1}^{5} \frac{a_i}{R_i + \mu} = 0. \] (40)

If \( u \to \infty \), ruin does not happen all the time and dividends are always paid at a constant rate \( \alpha \). So we have \( \lim_{u \to \infty} V_{1,2}(u, b) = \alpha/\delta \). We can found that \( \alpha/\delta \) is really a particular solution of (38). So Eq. (38) has a solution of the form
\[ V_{1,2}(u, b) = \frac{\alpha}{\delta} + \sum_{i=1}^{3} b_i e^{H_i u}, \] (41)

where \( \Re(H_i) \leq 0 \) (i = 1, 2, 3) are the roots of
\[
\frac{\sigma^4}{4} R^5 + \sigma^2 \left( c_2 + \frac{1}{4} \mu \sigma^2 \right) R^4 + \left( c_2^2 + \sigma^2 (c_2 \mu - \lambda - \delta) \right) R^3 + \left[ \mu c_2^2 - \left( \lambda + \delta \right) \left( 2c_1 + \mu \sigma^2 \right) \right] R^2
\]
\[
+ (\lambda + \delta)(\lambda + \delta - 2c_2 \mu) R - \mu \delta (2\lambda + \delta) = 0,
\]
or equivalently,
\[
\left( \frac{(\lambda + \delta) - c_2 R - \frac{\sigma^2}{2} R^2}{\lambda} \right)^2 = \hat{f}(R) = \frac{\mu}{\mu + R},
\]
and \( b_i \) (i = 1, 2, 3) are coefficients to be determined in the following. (The coefficients of the roots with positive real parts have to be zero because \( \lim_{u \to \infty} V_{1,2}(u, b) = \alpha/\delta \).)

Substituting (39) and (41) in (36), we obtain
\[ \sum_{i=1}^{3} b_i e^{H_i b} - \sum_{i=1}^{5} a_i e^{R_i b} + \frac{\alpha}{\mu \delta} = 0. \] (42)

From Eqs. (17)–(19), we obtain the following six boundary value conditions
\[ V_{1,1}(0, b) = 0, \quad V_{1,1}(b+, b) = V_{1,2}(b+, b), \quad V_{1,1}(b-, b) = V_{1,2}(b+, b), \]
\[ c_1 V'_{1,1}(0, b) + \frac{\sigma^2}{2} V''_{1,1}(0, b) = 0, \]
\[ c_1 V'_{1,1}(b-, b) + \frac{\sigma^2}{2} V''_{1,1}(b-, b) = c_2 V'_{1,2}(b-, b) + \frac{\sigma^2}{2} V''_{1,2}(b-, b) + \alpha, \]
\[ c_1 V'_{1,1}(b-, b) + \frac{\sigma^2}{2} V''_{1,1}(b-, b) = c_2 V'_{1,2}(b-, b) + \frac{\sigma^2}{2} V''_{1,2}(b-, b), \]
from which and Eqs. (40) and (42), we can determine the coefficients \( a_i \) (i = 1, \ldots, 5) and \( b_i \) (i = 1, 2, 3).

If setting \( \delta = 0 \) and \( \omega(x, y) = 1 \) for all \( x, y \geq 0 \), then, \( \phi_b(u) \) becomes the probability of ultimate ruin, which we denote by \( \psi_b(u) \).

With all the above assumptions, we can get from Eqs. (30) and (31)
\[ \frac{\sigma^4}{4} \psi^{(4)}_{b,1}(u) + c_1 \sigma^2 \psi^{(3)}_{b,1}(u) + (c_1^2 - \lambda \sigma^2) \psi^{(2)}_{b,1}(u) - 2c_1 \lambda \psi^{(1)}_{b,1}(u) + \lambda^2 \psi_{b,1}(u) \]
\[
+ \lambda^2 \int_0^u \psi_{b,1}(u - v) e^{-\mu v} + \lambda^2 e^{-\mu u} = 0, \quad 0 < u < b, \] (43)
and
\[ \frac{\sigma^4}{4} \psi_{b,2}^{(4)}(u) + c_2 \sigma^2 \psi_{b,2}^{(3)}(u) + (c_2^2 - \lambda \sigma^2) \psi_{b,2}''(u) - 2c_2 \lambda \psi_{b,2}'(u) + \lambda^2 \psi_{b,2}(u) + \lambda^2 \int_0^{u-b} \psi_{b,2}(u-v) \, \text{de}^{-\mu u} + \lambda^2 \int_u^b \psi_{b,1}(u-v) \, \text{de}^{-\mu u} + \lambda^2 e^{-\mu u} = 0, \quad b < u < \infty. \] (44)

