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Rotation Number Associated with Difference Equations Satisfied by Polynomials Orthogonal on the Unit Circle

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unit circle. It is assumed that the reflection coefficients associated with these polynomials form a stationary stochastic ergodic process. In particular, the techniques mentioned above are used to prove a gap labelling result.

1. INTRODUCTION

The purpose of this paper is to use the techniques of topological dynamics and differential dynamical systems to study solutions $\Phi_n(z)$ to the difference equation

$$\Phi_n(z) = T(z, n)\, \Phi_{n-1}(z) \qquad n \geqslant 1, \tag{1.1}$$

where the matrix $T(z, n)$ is given by

$$T(z, n) = a_n \begin{pmatrix} z & \alpha_n \\ \bar{\alpha}_n z & 1 \end{pmatrix} \tag{1.2}$$

with

$$a_n = (1 - |\alpha_n|^2)^{-1/2}. \tag{1.3}$$

Here the α_n are complex numbers of modulus strictly less than one for all n. Such difference equations arise in a number of mathematical and physical problems. For instance, if we take $\Phi_0(z) = \binom{1}{1}$ then it is well known that $\Phi(z, n)$ has the form $\Phi(z, n) = \binom{\phi_n(z)}{\phi_n^*(z)}$, $\phi_n^*(z) = z^n \bar{\phi}_n(1/z)$, where the $\phi_n(z)$ are polynomials in z of degree n orthonormal with respect to a unique positive probability measure σ supported on the unit circle K. That is,

$$\int_K \phi_n(z)\, \overline{\phi_m(z)}\, d\sigma(\theta) = \delta_{n, m} \qquad z = e^{i\theta}.$$

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This difference equation also arises in one-dimensional lossless layered medium models in seismology and transmission lines (Bruckstein and Kailath [3], Bube and Burridge [4], Robinson and Treitel [25]). In the layered media models acoustic waves are partially reflected at the interface between layers where there is a change in the acoustic impedance. If the acoustic impedance of the nth layer $Z_n > 0$ for all n then $\alpha_n = (Z_{n-1} - Z_n)/(Z_{n-1} + Z_n)$, which shows that $|\alpha_n| < 1$. For this problem the relevant quantities are $\binom{D'_0(z)}{U'_0(z)}$, where $D'_0(z)$ is the z transform of the downward waves at the bottom of the zeroth level and $U'_0(z)$ is the z transform of the upward waves at the bottom of the zeroth level. It is presumably the coefficients of $U'_0(z)$ that produce the seismic trace. If $D_n(z)$ and $U_n(z)$ are respectively the z transforms of the downward and upward waves at the top of the nth level, then [25, pp. 301–308]

$$\begin{pmatrix} D'_0(z) \\ U'_0(z) \end{pmatrix} = a_1 \begin{pmatrix} 1 & \alpha_1 \\ \alpha_1 & 1 \end{pmatrix} \begin{pmatrix} D_1(z) \\ U_1(z) \end{pmatrix}$$

and

$$\Psi(z, n) = \begin{pmatrix} \hat{D}_n(z) \\ \hat{U}_n(z) \end{pmatrix} = z^{(n+1)/2} \begin{pmatrix} D'_n(z) \\ U'_n(z) \end{pmatrix}$$

satisfies the equation

$$\Phi(z, n) = T^{-1}(z, n+1)\, \Psi(z, n+1).$$

If we consider $m_n(z) = \hat{U}_n(z)/\hat{D}_n(z) = U'_n(z)/D'_n(z)$ then $m_n(z)$ satisfies the equation

$$m_n(z) = \frac{1 - \alpha_{n+1} m_{n+1}(z)}{m_{n+1}(z) - \alpha_{n+1}}$$

and $m_n(z)$ is one of the Weyl m functions introduced below.

The problem we consider here is the study of the orthogonality measure σ and the Weyl m functions in the case when $\{\alpha_n\}$ form a stationary stochastic sequence. More precisely there are a probability space (Ω, μ), μ a Borel probability measure, a bimeasurable bijection $s\colon \Omega \to \Omega$, and a measurable function $g\colon \Omega \to \{z \in \mathbb{C} \mid |z| < 1\}$ = open unit disc such that

$$\alpha_n = g(s^n(\omega)) \qquad (-\infty < n < \infty) \tag{1.4}$$

for some $\omega \in \Omega$. Thus α_n depends on ω, and we pose questions involving the properties of $\{T(z, n)\}$ which are true for μ-almost all $\omega \in \Omega$. Hence we consider problems related to random layered media models.

We begin in Section 2 by recasting (1.1) as

$$\Phi_n(z) = (A_n + zB_n)\, \Phi_{n-1}(z) \tag{1.5}$$

and review the necessary spectral theory associated with (1.1) introducing the concepts that will be used in the sequel. Also, in this section we introduce the *Floquet exponent* $\tilde{\mathrm{w}}(z)$ associated with Eq. (1.1), which is a complex number,

$$\tilde{\mathrm{w}}(z)=\tilde{\gamma}(z)+i\,\frac{(\tilde{\rho}(z)+\operatorname{Arg} z)}{2} \qquad |z|\leqslant 1,$$

where $\tilde{\gamma}(z)$ is the so-called *Lyapunov exponent* and $\tilde{\rho}(z)$ is the *rotation number*. In this section the notion of exponential dicotomy is introduced and related to the boundedness of solutions (1.1). In order to study the deeper properties of $\tilde{\rho}(z)$ we apply (in Section 3) the well-known suspension construction (e.g., Ellis [8]) to the family of matrices $\{T(z,n)\}$ to obtain a two-dimensional system of Atkinson type

$$Ju' = Au + \lambda Bu \tag{1.6}$$

where $J=\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$, $\lambda=-i\log z$, A and B are symmetric 2×2 matrices, and B is negative semi-definite. If $\Phi(\lambda, t)$ is the fundamental matrix solution of (1.6) satisfying $\Phi(\lambda, 0)=2\times 2$ identity matrix, then

$$\Phi(-i\log z, n)=T(z,n)\,T(z,n-1)\cdots T(z,1)=\Delta(z,n) \tag{1.7}$$

for all positive integers n. This establishes a basic and simple connection between solutions of (1.6) and the solutions of (1.1). If α_n is a stationary stochastic ergodic process then the suspension of $\Delta(z,n)$, produces a random family of differential equations $(1.6)_{\hat{\omega}}$ indexed by $\hat{\omega}\in\hat{\Omega}$ where $(\hat{\Omega},\hat{\mu})$ is the suspension of (Ω,μ) as described below. The recent techniques developed by Johnson and Moser [17], Johnson and Nerurkar [18], Kotani [20], and others to study (1.6) when the coefficients are "random" can now be used to study (1.1). In particular there is a Floquet exponent $\mathrm{w}(\lambda)$ for Eq. $(1.6)_{\hat{\omega}}$ which is introduced in a way completely analogous to the way this quantity is introduced for the random Schrödinger operator (Johnson and Moser [17]). In this case the Floquet exponent is a complex number

$$\mathrm{w}(\lambda)=\gamma(\lambda)+i\rho(\lambda)$$

defined for $\operatorname{Im}\lambda\geqslant 0$. If λ is real, then $\gamma(\lambda)$ and $\rho(\lambda)$ are respectively the Lyapunov exponent and rotation number. The relation between the discrete Floquet exponent $\tilde{\mathrm{w}}$ and the continuous exponent w is

$$\tilde{\mathrm{w}}(z)=\mathrm{w}(-i\log z)+\frac{\log z}{2}, \tag{1.8}$$

which connects the Lyapunov exponent for the discrete problem with the one associated with (1.6). Furthermore the connection between the rotation number $\tilde{\rho}$, which appears as the conjugate function to the Lyapunov, $\hat{\gamma}$ and the rotation number ρ associated with the differential equation allows us to impart important dynamical information to $\tilde{\rho}$ not otherwise available. Next (in Section 4) we prove the following.

1. The complement of the set of isolated points in the topological support of the measure $\sigma(dz) = \sigma_\omega(dz)$ $(\omega \in \Omega)$ is independent of ω for $\mu - a.a.$ ω, and can be described as the set of non-constancy points of the monotone, non-increasing function $\rho(-i \log z)$ for $|z| = 1$. Thus this set Σ can be determined in a simple way.

2. (Gap-Labelling). The intervals in the open set $K - \Sigma$ are labelled by the values of $\rho(-i \log z)$, which for $z \in K - \Sigma$ lie in a countable subgroup of $\mathbb{R}$ determined by the topology of Ω (see below).

3. (Pastur and Ishii). If $\gamma(-i \log z)$ is positive on a Borel subset $B \subset K$, then for $\mu - a.a.$ ω there is no absolutely continuous component of $\sigma_\omega(dz)$ in B.

4. (Kotani). The absolutely continuous component $\sigma^{ac}_\omega(dz)$ is independent of ω for $\mu - a.a.$ ω, and the support of $\sigma^{ac}_\omega(dz)$ equals the set of $z \in K$ for which $\gamma(-i \log z) = 0$.

Most of the results listed above use the rotation number and Lyapunov exponent to characterize the boundary values of the associated m functions. For unity of presentation we state and prove our results in the case when (i) $|\alpha_n| \leqslant 1 - \delta$ for some positive δ $(-\infty < n < \infty)$; (ii) there exist positive numbers $\varepsilon > 0$, $T > 0$ such that for every n_0 there exits an n, $n_0 \leqslant n \leqslant n_o + T$ so that $|\alpha_n| \geqslant \varepsilon$. The first assumption allows us to give Ω a compact metric topology. On the other hand, the topology of Ω is only used to discuss the gap-labelling result but is not necessary for results (3) and (4) above. Indeed these results can be proved under a less restrictive assumption, namely that

$$\int_\Omega \log(1 - |\alpha_n(\omega)|)\, d\omega > -\infty.$$

These proofs use functional analytical techniques and are given in Geronimo [9] and Geronimo and Teplaev [11]. A version of (1) is also proven in these papers where the "integrated density of states" k replaces the rotation number ρ.

The relation between the rotation number and the integrated density of states is well known in the theory of the random Schrödinger operator and is in fact proved using classical Sturm oscillation theory. One of the contributions of the present paper is to define a rotation number for (1.1) which has the same relation to the integrated density of states k of [9] and [11] as the relation existing between these quantities for the random Schrödinger operator. A second contribution is to formulate and prove the gap-labelling result (2). Third, we point out that our approach to these results is based on techniques and results of dynamical systems, and thus complements the probabilistic and functional-analytic approach of [9, 11]. Finally, the Kotani-type result can be strengthened in a very useful way when Ω is compact: see Geronimo and Johnson [10] for an application to the inverse problem for orthogonal polynomials.

2. SPECTRAL THEORY

We begin this section with a quick review of spectral theory (see Atkinson [1, Chap. 3]. Consider (1.5) with

$$B_n = a_n \begin{pmatrix} 1 & 0 \\ \bar{\alpha}_n & 0 \end{pmatrix} \tag{2.1}$$

and

$$A_n = a_n \begin{pmatrix} 0 & \alpha_n \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

If $\tilde{J} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $B_n^\dagger \tilde{J} A_n = A_n^\dagger \tilde{J} B_n = 0$, $B_n^\dagger \tilde{J} B_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $A_n^\dagger \tilde{J} A_n = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus for any two solutions Φ_n and Ψ_n of (1.5) we find

$$\begin{aligned} &\Phi_{n+1}^\dagger(z_1)\, \tilde{J} \Psi_{n+1}(z_2) \\ &\quad = (\bar{z}_1 z_2 - 1)\, \Phi_n^\dagger(z_1)\, B_{n+1}^\dagger \tilde{J} B_{n+1}\, \Psi_n(z_2) + \Phi_n^\dagger(z_1)\, \tilde{J} \Psi_n(z_2). \end{aligned}$$

In particular let $Y(z, n)$ be a fundamental solution to (1.5) with $Y(z, 0) = I$; then

$$\begin{aligned} &Y^\dagger(z_1, N_1)\, \tilde{J} Y(z_2, N_1) - Y^\dagger(z_1, -M_1)\, \tilde{J} Y(z_2, -M_1) \\ &\quad = (\bar{z}_1 z_2 - 1) \sum_{k=-M_1}^{N_1 - 1} Y^\dagger(z_1, k-1)\, W Y(z_2, k-1), \end{aligned} \tag{2.3}$$

with $W = W_k = B_k^\dagger \tilde{J} B_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all k.

Consider the following boundary value problem. Let $L_1 = \left(\begin{smallmatrix} c_1 & 0 \\ d_1 & 0 \end{smallmatrix}\right)$ and $K_1 = \left(\begin{smallmatrix} 0 & e_1 \\ 0 & f_1 \end{smallmatrix}\right)$ where $|c_1| = |d_1| \neq 0$, $|e_1| = |f_1| \neq 0$, and $c_1/d_1 = \overline{(f_1/e_1)}$. Note that

$$L_1^\dagger \tilde{J} L_1 = K_1^\dagger \tilde{J} K_1 = 0. \tag{2.4}$$

We search for nonzero vectors v such that Φ_n is a solution of (1.5) with $\Phi_{N_1} = K_1 v$ and $\Phi_{-M_1} = L_1 v$. It is not difficult to see that the eigenfunctions associated with this boundary value problem are the roots of the polynomial $\det(K_1 - Y(z, N_1)\, Y^{-1}(z, -M_1)\, L_1)$. Furthermore it can be shown [1, Chap. 3] that these eigenvalues lie on the unit circle, are simple, and the eigenvectors associated with different eigenvalues are orthogonal. For $0 \leqslant \theta < 2\pi$ let $\{z_j = e^{i\theta_j}\}$ be the eigenvalues associated with the boundary value problem being considered and let $\{u_i\}$ be the corresponding normalized eigenvectors, i.e.,

$$\sum_{n=-M_1}^{N_1-1} u_i^\dagger(n)\, W u_j(n) = \delta_{ij}. \tag{2.5}$$

Let $\tilde{T}^{(1)}(\theta)$ be the spectral matrix associated with the boundary value problem being considered defined as follows: $\tilde{T}^{(1)}$ is constant on each arc between successive eigenvalues in $[0, 2\pi]$, is right continuous at each eigenvalue, and for each eigenvalue $e^{i\theta_j}$

$$\lim_{\theta \to \theta_j^+} \tilde{T}^{(1)}(e^{i\theta}) - \lim_{\theta \to \theta_j^-} \tilde{T}^{(1)}(e^{i\theta}) = u_i(0)\, u_i^\dagger(0). \tag{2.6}$$

We assume that $\{B_n\}$ is such that if $\{\Phi_n\}$ is any solution of (1.5) which is not identically zero then $\sum_{n=-M_1}^{N_1-1} \Phi_n^\dagger W \Phi_n \neq 0$. Let Ψ_n be a solution of the difference equation

$$\Psi_n(z) = (A_n + zB_n)\, \Psi_{n-1}(z) + B_n \chi_{n-1}, \tag{2.7}$$

with $\Psi_{-M_1} = L_1 v$ and $\Psi_{N_1} = K_1 v$ for some vector v. If z is not an eigenvalue of the homogeneous problem then Ψ_n is unique and v (which may be zero) is determined in terms of χ. Furthermore, Ψ_n can be represented as

$$\Psi_n \sim \int_0^{2\pi} Y(\theta, n)\, d\tilde{T}^{(1)}(\theta)\, \hat{\Psi}(\theta), \tag{2.8}$$

where $\hat{\Psi}(\theta)$ is the “Fourier Transform” of Ψ_n; i.e.,

$$\hat{\Psi}(\theta) = \sum_{n=-M_1}^{N_1-1} Y^\dagger(\theta, n)\, W \Psi_n. \tag{2.9}$$

Note that (2.8) may only faithfully reproduce the part of Ψ_n not in the kernel of W. Since $W \sum_i u_i(n)\, u_i^\dagger(m)\, W = \delta_{n,m} W$, [1, Eq. (7.3.7)], the meaning of (2.8) is

$$\| \Psi - \Phi \|_W = \left\{ \sum_{n=-M_1}^{N_1-1} (\Psi_n - \Phi_n)^\dagger\, W (\Psi_n - \Phi_n) \right\}^{1/2} = 0, \tag{2.10}$$

where

$$\Phi_n = \int_0^{2\pi} Y(\theta, n)\, d\tilde{T}^{(1)}(\theta)\, \hat{\Psi}(\theta). \tag{2.11}$$

We also find that

$$\|\hat{\Psi}\|^2 = \int_0^{2\pi} \hat{\Psi}^\dagger(\theta)\, d\tilde{T}^{(1)}(\theta)\, \hat{\Psi}(\theta) = \sum_{n=-M_1}^{N_1-1} \Psi_n^\dagger W \Psi_n = \|\Psi\|_W^2. \tag{2.12}$$

Let $\Psi_n \sim \sum_i c_i u_i(n)$, where

$$c_i = \sum_{n=-M_1}^{N_1-1} u_i^\dagger(n)\, W\Psi_n,$$

and let $d_i = \sum_{n=-M_1}^{N_1-1} u_i^\dagger(n)\, W\chi_n$. By routine manipulations using (2.7) and the boundary conditions satisfied by Ψ_n it is not difficult to show [1, Eq. (9.6.37)] that if z is not an eigenvalue of the boundary value problem being considered then $(\bar{z}_i - z)\, c_i = d_i$. Therefore $\sum_i |\bar{z}_1 - z|^2\, |c_i|^2 = \sum_i |d_i|^2$. Now (2.8), (2.9), and Bessel's inequality imply that

$$\int_0^{2\pi} |e^{i\theta} - z|^2\, \hat{\Psi}^\dagger(\theta)\, d\tilde{T}^{(1)}(\theta)\, \hat{\Psi}(\theta) \leqslant \|\chi\|_W^2. \tag{2.13}$$

Provided z is not an eigenvalue of the boundary value problem, any solution u of the equation

$$u(n) = T(z, n)\, u(n-1) + \psi(n)$$

with $u(-M_1) = L_1 v$ and $u(N_1) = K_1 v$ can be written as

$$u(n) = \sum_{i=-M_1+1}^{N_1} R(n, i)\, \psi(i),$$

where $R(n, m)$ is the Green's function given by

$$R(n, m) = \begin{cases} Y(z, n)(F^1(z) - 1/2I)\; Y(z, m)^{-1} & n < m \\ Y(z, n)(F^1(z) + 1/2I)\; Y(z, m)^{-1} & n \geqslant m, \end{cases}$$

where

$$F^1(z) = 1/2[\, Y(z, -M_1)^{-1} L_1 + Y(z, N_1)^{-1} K_1]$$
$$\times [\, Y(z, N_1)^{-1} K_1 - Y(z, -M_1)^{-1} L_1]^{-1},$$

is the characteristic function. Note that $R(n, m)$ satisfies the equation

$$R(n, m) - T(z, n)\, R(n-1, m) = \delta_{n,m} I. \tag{2.14}$$

If we set $\hat{F}^1(z) = F^1(z)\, J^{-1}$ then a simple calculation shows that $\operatorname{Re} \hat{F}^1(z) > 0$ for $|z| < 1$.

Let $\Psi_+(z, n, N_1) = \binom{\psi^1_+(z, n, N_1)}{\psi^2_+(z, n, N_1)}$ and $\Psi_-(z, n, -M_1) = \binom{\psi^1_-(z, n, -M_1)}{\psi^2_-(z, n, -M_1)}$ be solutions of (1.5) satisfying the boundary conditions

$$\Psi_+(z, N_1, N_1) = \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} \quad \text{and} \quad \Psi_-(z_1, -M_1, -M_1) = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}.$$

Since

$$\frac{\psi^2_+(z, n, N_1)}{\psi^1_+(z, n, N_1)} = z \left(\frac{\psi^2_+(z, n+1, N_1)/\psi^1_+(z, n+1, N_1) - \bar{\alpha}_{n+1}}{1 - \alpha_{n+1} \psi^2_+(z, n+1, N_1)/\psi^1_+(z, n+1, N_1)} \right), \tag{2.15}$$

and

$$\frac{\psi^1_-(z, n, -M_1)}{\psi^2_-(z, n, -M_1)} = \frac{z \dfrac{\psi^1_-(z, n-1, -M_1)}{\psi^2_-(z, n-1, -M_1)} + \alpha_n}{\bar{\alpha}_n z \dfrac{\psi^1_-(z, n-1, -M_1)}{\psi^2_-(z, n-1, -M_1} + 1}, \tag{2.16}$$

it follows from the inequalities $|(za + \alpha_n)/(\overline{\alpha_n} za + 1)| < 1$ and $|z(a - \overline{\alpha_n})/(1 - \alpha_n a)| < 1$ for $|z| < 1$ and $|a| \leqslant 1$ that $\psi^2_+(z, n, N_1)/\psi^1_+(z, n, N_1)$ and $\psi^1_-(z, n, -M_1)/\psi^2_-(z, n, -M_1)$ are both less than one in magnitude for $|z| < 1$. Set $\tilde{m}_+(z, N_1) = \psi^2_+(z, 0, N_1)/\psi^1_+(z, 0, N_1)$ and $\tilde{m}_-(z, -M_1) = \psi^1_-(z, 0, -M_1)/\psi^2_-(z, 0, -M_1)$. From (1.2) we find that

$$\begin{aligned} &\det(K_1 - Y(z, N_1)\, Y^{-1}(z, -M_1)\, L_1) \\ &\quad = z^{N_1} \det(Y(z, N_1)^{-1} K_1 - Y^{-1}(z, -M_1)\, L_1) \\ &\quad = z^{N_1} \psi^1_+(z, 0, N_1)\, \psi^2_-(z, 0, -M_1)(1 - \tilde{m}_+(z, N_1)\, \tilde{m}_-(z, -M_1)). \end{aligned} \tag{2.17}$$

Because $\psi^2_+(z, n, N_1)/\psi^1_+(z, n, N_1)$ and $\psi^1_-(z, n, -M_1)/\psi^2_-(z, n, -M_1)$ are bounded for $|z| \leqslant 1$ it follows from the uniqueness of the initial value problem that $\psi^1_+(z, n, N_1)$ and $\psi^2_-(z, n, -M_1)$ are non-zero for $|z| \neq 0$. Consequently (2.17) shows that the eigenvalues of the boundary value problem under consideration are at the points on the unit circle where $\tilde{m}_+(z, N_1) = \overline{\tilde{m}_-(z, -M_1)}$ with $|\tilde{m}_-(z, N_1)| = |\tilde{m}_-(z, -M_1)| = 1$. We now consider the coefficients in the power series for $\tilde{m}_-(z, -M_1)$. It follows from (2.16) that the contribution of $\psi^1_-(z, -i, -M_1)/\psi^2_-(z, -i, -M_1)$ to the power series coefficients of $\psi^1_-(z, 0, -M_1)/\psi^2_-(z, 0, -M_1)$ does not appear until the ith coefficient of the series for $\psi^1_-(z, 0, -M_1)/\psi^2_-(z, 0, -M_1)$. This implies that the contribution of $\psi^1_-(z, -M_1, -M_1)/\psi^2_-(z, -M_1, -M_1) = c_1/d_1$ does not appear until the M_1 coefficient in the series for $\tilde{m}_-(z, -M_1)$. Therefore all the coefficients c_m in the power series expansion for $\tilde{m}_-(z, -M_1)$ for $m < M_1$ are independent of M_1. Thus we can let $M_1 \to \infty$ and $\lim_{M_1 \to \infty} \tilde{m}_-(z, -M_1) = \tilde{m}_-(z)$ where the convergence is uniform on compact subsets of the open unit disk. An analogous argument shows $\tilde{m}_+(z, N_1) \to \tilde{m}_+(z)$. A geometric description of the process will be given in Section 4. Note that $|\tilde{m}_-(z)| < 1$ and $|\tilde{m}_+(z)| < 1$ for $|z| < 1$.

If we consider the projection operator whose range is $\left(\begin{smallmatrix} 1 \\ \tilde{m}_+(z, N_1) \end{smallmatrix}\right)$ and whose kernel is $\left(\begin{smallmatrix} \tilde{m}_-(z, -M_1) \\ 1 \end{smallmatrix}\right)$ we find

$$\tilde{P}^1 = \frac{\begin{pmatrix} 1 & -\tilde{m}_-(z, -M_1) \\ +\tilde{m}_+(z, N_1) & -\tilde{m}_-(z, -M_1)\, \tilde{m}_+(z, N_1) \end{pmatrix}}{1 - \tilde{m}_-(z, -M_1)\, \tilde{m}_+(z, N_1)}. \tag{2.18}$$

In terms of $\tilde{P}^1$ the characteristic function F^1 has the representation $F^1(z) = (\tilde{P}^1 - \frac{1}{2}I)$ and $\hat{F}^1(z) = (\tilde{P}^1 - \frac{1}{2}I)\, \tilde{J}^{-1}$ which gives

$$\hat{F}^1(z) = \frac{\begin{pmatrix} \frac{1}{2}(1 + \tilde{m}_-(z, -M_1)\, \tilde{m}_+(z, N_1)) & \tilde{m}_-(z, -_1) \\ \tilde{m}_+(z, N_1) & \frac{1}{2}(1 + \tilde{m}_-(z, -M_1)\, \tilde{m}_+(z, N_1)) \end{pmatrix}}{1 - \tilde{m}_-(z, -M_1)\, \tilde{m}_+(z, N_1)}. \tag{2.19}$$

Since $\hat{F}^1(z)$ has positive real part for $|z| < 1$ it has the representation

$$\hat{F}^1(z) = \frac{1}{2i} \begin{pmatrix} 0 & \alpha_0 \\ -\bar{\alpha}_0 & 0 \end{pmatrix} + \int_K \frac{e^{i\phi} + z}{e^{i\phi} - z}\, d\tilde{T}^{(1)}.$$

From the arguments above we find that $\tilde{T}^{(1)}$ converges weakly as N_1 and M_1 tend to infinity to a unique matrix measure T supported on the unit circle. Thus

$$\hat{F}^1(z) \to \hat{F}(z) = \frac{1}{2i} \begin{pmatrix} 0 & \alpha_0 \\ -\bar{\alpha}_0 & 0 \end{pmatrix} + \int_K \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\tilde{T}(\theta) \tag{2.20}$$

with $\int_K d\tilde{T} = \frac{1}{2}\left(\begin{smallmatrix} 1 & \bar{\alpha}_0 \\ \alpha_0 & 1 \end{smallmatrix}\right)$. Set

$$g(z) = tr\, \hat{F}(z) = \frac{1 + \tilde{m}_+ \tilde{m}_-}{1 - \tilde{m}_+ \tilde{m}_-}, \tag{2.21}$$

which is a Caratheodory function and has the representation

$$g(z) = \int_K \frac{e^{i\theta} + z}{e^{i\theta} - z}\, dv \tag{2.22}$$

where $g(0) = \int dv = 1$.