Applying the operator \( d/du + \mu \) to (43) and (44) we obtain
\[ \frac{\sigma^4}{4} \psi_{b,1}^{(5)}(u) + \sigma^2 \left( c_1 + \frac{1}{4} \mu \sigma^2 \right) \psi_{b,1}^{(4)}(u) + [c_1^2 + \sigma^2 (c_1 \mu - \lambda)] \psi_{b,1}^{(3)}(u) + [\mu c_1^2 - \lambda (2c_1 + \mu \sigma^2)] \psi_{b,1}''(u) + \lambda (\lambda - 2c_1 \mu) \psi_{b,1}'(u) = 0, \quad 0 < u < b, \] (45)

and
\[ \frac{\sigma^4}{4} \psi_{b,2}^{(5)}(u) + \sigma^2 \left( c_2 + \frac{1}{4} \mu \sigma^2 \right) \psi_{b,2}^{(4)}(u) + [c_2^2 + \sigma^2 (c_2 \mu - \lambda)] \psi_{b,2}^{(3)}(u) + [\mu c_2^2 - \lambda (2c_2 + \mu \sigma^2)] \psi_{b,2}''(u) + \lambda (\lambda - 2c_2 \mu) \psi_{b,2}'(u) = 0, \quad b < u < \infty. \] (46)

The solution of (45) is of the form
\[ \psi_{b,1}(u) = \sum_{i=1}^5 d_i e^{L_i u}, \quad 0 < u < b, \]

where \( L_i \) \((i = 1, 2, \ldots, 5)\) are the roots of
\[ \frac{\sigma^4}{4} L^5 + \sigma^2 \left( c_1 + \frac{1}{4} \mu \sigma^2 \right) L^4 + [c_1^2 + \sigma^2 (c_1 \mu - \lambda)] L^3 + [\mu c_1^2 - \lambda (2c_1 + \mu \sigma^2)] L^2 + \lambda (\lambda - 2c_1 \mu) L = 0. \]

Because \( \psi_{b,1}(0) = 1 \) and the above equation has an root 0, then,
\[ \psi_{b,1}(u) = 1 + \sum_{i=1}^4 d_i e^{L_i u}, \quad 0 < u < b, \] (47)

where \( L_i \) \((i = 1, 2, 3, 4)\) are the roots of
\[ \frac{\sigma^4}{4} L^4 + \sigma^2 \left( c_1 + \frac{1}{4} \mu \sigma^2 \right) L^3 + [c_1^2 + \sigma^2 (c_1 \mu - \lambda)] L^2 + [\mu c_1^2 - \lambda (2c_1 + \mu \sigma^2)] L + \lambda (\lambda - 2c_1 \mu) = 0, \] (48)

and \( d_i \) \((i = 1, 2, \ldots, 4)\) are coefficients to be determined in the following.

Substituting (47) into (43), a comparison of coefficients gives
\[ \sum_{i=1}^4 \frac{d_i}{L_i + \mu} + \frac{2}{\mu} = 0. \] (49)

On the other hand, Eq. (46) has solution of the form \((\psi_{b,2}(u) = 0\) is a particular solution of (46))
\[ \psi_{b,2}(u) = \sum_{i=1}^3 k_i e^{G_i u}, \quad b < u < \infty, \] (50)

where \( \Re(G_i) \leq 0 \) \((i = 1, 2, 3)\) are the roots of
\[ \frac{\sigma^4}{4} G^5 + \sigma^2 \left( c_2 + \frac{1}{4} \mu \sigma^2 \right) G^4 + [c_2^2 + \sigma^2 (c_2 \mu - \lambda)] G^3 + [\mu c_2^2 - \lambda (2c_2 + \mu \sigma^2)] G^2 + \lambda (\lambda - 2c_1 \mu) G = 0, \]
or equivalently,
\[
\left[ \frac{\lambda - c_2 G - \frac{\sigma^2}{2} \lambda^2}{\lambda} \right]^2 = \frac{\mu}{\mu + G}.
\]
and \( k_i \ (i = 1, 2, 3) \) are coefficients to be determined in the following. (Because \( \lim_{u \to \infty} \psi_{b,2}(u) = 0 \), so the coefficients of the roots with positive real parts have to be zero.)

Substituting (47) and (50) into (44), we obtain
\[
\sum_{i=1}^{3} \frac{k_i G_i \lambda b}{G_i + \mu} - \sum_{i=1}^{4} \frac{d_i e^{L_i b}}{L_i + \mu} - \frac{1}{\mu} = 0.
\]
From Eqs. (32)–(34), we obtain the following five boundary value conditions
\[
\psi_{b,1}(b) = \psi_{b,2}(b+), \quad \psi'_{b,1}(b) = \psi'_{b,2}(b+),
\]
\[
\frac{\sigma^2}{2} \psi''_{b,1}(0) + c_1 \psi'_{b,1}(0) = \frac{\sigma^2}{2} \psi''_{b,2}(0) + c_2 \psi'_{b,2}(0),
\]
\[
\frac{\sigma^2}{2} \psi''_{b,1}(b-), \quad \frac{\sigma^2}{2} \psi'_{b,1}(b-) = \frac{\sigma^2}{2} \psi''_{b,2}(b+) + c_2 \psi'_{b,2}(b+),
\]
\[
\frac{\sigma^2}{2} \psi_{b,1}(b-), \quad \frac{\sigma^2}{2} \psi_{b,1}(b-) = \frac{\sigma^2}{2} \psi_{b,2}(b+) + c_2 \psi''_{b,2}(b+),
\]
from which and Eqs. (49) and (51), we can determine the coefficients \( d_i \ (i = 1, 2, 3, 4) \) and \( k_i \ (i = 1, 2, 3) \).

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References