Let s be the bimeasurable bijection discussed in the Introduction and let α_n satisfy (1.4). Furthermore we will suppose that s is ergodic, i.e., $s^{-1}A = A \Rightarrow \mu(A) \in \{0, 1\}$. We assume that $\alpha_n \not\equiv 0$.

LEMMA 2.1. *Suppose $s\colon \Omega \to \Omega$ is an ergodic automorphism and $\{\alpha_n\}$ is not identically zero. Then there is a unique measure k such that for μ-almost every ω,*

$$\mathbb{E}(g) = \mathbb{E}\left(\frac{1 + \tilde{m}_+ \tilde{m}_-}{1 - \tilde{m}_+ \tilde{m}_-}\right) = \int_K \frac{e^{i\theta} + z}{e^{i\theta} - z}\, dk(\theta) \qquad |z| < 1.$$

Proof. The result follows from Fubini's Theorem.

LEMMA 2.2. *Suppose $s\colon \Omega \to \Omega$ is an ergodic automorphism, $\{\alpha_n\}$ is not identically zero and*

$$\mathbb{E}(\log(1 - |\alpha_1(\omega)|)) > -\infty. \tag{2.23}$$

Then

$$\begin{aligned}\lim_{n \to \infty} \frac{1}{n} \log \| Y(z, n) \| &= \lim_{n \to \infty} \frac{1}{n} \log \| Y^{-1}(z, -n) \| \\ &= \lim_{n \to \infty} \frac{1}{2n} \log \| Y(z, n)\, Y(z, -n)^{-1} \| = \tilde{\gamma}(z)\end{aligned} \tag{2.24}$$

exists and is independent of ω for each fixed $z \in \mathbb{C}$ and μ-almost every ω. Furthermore, $\tilde{\gamma}(z)$ is subharmonic and greater than or equal to zero.

Proof. The proof follows from Kingman's subadditive ergodic theorem (Krengel [21], Ruelle [26]; see also Geronimo and Teplaev [11], Geronimo [9], Craig and Simon [6], and Herman [13]).

$\tilde{\gamma}(z)$ is the Lyapunov exponent associated equations (1.5). Before deriving the so-called Thouless formula (Avron and Simon [2], Johnson and Moser [17]) we need

LEMMA 2.3. *With the hypothesis of the previous lemma*

$$\tilde{\gamma}(z)=\lim_{n\to\infty}\frac{1}{2n}\log|\phi(z,2n+1,-n)+\phi^*(z,2n+1,-n)|$$

uniformly on compact subsets of $\mathbb{C}\setminus K$.

Proof. Using the Hilbert–Schmidt norm we find

$$\|Y(z,n)\,Y^{-1}(z,-n)\|^2$$
$$=|A_+(z,n)|^2\left\{\left[1+\left|\frac{A_-(z,n)}{A_+(z,n)}\right|^2\right]+\left|\frac{B_+(z,n)}{A_+(z,n)}\right|^2\left[1+\left|\frac{B_-(z,n)}{B_+(z,n)}\right|^2\right]\right\},$$

where $A_\pm(z,n)=\phi(z,2n+1,-n)\pm\phi^*(z,2n+1,-n)$, and $B_\pm(z,n)=\phi_1(z,2n+1,-n)\pm\phi_2(z,2n+1,-n)$ with

$$\begin{pmatrix}\phi(z,2n,-n)\\ \phi^*(z,2n,-n)\end{pmatrix}=Y(z,n)\,Y(z,-n)^{-1}\begin{pmatrix}1\\1\end{pmatrix}$$

and

$$\begin{pmatrix}\phi_1(z,2n,-n)\\ \phi_2(z,2n,-n)\end{pmatrix}=Y(z,n)\,Y^{-1}(z,-n)\begin{pmatrix}1\\-1\end{pmatrix}.$$

Suppose $|z|<1$, then

$$\frac{A_-(z,n)}{A_+(z,n)}=\frac{1-\phi(z,2n,-n)/\phi^*(z,2n,-n)}{1+\phi(z,2n,-n)/\phi^*(z,2n,-n)}$$

since $\phi^*(z,2n,-n)$ has all its zeros outside the unit circle (Geronimus [12]). This implies that

$$\left|\frac{\phi(z,2n,-n)}{\phi^*(z,2n,-n)}\right|<1$$

for $|z|<1$ since $|\phi(z,2n,-n)|=|\phi^*(z,2n,-n)|$ for $|z|=1$. Thus, $A_-(z,n)/A_+(z,n)$ is a Caratheodory function and has the representation

$$\frac{A_-(z,n)}{A_+(z,n)}=iv_n+\int_{-\pi}^{\pi}\frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\sigma_n(\theta).$$

Hence we find that for each compact subset $\tilde{K}$ of the open unit disc there is constant c depending only on $\tilde{K}$ such that $(A_-(z,n)/A_+(z,n))<c/(1-|\alpha_n|)$. If $|z|>1$ then the same bound with a different constant is obtained if the fact that $\phi(z,2n,-n)$ has all its zeros inside the unit circle is used. With similar arguments it is not difficult to see (Geronimo [9]) that $|B_-(z,n)/B_+(z,n)| \leqslant c/(1-|\alpha_n|)$ and $|B_+(z,n)/A_+(z,n)| < c$, where c depends only upon the compact subset $\tilde{K} \in \mathbb{C} \backslash K$ and not on n. The result now follows since $(1/n)\log(1-|\alpha_n|) \to 0$ by (2.23). ▮

THEOREM 2.4. *Suppose $s: \Omega \to \Omega$ is an ergodic automorphism, $\{\alpha_n\}$ is not identically zero, and* (2.23) *holds. Then*

$$\tilde{\gamma}(z) = R + \int_K \log|z - e^{i\phi}|\, dk(\phi) \tag{2.25}$$

for all z where $R = \lim_{n\to 0}(1/n)\log \prod_{i=1}^{n} a_n < \infty$ for μ-almost all ω.

(Here we have adopted the convention that $\int_K \log|z-e^{i\phi}|\, dk$ is equal to $-\infty$ if the integral diverges to $-\infty$.)

Proof. If we return to the boundary value problem considered at the beginning of this section and set $M_1 = N_1$, $L_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $K_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ then

$$\det(K_1 - Y(z, N_1)\, Y^{-1}(z, -N_1)\, L_1) = \phi(z, 2N, -N_1) + \phi^*(z, 2N, -N_1).$$

Thus for n finite we have

$$\frac{1}{2n}\log|\phi^*(z, 2n-n) + \phi(z, 2n, -n)|$$
$$= \frac{1}{2n}\sum_{j=-n+1}^{n} \ln a_j + \int_K \log|z-e^{i\phi}|\, dk^{(n)}(\theta).$$

Equation (2.22), Lemma 2.2, and the fact that $\ln|z-e^{i\phi}|$ is continuous for z not on the unit circle now give the result for $|z| \neq 1$. The result for $|z|=1$ follow since both sides of (2.25) are subharmonic and equal for all $|z| \neq 1$. ▮

We now define the Floquet exponent associated with the system. To this end set

$$\tilde{w}(z) \equiv R + \int_K \ln(z - e^{i\phi})\, dk(\phi)$$
$$\equiv \tilde{\gamma}(z) + i\,\frac{\tilde{\rho} + \operatorname{Arg} z}{2}, \qquad |z| < 1, \tag{2.26}$$

where we use the principal branch of the logarithm. From (2.26) we find that

$$\frac{1}{i}\frac{\partial \tilde{w}}{\partial \theta} - \frac{1}{2} = -\frac{1}{2}\int_K \frac{e^{i\phi}+z}{e^{i\phi}-z}\, dk(\phi) = \frac{1}{i}\frac{\partial \tilde{\gamma}}{\partial \theta} + \frac{1}{2}\frac{\partial \tilde{\rho}}{\partial \theta}, \qquad z = re^{i\theta}, \quad r<1.$$

This implies that

$$\frac{\partial\tilde{\rho}}{\partial\theta}=-\int_K \frac{1-|z|^2}{|z-e^{i\phi}|^2}\,dk(\phi).$$

Therefore $\partial\tilde{\rho}/\partial\theta$ has radial boundary values Lebesgue almost everywhere and

$$\lim_{r\to 1}\int_K f(e^{i\theta})\frac{\partial\tilde{\rho}}{\partial\theta}\,d\theta=\int_K f(e^{i\theta})\,d\tilde{\rho}=-\int_K f(e^{i\theta})\,dk.$$

Since k is log Holder continuous and $\tilde{\rho}$ is continuous (see Section 4) we find for any sub arc I of the unit circle

$$\int_I d\tilde{\rho}(\theta)=-\int_I dk(\theta)=-\int_I E(tr\,d\tilde{T}(\theta)), \tag{2.27}$$

where the last formula follows from (2.18), (2.19), and Lemma 2.2.

It is convenient at this point to topologize the set Ω. We do so by first identifying $\omega\in\Omega$ with the biinfinite sequence $i(\omega)=\{g(s^n(\omega))\mid -\infty<n<\infty\}$, which we view as a element of the biinfinite product $\mathbb{C}^\infty$. Let $\tilde{\Omega}=\mathrm{cls}\{i(\omega)\mid\omega\in\Omega\}$ where the closure is taken in the topology of pointwise convergence. Then $\tilde{\Omega}$ is compact and the image measure $i(\mu)=\tilde{\mu}$ is ergodic with respect to the shift transformation on $\tilde{\Omega}$. We suppose without loss of generality that $\tilde{\Omega}$ is the topological support of the measure $\tilde{\mu}$. In what follows we identify (Ω,μ) with $(\tilde{\Omega},\tilde{\mu})$ and drop the tilde.

Another useful concept is that of *exponential dichotomy* (ED) (Coppel [5]).

Definition 2.5. Let $z\in\mathbb{C}-\{0\}$ be fixed. Equations (1.5) are said to have ED over Ω if there are constants $L>0$, $\beta>0$ and a continuous projection-valued function $P\colon\omega\to P_\omega$ (thus each $P_\omega\colon\mathbb{C}^2\to\mathbb{C}^2$ is linear and $P_\omega^2=P_\omega$) such that

$$\|Y(z,n)(I-P_\omega)\,Y(z,m)^{-1}\|\leqslant Le^{\beta(n-m)}\qquad n<m,$$

$$\|Y(z,n)\,P_\omega\,Y(z,m)^{-1}\|\leqslant Le^{-\beta(n-m)}\qquad n\geqslant m.$$

Thus Eq. (1.5) has ED if and only if the space of solutions of these equations admits a hyperbolic splitting. Note that the dimension of P_ω is a constant on each connected component of Ω.

The importance of exponential dichotomy derives from the fact that it is a very robust property [5] and from the fact that P is continuous resp. smooth in parameters when the difference equation with which it is associated is continuous resp. smooth in these same parameters

We now prove the main result of this section.

THEOREM 2.6. *Suppose* $s\colon \Omega \to \Omega$ *is an ergodic automorphism,* $\operatorname{ess\,sup} |g(\omega)| < 1$ (*see Eq.* (1.4)), *and that there exist positive numbers* ε, T *such that for each* $\omega \in \Omega$, *every interval* $[n_0 + 1, n_0 + T] \subset \mathbb{Z}$ *of length* T *contains a number* n *such that* $|g(s^n(\omega))| \geqslant \varepsilon$. *Let* $I = (\theta_1, \theta_2)$, $0 \leqslant \theta_1 < \theta_2 < 2\pi$. *Then Eq.* (1.5) *has ED over* Ω *for all* $z = e^{i\theta}$, $\theta \in I$, *if and only if the rotation number* $\tilde{\rho}(\theta)$ *is constant on* I.

The proof, which is based on arguments found in Johnson and Nerurkar [18] and is given for the convenience of the reader, will be presented after the following preparatory lemmas.

LEMMA 2.7. *Under the hypothesis of the above theorem, let* Φ_n *be a solution of* (1.5). *Then* $\sum_{n \in \mathbb{Z}} \Phi_n^\dagger W \Phi_n = 0$ *if and only if* $\Phi_n = 0$ *for all* n.

Proof. From the definition of W we see that $\sum_{n \in \mathbb{Z}} \Phi_n^\dagger W \Phi_n = \sum_{n \in \mathbb{Z}} |\phi_n^1|^2$, where $\Phi_n = \binom{\phi_n^1}{\phi_n^2}$. Therefore if $\sum_{n \in \mathbb{Z}} \Phi_n^1 W \Phi_n = 0$, $\phi_n^1 = 0$ for all n. From (1.5) we see that this implies that $\alpha_n \phi_0^2(n-1) = 0$ for all n. But there is an n_1 such that $\alpha_{n_1} \neq 0$ so $\phi_{n_1-1}^2 = 0$; hence $\Phi_{n_1-1} = 0$ which implies that $\Phi_n(z_1) = 0$ for all n. ∎

Suppose that $z_1 = e^{i\theta_1}$ is not an eigenvalue of (1.5). (Here we have let N_1 and M_1 go to infinity.) Suppose $\Phi_n(z_1)$ is a bounded solution of (1.5) such that $\sum \Phi_n^\dagger W \Phi_n = \infty$. Let $K_0 = [m_0, n_0]$ be a large interval containing zero and consider the following control problem. We look for a function χ with the following properties, (1) support $\chi \subset [m_0, m_0+1] \cup [n_0, n_0+1]$, (2) if h is a solution of $h(n) = (A_n + z_1 B_n)\, h(n-1) + B_n \chi(n)$ with $h(0) = \Phi_0$ then support $h = [m_0 - 1, n_0 + 2]$, and (3) $\chi(n)$ is bounded independently of the interval K_0.

DEFINITION 2.8. An invariant compact subset $M \subset \Omega$ is said to be *minimal* if every $\omega \in M$ has a dense orbit: $\operatorname{cls}\{s^n(\omega) - \infty < n < \infty\} = M$ for all $\omega \in M$.

LEMMA 2.9. *Suppose that* $s\colon \Omega \to \Omega$ *is an ergodic automorphism,* $\operatorname{ess\,sup} |g(\omega)| < 1$, *and that there exist positive numbers* ε, T *such that for each* $\omega \in \Omega$, *every interval* $[n_0, n_0 + T] \subset \mathbb{Z}$ *of length* T *contains a number* n *such that* $|g(s^n(\omega))| \geqslant \varepsilon$. *Let* $\omega \in \Omega$ *and write* $\alpha_n = \alpha_n(\omega)$ $(-\infty < n < \infty)$. *Then there are a control* χ *and a solution* h *satisfying* (1), (2), *and* (3).

Proof. Let $[m_0, n_0], [m_1, n_1], ..., [m_i, n_i]...$ be an infinite set of increasing intervals such that $0 < c_1 < |\alpha_{n_j}|$, $|\alpha_{m_j}| < c_2 < 1$ for fixed c_1 and c_2. Such intervals exist because of the assumptions on $\{\alpha_n\}$. If we choose $\chi(n) = \binom{\chi_1(n)}{0}$ with $\Phi_n(z) = \binom{\phi_n^1(z)}{\phi_n^2(z)}$ then $\chi_1(m_0+1) = z_1(\phi_{m_0}^1(z_1) - \phi_{m_0}^2(z_1)/\bar{\alpha}_{m_0})$,

$\chi_1(m_0)=\phi^2_{m0}(z_1)/\alpha_{m_0}$, $\chi_1(n_0)=-\phi^2_n(z_1)/a_{n0}\bar{\alpha}_{n_0}$, $\chi_1(n_0+1)=-z_1(\phi^1_{n0}(z_1)-\phi^2_{n0}(z_1)/\bar{\alpha}_{n_0})$ and $\chi(n)=0$ otherwise is the control having the desired properties. ▮

LEMMA 2.10. *With the hypotheses of Theorem* 2.6 *the map* $\omega \to d\tilde{T}_\omega$ *is weakly continuous, i.e.,* $\omega \to \int_K \Psi^\dagger(\theta)\, d\tilde{T}_\omega(\theta)\, \Psi(\theta)$ *is continuous.*

Proof. This is most easily seen by using the topology on Ω introduced above and recalling that the shift map is continuous. This plus the continuity of the m functions in terms of ω gives the result. ▮

Now we turn to the

Proof of Theorem 2.6. Let $\omega \in \Omega$. We begin by showing that if ρ is constant on I, that is, if $\rho(\theta_1)-\rho(\theta_2)=0$ (ρ is a monotonic function on $0 \leqslant \theta < 2\pi$) then the only bounded solution of (1.5) for all $\theta \in I$ is the zero solution. A theorem of Selgrade [29] then implies that, if $\omega \in M$ and M is a minimal subset of Ω, then ED holds over M for all $\theta \in I$. Let $\omega_0 \in M$, M a minimal set, and $\theta_0 \in I$ such that $\Phi_0(n)=\Phi_{n,\,\omega 0}(e^{i\theta_0}, n)$ is a bounded solution of (1.5) for all n. Note that the constancy of ρ on I, (2.27), and Lemma 2.10 show that

$$\int_{\theta_1}^{\theta_2} tr\; d\tilde{T}_{\omega 0}(\theta)=0. \tag{2.28}$$

This follows by approximating the characteristic function of I by continuous functions and using the dominated convergence theorem. We assume that $\sum \Phi^\dagger_0(n)\, W\Phi_0(n)=\infty$ for otherwise $\phi_0(n)$ would be an eigenvalue for (1.5) violating (2.28). Let χ be the control given in Lemma 2.9 and note that $h_n=\Phi_0(n)$ for $n \in [m_0, n_0]$. Let $[m_1, n_1]$, $[m_2, n_2] \cdots [m_i, n_i]$ with $m_1 < m_0-1$, $n_1 > n_0+1$ be a sequence of nested intervals such that $n_i \to \infty$ and $m_i \to -\infty$. Let matrices L_i and K_i be chosen so that the boundary value problems on each interval $[n_i, m_i]$, $i=0, 1, \ldots$ do not admit $z_0=e^{i\theta_0}$ as an eigenvalue and let T_i be the spectral matrix associated with the boundary value problem on $[m_i, n_i]$. Set

$$\hat{h}(\theta)=\sum_{n=m_i}^{n_i-1} Y^\dagger(\theta, n)\, Wh_n=\sum_{n=m_1}^{n_1-1} Y^\dagger(\theta, n)\, Wh_n.$$

The last inequality follows since support $h_n \subset [n_1, m_1]$. $Y(\theta, n)$, $z=e^{i\theta}$ is the fundamental matrix for (1.5) with $Y(\theta, 0)=I$. From (2.4) we find

$$\int_I \hat{h}(\theta)^\dagger\, dT^j \hat{h}(\theta)=\sum_{\theta_i \in I} \left| \sum_{n=m_j}^{n_j-1} u^\dagger_i(n)\, Wh_n \right|^2=\sum_{\theta_i \in I} \left| \sum_{n=m_1}^{n_1-1} u^\dagger_i(n)\, Wh_n \right|^2,$$

where $u_i(n) = Y(\theta_1, n)\, u_i(\theta)$. By Schwarz's inequality we get

$$\sum_{\theta_i \in I} \left| \sum_{n=m_1}^{n_1-1} u_i^\dagger(m)\, Wh_n \right|^2 \leqslant \|h\|_W^2 \sum_{\theta_i \in I} \left| \sum_{n=m_1}^{n_1-1} u_i^\dagger(n)\, Wu_i(n) \right|$$

with $\|h\|_W^2 = \sum_{n=m_1}^{n_1-1} h_n^\dagger W h_n$. In order to recast

$$\sum_{n=m_1}^{n_1-1} u_i^\dagger(n)\, Wu_i(n)$$

in a suitable form we make the following observation. Suppose that ω_0 is replaced by $s^l(\omega_0)$ and the intervals $[m_i, n_i]$ are replaced by $[m_i - l, n_i - l]$. Then the boundary value problem considered above becomes $x(n) = [A_{n+l} + zB_{n+l}]\, x(n-1)$ with $x(n_i - l) = K_i v$, $x(m_i - l) = L_i v$, $v \neq 0$. Let $\tilde{T}_l^i = \tilde{T}^i_{s^l(\omega_0)}$ be the spectral matrix corresponding to this boundary value problem. Then $\tilde{T}_l^i$ is constant except at the eigenvalues $z_j = e^{i\theta_j}$ with θ_j the same as when $l = 0$ and $\lim_{\theta \to \theta_j^+} \tilde{T}_l^i(\theta) - \lim_{\theta \to \theta_j^-} \tilde{T}_l^i(\theta) = u_j(l)\, u_j(l)^\dagger$. With this in mind we see that

$$\sum_{\theta_j \in I} \sum_{n=m_i}^{n_i-1} u_j^\dagger(n)\, Wu_j(n) = \sum_{n=m_i}^{n_i-1} \int_I d(tr\; W\tilde{T}_n^i(\theta)). \tag{2.29}$$

Since (2.28) also applies for $\tilde{T}_n$ we find from the dominated convergence theorem that $\lim_{i \to \infty} \sum_{n=m_i}^{n_i-1} \int_I d(tr(W\tilde{T}_n^i)) = 0$. Therefore

$$\lim_{j \to \infty} \int_I \hat{h}^\dagger(\theta)\, d\tilde{T}^j_{s^n(\omega_0)} \hat{h}(\theta) = 0. \tag{2.30}$$

If $\varepsilon = \min \frac{1}{2}(|e^{i\theta_1} - e^{i\theta_0}|, |e^{i\theta_2} - e^{i\theta_0}|)$ then the fact that $\|h\|_W > 0$ and (2.30) show that

$$\begin{aligned}\int_0^{2\pi} |e^{i\theta} - z_0|^2\, \hat{h}^\dagger\, d\tilde{T}^j \hat{h} &= \int_{|e^{i\theta} - z_0| > 2\varepsilon \,\cup\, |e^{i\theta} - z_0| \leqslant 2\varepsilon} |e^{i\theta} - z_0|^2\, \hat{h}^\dagger\, d\tilde{T}^j h \\ &\geqslant \varepsilon^2 \int_0^{2\pi} \hat{h}^\dagger\, d\tilde{T}^j \hat{h},\end{aligned}$$

for j sufficiently large. However, (2.12) and (2.13) imply that

$$\|\chi\|_W^2 \geqslant \varepsilon^2 \int_0^{2\pi} \hat{h}^\dagger\, d\tilde{T}^i h = \varepsilon^2\, \|h\|_W^2,$$

for large enough i. This yields a contradiction since $\|\chi\|_W$ is independent of the size of $[m_0, n_0]$ while by hypothesis $\|h\|_W \to \infty$ as this interval grows.

Thus $\Phi_n = 0$ and we have ED over the minimal set M. Now however the projections P_ω must have constant rank $= 1$, so a result of Sacker and Sell [27] now implies that (1.5) has ED over all Ω which completes the proof. ▮

3. THE SUSPENSION

We now study (1.5) by introducing a differential system whose solutions reflect behavior of iterates of (1.5). We do this by "suspending" the probability space (Ω, μ) and the difference equation (1.5). We first consider the suspension of a measure-preserving automorphism s of an abstract probability space (Ω, μ). Consider the product space $\Omega \times \mathbb{R}$, and define $\hat{s}: \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$: $\hat{s}(\omega, t) = (s(\omega), t-1)$. The group of mappings $\sigma = \{\hat{s}^n \mid -\infty < n < \infty\}$ acts freely on $\Omega \times \mathbb{R}$ in the sense that if $\hat{s}^n(\omega, t) = (\omega, t)$ for some point $(\omega, t) \in \Omega \times \mathbb{R}$ and some integer n, then $n = 0$ and $\hat{s}^n$ is the identity.

Let $\hat{\Omega}$ be the quotient space

$$\hat{\Omega} = (\Omega \times \mathbb{R})/\sigma.$$

Thus $\hat{\Omega}$ is the set of equivalence classes $[\omega, t]$ of pairs (ω, t) under the equivalence relation

$$(\omega_1, t_1) \sim (\omega_2, t_2) \Leftrightarrow \omega_2 = s^n \omega_1 \qquad \text{and} \qquad t_2 = t_1 - n$$

for some $n \in \mathbb{Z}$. Clearly each element of $\hat{\Omega}$ has a unique representative $[\omega, t]$ where $0 \leqslant t < 1$. We see further that $[\omega, 1] = [s(\omega), 0]$.

There is a Borel structure $\hat{B}$ on $\hat{\Omega}$ induced by the product Borel structure $\mathscr{B}$ on $\Omega \times \mathbb{R}$: thus $B \in \hat{\mathscr{B}}$ if and only if $\pi^{-1}(B) \in \mathscr{B}$ where $\pi: \Omega \times \mathbb{R} \to \hat{\Omega}$: $(\omega, t) \to [\omega, t]$ is the projection. We define a probability measure $\hat{\mu}$ on $\hat{\Omega}$ as follows. Fix $t \in \mathbb{R}$, and let μ_t be the measure on $\{[\omega, t] \mid \omega \in \Omega\} \subset \hat{\Omega}$ induced by μ. More precisely: the map $i_t: \Omega \to \hat{\Omega}: \omega \to [\omega, t]$ is a measure isomorphism onto its image, and we define

$$\mu_t(B) = \mu(i_t^{-1}(B)) \qquad (B \in \hat{\mathscr{B}}).$$

The measure μ_t is well defined because of the invariance of μ with respect to s. Now define

$$\hat{\mu}(B) = \int_0^1 \mu_t(B)\, dt \qquad (B \in \hat{\mathscr{B}}).$$

In this way we obtain a σ-additive measure $\hat{\mu}$ on $\hat{\mathscr{B}}$.

Next note that, for each $t \in \mathbb{R}$, the map $\tau_t\colon \hat{\Omega} \to \hat{\Omega}\colon \tau_t[\omega, s] = [\omega, s+t]$ is a bimeasurable bijection. Moreover, $\tau_0 = i\,dy$ and $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \in \mathbb{R}$. That is, $\{\tau_t \mid t \in \mathbb{R}\}$ is a one-parameter group of measure automorphisms of $(\hat{\Omega}, \hat{\mathscr{B}})$.

Using the invariance of μ with respect to s, one can now show:

PROPOSITION 3.1. *For each $t \in \mathbb{R}$ and $B \in \hat{\mathscr{B}}$, one has*

$$\hat{\mu}(\tau_t(B)) = \hat{\mu}(B).$$

Thus the measure $\hat{\mu}$ is invariant with respect to the one-parameter group $\{\tau_t \mid t \in \mathbb{R}\}$.

We turn to the construction of the suspension of the matrix $T(z, n)$ defined in (1.2). Fix a complex number $z \neq 0$ and consider the map $T_z\colon \Omega \times \mathbb{Z}^+ \to GL(2, \mathbb{C})$ given by (1.2). We suppose that α_n is given by (1.4). The map T_z defines an *integer cocycle* Δ_z on $\Omega \times \mathbb{Z}$ as we now explain. First of all, an integer cocycle is a μ-measurable map Δ which satisfies the conditions

$$\begin{aligned} \Delta(\omega, 0) &= I = \text{identity matrix}, \\ \Delta(\omega, n_1 + n_2) &= \Delta(s^{n_1}(\omega), n_2)\, \Delta(\omega, n_1) \end{aligned} \tag{3.1}$$

for $\omega \in \Omega$, $n_1, n_2 \in \mathbb{Z}$. A measurable map $T\colon \Omega \to GL(2, \mathbb{C})$ generates a cocycle Δ in a natural way:

$$\begin{aligned} \Delta(\omega, 0) &= I \\ \Delta(\omega, n) &= T(s^{n-1}(\omega)) \cdots T(s(\omega))\, T(\omega) \qquad (n \geqslant 1), \\ \Delta(\omega, -n) &= T(s^{-n}(\omega))^{-1} \cdots T(s^{-1}(\omega))^{-1} \qquad (n \geqslant 1). \end{aligned} \tag{3.2}$$

Now set $T(\omega) = T_z(\omega, 1) = a_1\begin{pmatrix} z & \alpha_1 \\ \bar{\alpha}_1 z & 1 \end{pmatrix}$ and let $\Delta_z(\omega, n)$ be the cocycle defined by the formulas (3.2).

Now we will "suspend" the cocycle Δ_z so as to obtain a ("real") cocycle $\Phi_z\colon \hat{\Omega} \times \mathbb{R} \to GL(2, \mathbb{C})$, i.e., a mapping which is measurable with respect to the natural Borel structures and which satisfies

$$\begin{aligned} \Phi_z(\hat{\omega}, 0) &= I \\ \Phi_z(\hat{\omega}, t_1 + t_2) &= \Phi_z(\tau_{t_1}(\hat{\omega}), t_2)\, \Phi_z(\hat{\omega}, t_1) \end{aligned} \tag{3.3}$$

for $\hat{\omega} \in \hat{\Omega}$ and $t_1, t_2 \in \mathbb{R}$. Actually we will see that Φ_z is naturally a function of $\log z$ and not of z, and hence can (and will) be viewed as defined on the Riemann surface of

$$\lambda = -i \log z. \tag{3.4}$$

The key observation is that the map $\omega \to T_z(\omega, 1)$ is homotopic to the identity map $\omega \to I$. In fact, let $f\colon [0, 1] \to [0, 1]$ be a motonically increasing C^∞ function such that $f(0) = 0$, $f(1) = 1$, and $D^n f(0) = D^n f(1) = 0$ for all derivatives of order $n \geqslant 1$. Define $\hat{T}_\lambda\colon \Omega \times [0, 1] \to GL(2, \mathbb{C})$,

$$\hat{T}_\lambda(\omega, t) = c_0(t) \begin{pmatrix} z^{f(t)} & f(t)\, \alpha_1(\omega) \\ f(t)\, \overline{\alpha_1(\omega)}\, z^{f(t)} & 1 \end{pmatrix}, \tag{3.5}$$

where $\lambda = -i \log z$ and we have written $\alpha_1(\omega) = g(\omega)$. The quantity $c_0(t)$ interpolates a_1: $c_0(t) = (1 - f^2(t)\, |\alpha_1(\omega)|^2)^{-1/2}$. Clearly

$$\hat{T}_\lambda(\omega, 1) = a_1 \begin{pmatrix} z & \alpha_1(\omega) \\ \overline{\alpha_1(\omega)}\, z & 1 \end{pmatrix} = T_z(\omega, 1).$$

Observe that $\hat{T}_\lambda$ is a function of λ and not of z because of the factor $z^{f(t)}$ in its definition.

We now define the cocycle $\Phi_\lambda(\hat{\omega}, t)$ as the fundamental matrix solution of a differential equation. If $\hat{\omega} = [\omega, s] \in \hat{\Omega}$ where $\omega \in \Omega$ and $0 \leqslant s < 1$, define

$$Q(\hat{\omega}) = \frac{d}{dt} \hat{T}_\lambda(\omega, t)|_{t=s} \cdot T_\lambda(\omega, s)^{-1}. \tag{3.6}$$

Then Q is well defined, as one easily verifies, in fact using the flatness of f at the edges $Q([\omega, 0]) = 0 = Q([\omega, 1])$ for each $\omega \in \Omega$. Furthermore Q is measurable in $\hat{\omega}$, and the map $t \to Q(\tau_t(\hat{\omega}))$ is continuous for all $\hat{\omega} \in \hat{\Omega}$.

Define $\Phi_\lambda(\hat{\omega}, t)$ to be the fundamental matrix solution of the differential equation

$$x' = Q(\tau_t(\hat{\omega}))\, x \tag{3.7}$$

which satisfies $\Phi_\lambda(\hat{\omega}, 0) = I$. Then Φ_λ automatically satisfies the cocycle condition (3.3). Furthermore, if $\hat{\omega} = [\omega, 0] \in \hat{\Omega}$, then

$$\Phi_\lambda(\hat{\omega}, t) = \hat{T}_\lambda(\omega, t) \qquad (0 \leqslant t \leqslant 1), \tag{3.8}$$

and so $\Phi_\lambda(\hat{\omega}, 1) = T_z(\omega)$ if $\lambda = -i \log z$. It follows that

$$\Phi_\lambda(\hat{\omega}, n) = \Delta_z(\omega, n) \qquad (n \in \mathbb{Z}) \tag{3.9}$$

for each point $\hat{\omega} \in \hat{\Omega}$ of the form $\hat{\omega} = [\omega, 0]$. Thus Φ_λ interpolates Δ_z in a natural way.

The real cocycle $\Phi_\lambda\colon \hat{\Omega} \times \mathbb{R} \to GL(2, \mathbb{C})$ just defined is the suspension of $T(z, n)$, or more precisely of the integer cocycle Δ_z. We now observe that if $|z| = 1$, then $\Phi_\lambda(\hat{\omega}, t)$ preserves the indefinite form B on $\mathbb{C}^2$ defined by

$$B(u, v) = \langle u, Jv \rangle \qquad (u, v \in \mathbb{C}^2),$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product and

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

That is,

$$\langle \Phi_\lambda(\hat{\omega}, t)\, u, J\Phi_\lambda(\hat{\omega}, t)\, v \rangle = \langle u, Jv \rangle$$

if $\lambda \in \mathbb{R}$. This can be checked as follows: if $|z| = 1$ (i.e., if $\lambda \in \mathbb{R}$), then (3.5) implies that $\hat{T}_\lambda(\omega, t)$ preserves B, and it follows from (3.6) that $\Phi_\lambda(\hat{\omega}, t)$ preserves B as well. In group-theoretic language, $\Phi_\lambda(\hat{\omega}, t)$ belongs to the group $U(1, 1)$ when $\lambda \in \mathbb{R}$.

Finally we note that the random family of differential equations can be written out explicitly. In fact, let $\hat{\omega} = [\omega, s] \in \hat{\Omega}$ where $0 \leqslant s < 1$, and define

$$\begin{aligned} \beta(\hat{\omega}) &= c_0^2(s) \begin{pmatrix} 0 & if'(s)\, \alpha_1(\omega) \\ -if'(s)\, \overline{\alpha_1(\omega)} & 0 \end{pmatrix} \\ \Gamma(\hat{\omega}) &= c_0^2(s) \begin{pmatrix} -f'(s) & f(s)\, f'(s)\, \alpha_1(\omega) \\ f(s)\, f'(s)\, \overline{\alpha_1(\omega)} & -f^2(s)\, f'(s)\, |\alpha_1(\omega)|^2 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Using (3.6), one can calculate that $\Phi_\lambda(\hat{\omega}, t)$ is the fundamental matrix solution of the differential equation

$$Ju' = [\beta(\tau_t(\hat{\omega})) + \lambda\Gamma(\tau_t(\hat{\omega}))]\, u \qquad (3.11)_{\hat{\omega}}$$

satisfying $\Phi_\lambda(\hat{\omega}, 0) = I$ $(\hat{\omega} \in \hat{\Omega})$. It is easily seen that $t \to \Gamma(\tau_t(\hat{\omega}))$ is negative semi-definite when $f(s)\, f'(s) > 0$.

Thus we can study the behavior of $T_z(\omega, n)$, and ultimately the corresponding orthogonal polynomials, by studying the solutions of the differential equations $(3.11)_{\hat{\omega}}$.

4. THE m-FUNCTIONS AND THE FLOQUET EXPONENT

In this section, we will apply methods of Johnson [15] and Johnson and Nerurkar [18] to the differential equations $(3.11)_{\hat{\omega}}$. We summarize the facts which form the starting point of this section. The cocycle $\Phi_\lambda(\hat{\omega}, t)$ is the fundamental matrix solution of

$$J\frac{du}{dt} = (\beta(\tau_t(\hat{\omega})) + \lambda\Gamma(\tau_t(\hat{\omega}))\, u \qquad (4.1)_{\hat{\omega}}$$

where $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the matrices β, Γ are given in (3.10). The quantity λ equals $-i \log z$, hence is real exactly when $|z| = 1$. If $\lambda \in \mathbb{R}$, then $\Phi_\lambda(\hat{\omega}, t) \in U(1, 1)$, the group of 2×2 complex matrices preserving the form

$$B(u, v) = \langle u, Jv \rangle \qquad (u, v \in \mathbb{C}^2).$$

In applying the theory of [15], and [18] it is convenient to make the following

4.1. Assumption. *The coefficient function* $\alpha_1(\omega) = g(\omega)$ *satisfies*

$$\|g\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} |g(\omega)| < 1,$$

and there exist numbers $\varepsilon, T > 0$ *such that, for each* $\omega \in \Omega$, *every interval* $[n_0 + 1, n_0 + T] \subset \mathbb{Z}$ *of length* T *contains a number* n *such that* $|\alpha_1(s^n(\omega))| \geqslant \varepsilon$.

As we noted in Section 1, all of our results except the gap-labelling result 5.6 can be formulated and proved under the weaker assumption

$$\int_\Omega \log(1 - |\alpha_1(\omega)|)\, \mu(d\omega) > -\infty.$$

We make Assumption 4.1 because it allows for a direct application of the theory of [15], and [18] and because we can prove our gap-labelling result (2).

Because of 4.1, the functions

$$t \to \beta(\tau_t(\hat{\omega}))$$
$$t \to \Gamma(\tau_t(\hat{\omega}))$$

are uniformly bounded and uniformly continuous functions of t, as can easily be verified. We may thus compactify $\hat{\Omega}$. The details of the construction are carried out in [16] and we outline them here. Each $\hat{\omega} \in \hat{\Omega}$ defines a function of t by the formula $t \to (\beta(\tau_t(\hat{\omega}), \Gamma(\tau_t(\hat{\omega}))$. This function takes values in the set $H = L^\infty(\mathbb{R}, \text{Her}) \times L^\infty(\mathbb{R}, \text{Her})$ where Her is the set of hermitian 2×2 complex matrices. There is a flow $\tau^{(1)}$ defined on H by translation:

$$\tau_t^{(1)}(g, \gamma)(s) = (g(t+s), \gamma(t+s)) \qquad (t, s \in \mathbb{R}).$$

The measure $\hat{\mu}$ on $\hat{\Omega}$ induces a measure $\hat{\mu}_1$ on H in a natural way [16]. We agree to identify $\hat{\mu}$ with $\hat{\mu}_1$ and $\hat{\Omega}_1$ with the topological support of $\hat{\mu}_1$ in H (the topological support is compact).

Summing up, we agree to identify $(\hat{\Omega}, \hat{\mu})$ with the space $\hat{\Omega}_1$ and Radon measure $\hat{\mu}_1$ on $\hat{\Omega}_1$ constructed above. We note that the functions β and Γ in $(4.1)_{\hat{\omega}}$ correspond to *continuous* matrix-valued functions β_1 and Γ_1 on $\hat{\Omega}_1$ via the above correspondence. We drop the subscripts in what follows.

The next step is to define the Weyl m-functions. We will do this in two equivalent ways and show that the results obtained coincide.

It is convenient to take a geometric point of view. Let $\mathbb{P}^1_{\mathbb{C}}$ be the usual complex one-dimensional projective space of (complex) lines through the origin in $\mathbb{C}^2$. We can identify $\mathbb{P}^1_{\mathbb{C}}$ with the usual Riemann complex number sphere in the following way: let $(a, b) \in \mathbb{C}^2$ be a non-zero vector, and consider the line $l = \{(ca, cb) \mid c \in \mathbb{C}\}$ which contains (a, b). Then $m = b/a$ parametrizes l, and each m in the Riemann sphere parametrizes exactly one line $l \in \mathbb{P}^1_{\mathbb{C}}$. We will use this parametrization m of $\mathbb{P}^1_{\mathbb{C}}$ in our subsequent considerations.

Fix $\lambda = -i \log z \in \mathbb{C}$ and $\hat{\omega} \in \hat{\Omega}$. The cocycle $\Phi_\lambda(\hat{\omega}, t)$ induces a one-parameter group of homeomorphisms $\{\tilde{\tau}_t \mid t \in \mathbb{R}\}$ of $\hat{\Omega} \times \mathbb{P}^1_{\mathbb{C}}$,

$$\tilde{\tau}_t(\hat{\omega}, l) = (\tau_t(\hat{\omega}), \Phi_\lambda(\hat{\omega}, t) \cdot l),$$

where $\Phi_\lambda(\hat{\omega}, t) \cdot l$ is the image of the complex line $l \subset \mathbb{C}^2$ under the linear map $\Phi_\lambda(\hat{\omega}, t)\colon \mathbb{C}^2 \to \mathbb{C}^2$. In terms of the coordinate m, we have

$$\tilde{\tau}_t(\hat{\omega}, m) = (\tau_t(\hat{\omega}), m(t)),$$

where m satisfies a Riccati equation. Indeed, writing

$$J^{-1}[\beta(\tau_t(\hat{\omega})) + \lambda \Gamma(\tau_t(\hat{\omega}))] = \begin{pmatrix} \tilde{a}(t) & \tilde{b}(t) \\ \tilde{c}(t) & \tilde{d}(t) \end{pmatrix},$$

we have

$$m' = \tilde{c}(t) + (\tilde{d}(t) - \tilde{a}(t))\, m - \tilde{b}(t)\, m^2. \tag{4.2}$$

Alternatively, write

$$\Phi_\lambda(\hat{\omega}, t) = \begin{pmatrix} \hat{a}(t) & \hat{b}(t) \\ \hat{c}(t) & \hat{d}(t) \end{pmatrix}.$$

Then

$$m(t) = \frac{\hat{c}(t) + \hat{d}(t)\, m(0)}{\hat{a}(t) + \hat{b}(t)\, m(0)}, \tag{4.3}$$

i.e., $m(t)$ is related to $m(0)$ via a linear fractional transformation.

Next let K_0 be the unit circle in m-space:

$$K_0 = \{m \in \mathbb{P}^1_{\mathbb{C}} \mid |m| = 1\}.$$

Let $\lambda \in \mathbb{R}$. Since $\Phi_\lambda(\hat{\omega}, t) \in U(1, 1)$, one checks that

$$\Phi_\lambda(\hat{\omega}, t) \cdot K_0 = K_0 \qquad (\hat{\omega} \in \hat{\Omega},\ t \in \mathbb{R}).$$

Let $D_\pm$ be the discs $D_+ = \{m \in \mathbb{P}^1_{\mathbb{C}} \mid |m| < 1\}$, $D_- = \{m \in \mathbb{P}^1_{\mathbb{C}} \mid |m| > 1\}$.

To define the Weyl functions, let $\operatorname{Im} \lambda > 0$; i.e., $|z| < 1$. Pick t large positive, and consider the image

$$\Phi_\lambda(\tau_t(\hat{\omega}),\ -t)\ K_0 \equiv K_t \subset \mathbb{P}^2_{\mathbb{C}}.$$

Since Φ_λ acts as a linear fractional transformation, the image K_t is a circle in m-space, and it can be checked that K_t lies in D_+. Using the boundedness of β and Γ, it can further be checked that, as $t \to \infty$, K_t shrinks to a point in D_+. This point is denoted $m_+(\hat{\omega}, \lambda)$ and defines one of the Weyl functions. Of course, we have just copied the Weyl limit point construction. In an analogous way, we choose t large positive and consider the circle

$$\Phi_\lambda(\tau_{-t}(\hat{\omega}),\ \ t) \cdot K_0 \subset D_-.$$

Once again the image circle shrinks to a point as $t \to \infty$; this point is denoted $m_-(\hat{\omega}, \lambda)$. Let $H^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0\}$ be the upper half-plane. One checks that $m_\pm(\hat{\omega}, \cdot)\colon H^+ \to D_\pm$ are holomorphic. Moreover these functions are continuous in $\hat{\omega}$ for each $\lambda \in H^+$. We can define the m-functions for $\lambda \in H^- = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda < 0\}$ as well; we find that $m_\pm(\hat{\omega}, \cdot)\colon H^- \to D_\mp$ in this case.

It is important to note that we can also define the m-functions using the integer cocycle $\Delta_z(\omega, n)$ (see Section 2; recall that $\Delta_z(\omega, n)$ was defined using the transfer matrix $T(z, n)$). For this, recall that, by (3.9),

$$\Phi_\lambda(\hat{\omega}, n) = \Delta_z(\omega, n)$$

when $\hat{\omega} = [\omega, 0]$ and $n \in \mathbb{Z}$. Recalling further that $\tau_n(\hat{\omega}) = [s^n(\omega), 0]$, we have

$$\begin{aligned} \Phi_\lambda(\tau_n(\hat{\omega}),\ -n) \cdot K_0 &= \Delta_z(s^n(\omega),\ -n) \cdot K_0 \\ \Phi_\lambda(\tau_{-n}(\hat{\omega}), n) \cdot K_0 &= \Delta_z(s^{-n}(\omega), n) \cdot K_0. \end{aligned} \tag{4.4}$$

Thus $m_\pm(\hat{\omega}, \lambda) = \lim_{n \to \infty} \Delta_z(s^{\pm n}(\omega), \mp n) \cdot K_0$ when $\hat{\omega} = [\omega, 0]$ which from (2.15) and (2.16) means $\tilde{m}_+(z) = m_+(\hat{\omega}, \lambda)$ and $\tilde{m}_-(z) = 1/m_-(\hat{\omega}, \lambda)$.

There is a different way to define the m-functions. For this, we need the concept of *exponential dichotomy* (ED for short), as developed by Coppel [5], Palmer [23], Selgrade [29], and others. This concept was discussed for the difference equations (1.5) in Section 2.

DEFINITION 4.2. Equations $(4.1)_{\hat{\omega}}$ are said to have an exponential dichotomy if there are constants $C>0$, $\delta>0$ and a projection-valued function $P=P_\lambda(\hat{\omega})$: $\mathbb{C}^2\to\mathbb{C}^2$ on $\hat{\Omega}$ such that

$$\|\Phi_\lambda(\hat{\omega},t)\,P_\lambda(\hat{\omega})\,\Phi_\lambda(\hat{\omega},s)^{-1}\|\leqslant Ce^{-\delta(t-s)}\qquad (t\geqslant s)$$
$$\|\Phi_\lambda(\hat{\omega},t)(I-P_\lambda(\hat{\omega}))\,\Phi_\lambda(\hat{\omega},s)^{-1}\|\leqslant Ce^{\delta(t-s)}\qquad (t\leqslant s).$$

The following result is a special case of Theorem 3.1 of Johnson [15].

THEOREM 4.3. *Suppose* $\operatorname{Im}\lambda\neq 0$. *Then Eqs.* $(4.1)_{\hat{\omega}}$ *have an ED, and thed imension of the range of* $P_\lambda(\hat{\omega})$ *(hence the dimension of the kernel of* $P_\lambda(\hat{\omega})$*) equals one for all* $\hat{\omega}\in\hat{\Omega}$.

It is shown in [15] that the m-coordinate of the complex line range $P_\lambda(\hat{\omega})$ is just $m_+(\hat{\omega},\lambda)$, while the m-coordinate of kernel $P_\lambda(\hat{\omega})$ is $m_-(\hat{\omega},\lambda)$.

We thus see that the Weyl functions $m_\pm(\hat{\omega},\lambda)$ are defined by the hyperbolic splitting of the space of solutions of equations $(4.1)_{\hat{\omega}}$. They can thus be interpreted from a “dynamical” point of view.

Now we define the Floquet exponent $\mathrm{w}=\mathrm{w}(\lambda)$ if $\operatorname{Im}\lambda>0$. Fix $\hat{\omega}\in\hat{\Omega}$, $\lambda\in H^+$, and let $m_0(t)=m_+(\tau_t(\hat{\omega}),\lambda)$. Thus $m_0(t)$ is a solution of the Riccati equation (4.2):

$$m'=\tilde{c}+(\tilde{d}-\tilde{a})\,m-\tilde{b}m^2.$$

We explicitly write out the coefficients $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$:

$$\begin{aligned}\tilde{a}&=ic_0^2\lambda f'\\ \tilde{b}&=c_0^2\alpha_1 f'[1-i\lambda f]\\ \tilde{c}&=c_0^2\bar{\alpha}_1 f'[1+i\lambda f]\\ \tilde{d}&=-ic_0^2\lambda f^2 f'\,|\alpha_1|^2.\end{aligned}$$

In particular, if $\lambda=-i\log z$ where $z=e^{i\theta}\in K$, we have

$$\tilde{a}-\tilde{d}=i\theta c_0^2 f'[1+f^2\,|\alpha_1|^2],\tag{4.5}$$

so that $\tilde{a}-\tilde{d}$ is pure imaginary when $\lambda\in\mathbb{R}$.

We linearize (4.2) around the solution $m_0(t)=m_+(\tau_t(\hat{\omega}),\lambda)$, and obtain

$$(\delta m)'=[(\tilde{d}-\tilde{a})-2\tilde{b}m_0(t)]\,\delta m.$$

DEFINITION 4.4. *The* Floquet exponent $\mathrm{w}(\lambda)$ *is*

$$\mathrm{w}(\lambda) = \int_{\hat{\Omega}} [\tilde{d}(\hat{\omega}) - \tilde{a}(\hat{\omega}) - 2\tilde{b}(\hat{\omega})\, m_+(\hat{\omega}, \lambda)]\, \hat{\mu}(d\hat{\omega}). \tag{4.6}$$

Thus $\mathrm{w}(\lambda)$ is the space-average of the coefficient in the variational equation for δm. Using the Birkhoff ergodic theorem, we have

$$\mathrm{w}(\lambda) = \lim_{T \to \infty} \frac{1}{T} \int_0^T [\tilde{d}(\tau_t(\hat{\omega})) - \tilde{a}(\tau_t(\hat{\omega})) - 2\tilde{b}(\tau_t(\hat{\omega}))\, m_+(\tau_t(\hat{\omega}), \lambda)]\, dt \tag{4.7}$$

for $\hat{\mu}$-a.a. $\hat{\omega} \in \hat{\Omega}$.

We see that $\mathrm{Re}\, \mathrm{w}(\lambda)$ measures the average exponential rate of growth of δm, while $\mathrm{Im}\, \mathrm{w}(\lambda)$ measures the average rate of rotation "around" m_+. From Theorem 4.3 we see that the equations $(4.1)_{\hat{\omega}}$ have ED for $\mathrm{Im}\, \lambda > 0$. Hence the definition of m_+ shows that solutions move away from it at a non-negative exponential rate. Thus $\mathrm{Re}\, \mathrm{w}(\lambda) \geqslant 0$, and since $\mathrm{Re}\, \mathrm{w}(\lambda)$ is harmonic in H^+, we have

$$\mathrm{Re}\, \mathrm{w}(\lambda) > 0 \qquad (\mathrm{Im}\, \lambda > 0). \tag{4.8}$$

Next we interpret $\mathrm{Re}\, \mathrm{w}(\lambda)$ in terms of the (upper) *Lyapounov exponent* of equations $(4.1)_{\hat{\omega}}$.

PROPOSITION 4.5. *Fix $\lambda \in \mathbb{C}$. For $\hat{\mu}$-a.e. $\hat{\omega} \in \hat{\Omega}$, the limit*

$$\lim_{t \to \infty} \frac{1}{t} \ln \|\Phi_\lambda(\hat{\omega}, t)\|$$

exists and is independent of $\hat{\omega}$.

This limit (which is constant $\hat{\mu}$-a.e.) is called the *Lyapounov exponent* of equations $(4.1)_{\hat{\omega}}$. From (2.24), (3.9), and the above proposition we see that the Lyapunov exponent equals $\tilde{\gamma}(z)$.

Write $\gamma(\lambda) = \mathrm{Re}\, \mathrm{w}(\lambda)$. The relation between γ and $\tilde{\gamma}$ is given by

Observation 4.6 [12, p. 235]. $\tilde{\gamma}(z) = \gamma(\lambda) + \mathrm{Re}(i\lambda/2) = \gamma(\lambda) + \frac{1}{2}\log |z|$ if $\lambda = -i \log |z|$ and $\mathrm{Im}\, \lambda > 0$.

We take Observation 4.6 as motivation for correcting an error on p. 235 of [12]. There an assertion is made which implies that the Lyapunov exponent of Eqs. $(4.1)_{\hat{\omega}}$ for $\mathrm{Im}\, \lambda > 0$ equals $\gamma(\lambda)$. However formula (26) on the same p. 235 of [12] shows that the correct relation is $\hat{\gamma} = \mathrm{Re}(\mathrm{w} + \mathrm{M})$, where M is the mean value of the trace of the matrix $\left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ of (4.2). Now

$$\tilde{a} + \tilde{d} = \frac{i\lambda}{2} f', \qquad 0 \leqslant t < 1$$

if $\hat{\omega} = [\omega, t] \in \hat{\Omega}$ is a point such that $0 \leqslant t < 1$. The mean value of the trace can then be computed as

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t [\tilde{a}(s) + \tilde{d}(s)]\, ds = \frac{i\lambda}{2} \lim_{t \to \infty} \frac{[t] + f(t - [t])}{t} = \frac{i\lambda}{2},$$

where $[t]$ denotes the integer part of t. This hold for $\hat{\omega} = [\omega, 0] \in \hat{\Omega}$, for a.a $\omega \in \Omega$. Hence we obtain 4.6.

Now we discuss the *rotation number* for Eqs. $(4.1)_{\hat{\omega}}$ when λ is real. As noted earlier, $\Phi_\lambda(\hat{\omega}, t) \cdot K_0 = K_0$ where $K_0 = \{m \in \mathbb{P}^2_{\mathbb{C}} \mid |m| = 1\}$. Let

$$\psi = \operatorname{Arg} m,$$

so that ψ is an angular coordinate on K_0. If $m_0 \in K_0$, let $m(t)$ be the solution of the Riccati equation (4.2) satisfying $m(0) = m_0$, and let $\psi(t) = \operatorname{Arg} m(t)$ be a continuous determination of the argument. Define

$$\rho(\lambda) = \lim_{t \to \infty} \frac{1}{2} \frac{\psi(t)}{t}. \tag{4.9}$$

It can be shown (Johnson and Moser [17]) that, for fixed $\lambda \in \mathbb{R}$, the limit in (4.9) exists for $\hat{\mu}$-almost all $\hat{\omega}$ and is constant $\hat{\mu}$-a.e.

THEOREM 4.7 (Johnson and Moser [17]). *The function $\lambda \to \rho(\lambda)$ is continuous and monotone non-increasing. Furthermore, if $\lambda \in \mathbb{R}$, then*

$$\lim_{\varepsilon \to 0^+} \operatorname{Re} \mathrm{w}(\lambda + i\varepsilon) = \gamma(\lambda) \qquad \textit{Lebesgue a.e.};$$

$$\lim_{\varepsilon \to 0^+} \operatorname{Im} \mathrm{w}(\lambda + i\varepsilon) = \rho(\lambda) \qquad \textit{for all} \quad \lambda \in \mathbb{R}.$$

It can actually be shown that $\operatorname{Re} \mathrm{w}(\lambda + i\varepsilon) \to \gamma(\lambda)$ for all $\lambda \in \mathbb{R}$, though one needs a more sophisticated argument than that given in Johnson [15].

We turn to the spectral theory of the Eqs. $(4.1)_{\hat{\omega}}$. Write $u = \binom{u_1}{u_2}$, and consider the boundary value problem

$$\begin{gathered} \left[J \frac{d}{dt} - \beta(\tau_t(\hat{\omega})) \right] u = \lambda \Gamma(\tau_t(\hat{\omega}))\, u \\ u_2(-a) = u_2(a) = 0. \end{gathered} \tag*{$(4.10)_{\hat{\omega}}$}$$

Here a is a fixed positive real number. Let $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$ be an enumeration of the equivalues of $(4.10)_{\hat{\omega}}$, and let $T(\lambda)$ be the corresponding 2×2 *spectral matrix*. Thus $T(0) = 0$, $T(\lambda)$ is hermitian nondecreasing for real λ, and

$$T(\lambda_i^+) - T(\lambda_i^-) = e_i(0)\, e_i^*(0)$$

where * means adjoint and $e_i(t)$ is an eigenvector of $(4.10)_{\hat{\omega}}$, normalized so that

$$\int_{-a}^{a} \langle e_i(t), \Gamma(\tau_t(\hat{\omega}))\, e_i(t)\rangle = 1.$$

Note that $e_i e_i^*$ is a 2×2 matrix because e_i is a column vector, and note also that $e_i(t)$ is unique up to a complex factor of modulus 1.

Following Johnson [15] (which starts from [1]), we introduce the "characteristic function"

$$F(\hat{\omega}, \lambda) = (Q(\hat{\omega}, \lambda) - \tfrac{1}{2} I)\, J^{-1}, \tag{4.11}$$

where $Q(\hat{\omega}, \lambda)$ is the projection on $\mathbb{C}^2$ whose range is the complex line $m_+(\hat{\omega}, \lambda)$ and whose kernel is the complex line $m_-(\hat{\omega}, \lambda)$. Explicitly: if $N = \left(\begin{smallmatrix} 1 & 1 \\ m_+ & m_- \end{smallmatrix}\right)$, then $F(\hat{\omega}, \lambda) = (N \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) N^{-1} - \tfrac{1}{2} I)\, J^{-1}$, or

$$F(\hat{\omega}, \lambda) = \frac{i}{m_+ - m_-} \begin{pmatrix} m_- & 1 \\ m_+ m_- & m_+ \end{pmatrix} + \begin{pmatrix} +i/2 & 0 \\ 0 & -i/2 \end{pmatrix}.$$

We have [15]

$$-Tr\, \frac{\operatorname{Im} F(\hat{\omega}, \lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} Tr\, \frac{dT_{\hat{\omega}}(t)}{|t - \lambda|^2}. \tag{4.12}$$

A proof of (4.12) will be given at the end of this section for completeness.

The following formula is fundamental [15, Eq. 28]:

$$\frac{dw}{d\lambda} = -\int_{\hat{\Omega}} Tr\{F(\hat{\omega}, \lambda)\, \Gamma(\hat{\omega})\}\, d\hat{\mu}(\hat{\omega}) \qquad (\operatorname{Im} \lambda > 0). \tag{4.13}$$

Using (4.13) together with Theorem 4.7 we obtain another basic relation (see Johnson [15]),

$$\frac{1}{\pi} \int_I d\rho(t) = \int_{\hat{\Omega}} \left(\int_I Tr\{dT_{\hat{\omega}}(t)\, \Gamma(\hat{\omega})\} \right) d\hat{\mu}(\hat{\omega}) \tag{4.14}$$

for each finite interval $I \subset \mathbb{R}$.

We will use a final result, proved in the generality needed here in [18] (see also Theorem 2.6). Recall that $\hat{\Omega}$ is the topological support of the measure $\hat{\mu}$.

THEOREM 4.8. *Let $I = (a, b)$ be a finite open interval in $\mathbb{R}$. Equations $(4.10)_{\hat{\omega}}$ have an exponential dichotomy for all $\lambda \in I$ if and only if the rotation number $\rho(\cdot)$ is constant on I, i.e., if and only if*

$$\rho(a) = \rho(b).$$

We finish this section with a proof of formula (4.12). First of all, it follows from [1, Ch. 9] and standard arguments that $\lambda \to -Tr\, F(\hat{\omega}, \lambda)$ is holomorphic in $H^+ = \{\lambda \mid \operatorname{Im} \lambda > 0\}$ with positive imaginary part, and has the representation

$$-Tr\, F(\hat{\omega}, \lambda) = a(\hat{\omega}) + b(\hat{\omega})\, \lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) Tr\, dT_{\hat{\omega}}(t),$$

for real functions $a(\hat{\omega})$ and $b(\hat{\omega}) \geqslant 0$. Formula (4.12) is equivalent to the statement $b(\hat{\omega}) = 0$, hence we show that indeed $b(\hat{\omega}) = 0$ for all $\hat{\omega} \in \hat{\Omega}$.

For this, we use the Floquet exponent $\mathrm{w}(\lambda)$. First of all, by (4.13) (whose proof does not use (4.12); see Johnson [15]) we have $\operatorname{Im} \mathrm{w}'(\lambda) < 0$ if $\lambda \in H^+$. Furthermore, by (4.8), $i\mathrm{w}(\lambda)$ has positive imaginary part for $\lambda \in H^+$. From basic theory of functions holomorphic in H^+ with positive imaginary part, we have

$$-\mathrm{w}'(\lambda) = a_0 + b_0 \lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\rho(t), \tag{4.15}$$

where $a_0 = \int_{\hat{\Omega}} a(\hat{\omega})\, d\hat{\mu}(\hat{\omega})$ and $b_0 = \int_{\hat{\Omega}} b(\hat{\omega})\, d\hat{\mu}(\hat{\omega})$. It is easy to check that the integral in (4.15) is $o(\lambda)$ in any closed subsector of H^+. Hence integrating (4.15), we get

$$\mathrm{w}(\lambda) = \tilde{w} - a_0 \lambda - \frac{b_0}{2} \lambda^2 + o(\lambda^2),$$

where $\tilde{w}$ is a constant. But $i\mathrm{w}(\lambda)$ has a representation analogous to (4.15), hence $\mathrm{w}(\lambda) = O(\lambda)$ in closed subsectors of H^+. We conclude that $b_0 = 0$ everywhere on the topological support of $\hat{\mu}$ because b is certainly continuous. This proves (4.12).

5. RESULTS

We begin with a preliminary result which is a corollary of (4.14) and Theorem 4.8. Define the measure

$$d\theta_{\hat{\omega}}(\lambda) = Tr\, dT_{\hat{\omega}}(\lambda) \qquad (\lambda \in \mathbb{R}),$$

where $dT_{\hat{\omega}}(\cdot)$ is the spectral matrix of $(4.10)_{\hat{\omega}}$. The *support* of $d\theta_{\hat{\omega}}$ is the complement of the largest open set $A \subset \mathbb{R}$ satisfying $\int_A d\theta_{\hat{\omega}}(\lambda) = 0$. Write $\Theta_{\hat{\omega}}$ for the support of $d\theta_{\hat{\omega}}$.

THEOREM 5.1. *There is a set $\hat{\Omega}_0 \subset \hat{\Omega}$ satisfying $\hat{\mu}(\hat{\Omega} - \hat{\Omega}_0) = 0$ such that, if $\hat{\omega} \in \hat{\Omega}_0$, then*

$$\mathbb{R} - \Theta_{\hat{\omega}} = \{\lambda \in \mathbb{R} \mid \textit{equations } (4.10)_{\hat{\omega}} \textit{ have an ED}\} \equiv E.$$

In particular, $\Theta_{\hat{\omega}}$ is independent of $\hat{\omega} \in \hat{\Omega}_0$.

Proof. Let $I \subset \mathbb{R}$ be an open interval on which Eqs. $(4.10)_{\hat{\omega}}$ have an ED. By (4.14) and Theorem 4.8, $\int_I d\theta_{\hat{\omega}}(\lambda) = 0$ for $\hat{\mu}$-a.a. $\hat{\omega}$, and it follows that there is a set $\hat{\Omega}_1 \subset \hat{\Omega}$ satisfying $\hat{\mu}(\hat{\Omega} - \hat{\Omega}_1) = 0$ with the property that, if $\hat{\omega} \in \hat{\Omega}_1$, then $E \subset \mathbb{R} - \Theta_{\hat{\omega}}$.

We prove that $\mathbb{R} - \Theta_{\hat{\omega}} \subset E$ for $\hat{\mu}$-a.a. $\hat{\omega}$. Let $\hat{\omega}_1 \in \hat{\Omega}_1$ be a point with dense orbit; that is $\mathrm{cls}\{\tau_t(\hat{\omega}) \mid t \in \mathbb{R}\} = \hat{\Omega}$. Since $\mathrm{Supp}\,\hat{\mu} = \hat{\Omega}$, almost all points in $\hat{\Omega}$ have this property. It is easy to check that $\Theta_{\tau_t(\hat{\omega}_1)} = \Theta_{\hat{\omega}_1}$ for all $t \in \mathbb{R}$ (for example see the proof of Thm. 2.6). Using weak-* continuity in $\hat{\omega}$ of the measures $d\theta_{\hat{\omega}}$, we see that

$$\mathbb{R} - \Theta_{\hat{\omega}_1} \subset \mathbb{R} - \Theta_{\hat{\omega}}$$

for *all* $\hat{\omega} \in \hat{\Omega}$.

Now, if $E \subsetneqq \mathbb{R} - \Theta_{\hat{\omega}_1}$, then there is an open nonempty interval $I \subset \mathbb{R} - \Theta_{\hat{\omega}}$ for *all* $\hat{\omega} \in \hat{\Omega}$ which satisfies $I \cap E = \emptyset$. But now we have a contradiction with (4.14) and Theorem 4.8.

We can restate Theorem 5.1 as follows: for $\hat{\mu}$-a.e. $\hat{\omega}$, the support of the spectral measure $d\theta_{\hat{\omega}}$ equals the support of the non-negative measure $-d\rho$ (see Theorem 4.8). By continuity of $\rho(\cdot)$ (Theorem 4.7), the support of $-d\rho$ has no isolated points, hence we have

COROLLARY 5.2. *The support $\Theta_{\hat{\omega}}$ has no isolated points for $\hat{\mu}$-a.a. $\hat{\omega} \in \hat{\Omega}$.*

We now consider the relation between the measures $d\theta_{\hat{\omega}}(\lambda)$ and the orthogonality measures $d\sigma_\omega(z)$. We will need the following.

Remark 5.3. Let $\hat{\Omega}_1 \subset \hat{\Omega}$ be a set of full $\hat{\mu}$-measure which is *invariant*: if $\hat{\omega} \in \hat{\Omega}_1$ and $t \in \mathbb{R}$ then $\tau_t(\hat{\omega}) \in \hat{\Omega}_1$. Let $\Omega_1 = \{\omega \in \Omega \mid [\omega, 0] \in \hat{\Omega}_1\}$. Then by ergodicity $\mu(\Omega_1) = 1$.

Let $\omega \in \Omega$, and identify ω with $[\omega, 0] \in \hat{\Omega}$ for the moment. Recall that, by (4.3) and (4.8), $m_\pm(\omega, \lambda)$ are 2π-periodic in λ, hence define functions of $z = e^{-i\lambda}$. We will write $m_\pm(\omega, z)$ for these functions. It follows from (4.12) that $d\theta_\omega(\lambda)$ is also 2π-periodic, and thus we can write $d\theta_\omega = d\theta_\omega(z)$ where now $z \in K = \{z \in \mathbb{C} \mid |z| = 1\}$.

Our goal is to show that, for μ-a.a. ω, the set Θ_ω coincides with the set of *non-isolated* points of increase of σ_ω. We call this latter set Σ_ω.

The first step is to return to the definition of the coordinate m. Recall that $m=1$ parametrizes the complex line containing $\binom{1}{1}$. And, the orthogonal polynomials $\phi_n(z)$ arise by substituting the initial condition $\psi(z,0)=\binom{1}{1}$ in equation (1.1). With an eye to [1, Chap. 7, Sect. 6], we wish to define a function holomorphic in $|z|<1$, with values in the upper half-plane, having poles when $m=1$. Such a function is

$$\underline{m}(\omega, z) = i\,\frac{1+m_+(\omega, z)}{1-m_+(\omega, z)}. \tag{5.1}$$

Compare now with Atkinson: we see that, except for a factor of $-i$, the function in (5.1) coincides with the "characteristic function" of [1, p. 188]. As in [1, Sect. 7.6] or [11, Eq. 6.4], we see that

$$\underline{m}(\omega, z) = i\int_K \frac{z+\rho}{\rho-z}\,d\sigma_\omega(\rho). \tag{5.2}$$

This gives us a relation between the Weyl function $m_+(\omega, z)$ and the orthogonality measure $d\sigma_\omega(z)$.

To begin the second step, choose $\omega_0\in\Omega$ such that the orbit $\{\tau_t(\hat{\omega}_0)\mid t\in\mathbb{R}\}$ of $\hat{\omega}_0=[\omega_0, 0]$ is dense in $\hat{\Omega}$ (see Remark 5.3). Suppose that $\sigma_{\omega_0}(z)$ has only isolated discontinuities on an open interval $I\subset K$. Using (5.1) and (5.2), we see that $m_+(\omega_0, z)$ extends holomorphically across I and that $|m_+(\omega_0, z)|=1$ if $z\in I$. The geometric significance of these facts is that (roughly speaking) "if σ_{ω_0} has l jumps in I, then $m_+(\omega_0, I)$ covers K l times." We leave it to the reader to make this precise, and remark that the above statement and (3.3) imply that $m_+(\tau_t(\hat{\omega}_0), z)$ also extends holomorphically across I for all $t\in\mathbb{R}$. Using continuity in $\hat{\omega}$ of $m_+(\hat{\omega}, z)$ and the density of $\{\tau_t(\hat{\omega}_0)\mid t\in\mathbb{R}\}$, we can find a fixed open set D_0 in the z-plane which contains I such that $z\to m_+(\hat{\omega}, z)$ is holomorphic in D_0 for each $\hat{\omega}\in\hat{\Omega}$, and such that $|m_+(\hat{\omega}, z)|=1$ if $z\in I$, $\hat{\omega}\in\hat{\Omega}$.

The third step is to use Definition 4.4 to see that the Floquet exponent $\mathrm{w}(\lambda)$ extends holomorphically across each interval $\tilde{I}=\{\lambda=-i\log z\mid z\in I\}$. Hence the quantity $\gamma(z)=\operatorname{Re}\mathrm{w}(-i\log z)$ extends harmonically through I. Since $\gamma(z)>0$ for $|z|<1$, the zeroes of γ on I are isolated unless γ vanishes on I. We will see later, in our Kotani-type result 5.9 (whose proof is independent of the present considerations) that in the latter case $|m_+(\hat{\omega}, z)|<1$ for $z\in I$. Thus the zeroes of γ on I are isolated, and $\gamma>0$ except at the zeroes.

Suppose $|z|<1$. The quantity $m_+(\hat{\omega}, z)$ parametrizes the unique complex line $l_{\hat{\omega}}\subset\mathbb{C}^2$ with the following property: if $0\neq u_0\in l_\omega$ and $u(t)$ is the solution of $(4.10)_{\hat{\omega}}$ satisfying $u(0)=u_0$, then $u(t)\to 0$ exponentially as $t\to\infty$.

Now suppose that $z \in I$, so that in particular $|z| = 1$. Suppose that $\gamma(z) > 0$. Then using Observation 4.6, one can show that, if $l_{\hat{\omega}}$ is the complex line parameterized by $m_+(\hat{\omega}, z)$ and if $0 \neq u_0 \in l_{\hat{\omega}}$, then

$$-\gamma(z) = \lim_{t \to \infty} \frac{1}{t} \ln \|u(t)\|.$$

Thus if $\gamma(z) > 0$, then there is a *continuous* family $\{l_{\hat{\omega}} \mid \hat{\omega} \in \hat{\Omega}\}$ of lines giving rise to exponentially decaying solutions of Eqs. $(4.10)_{\hat{\omega}}$. The continuity in $\hat{\omega}$ implies, via Oseledec theory (Oseledec [22], Johnson *et al.* [19]) that Eqs. $(4.10)_{\hat{\omega}}$ have ED if $z \in I$, $\lambda = -i \log z$, and $\gamma(z) > 0$.

So far we have shown that $K - \Sigma_{\omega 0}$ consists (except perhaps for a discrete set) of points z such that, if $\lambda = -i \log z$, then Eqs. $(4.10)_{\hat{\omega}}$ have an ED. By Theorem 4.8 and continuity of the rotation number, the discrete set is empty. Let us slightly redefine E to be $E = \{z \in K \mid \text{if } \lambda = -i \log z, \text{ then Eqs. } (4.10)_{\hat{\omega}} \text{ have an ED}\}$. Then we can conclude that, for μ-a.a. ω, $K - \Sigma_\omega \subset E$.

Suppose on the other hand that $I \subset E$ is an interval. It is then easy to see that $m_+(\hat{\omega}, z)$ is holomorphic across I for all $\hat{\omega} \in \hat{\Omega}$ (e.g., De Concini and Johnson [7]). So we can conclude that

$$K - \Sigma_\omega = E \qquad (\mu\text{-a.a. } \omega \in \Omega).$$

Finally, note that the set $\hat{\Omega}_1$ of Theorem 5.1 is clearly invariant. If $\hat{\omega} = [\omega, 0] \in \hat{\Omega}$, define $\Theta_\omega = \{z \in K \mid \lambda = i \log z \in \Theta_{\hat{\omega}}\}$. Using Remark 5.3, we get

THEOREM 5.4. *For μ-a.a. $\omega \in \Omega$:*

$$\Theta_\omega = K - E = \Sigma_\omega.$$

In words, the spectrum of $d\theta_\omega$ equals the complement of the dichotomy set equals the essential spectrum of $d\sigma_\omega$.

We remark that one can prove as in, e.g., De Concini and Johnson [7], that each subinterval of E contains at most one increase point of σ_ω.

By Theorems 4.8 and 5.4, we have

COROLLARY 5.5. *For μ-a.a. ω, the set Σ_ω equals the set of increase points of $\rho = \rho(-i \log z)$.*

Let us write Σ for the common value of Σ_ω for μ-a.a. ω. Then Σ is a closed subset of K, and in general its complement $K - \Sigma$ may contain an infinite number of intervals (or gaps). Our next result shows how to label these gaps.

THEOREM 5.6 (Gap-labelling). *There is a countable subgroup N of $\mathbb{R}$, which depends only on the topology of Ω, with the following property. If I is a subinterval of $K-\Sigma$, then $2\rho(\lambda)\in N$ for $\lambda\in I$.*

Proof. We first introduce the *Schwartzmann homomorphism h*, which maps the first Čech cohomology group $\check{H}^1(\hat{\Omega}, \mathbb{R})$ with real coefficients into $\mathbb{R}$. The map h is defined as follows (Schwartzmann [28]). We can represent an element c of $\check{H}^1(\hat{\Omega}, \mathbb{R})$ by a continuous map $f_c\colon \hat{\Omega}\to K$. Letting Arg denote the argument function on K, define

$$h(c)=\lim_{t\to\infty}\frac{\operatorname{Arg} f_c(\tau_t(\hat{\omega}))}{t}.$$

By Schwartzmann [28], the limit exists and depends only on c for $\hat{\mu}$-a.a. $\hat{\omega}\in\hat{\Omega}$, and defines a group homomorphism of $\check{H}^1(\hat{\Omega}, \mathbb{R})$ into $\mathbb{R}$. Let N be the image of h.

Next let $z_0\in K-\Sigma$, so that Eqs. $(4.10)_{\hat{\omega}}$ have an ED if $\lambda=-i\log z_0$ (Theorem 5.4). From the arguments used in proving Theorem 5.4, we see that $m_+(\hat{\omega}, z_0)\in K$ for each $\hat{\omega}\in\hat{\Omega}$. So the map

$$c_0\colon \hat{\Omega}\to K\colon \hat{\omega}\to m_+(\hat{\omega}, z_0)$$

defines an element of $\check{H}^1(\hat{\Omega}, \mathbb{R})$. By (4.5), we have

$$2\rho(\lambda)=h(c_0)\in N.$$

This completes the proof of Theorem 5.6.

Remark 5.7. (1) Note that, if $c([\omega, t])=e^{2\pi it}$ $(0\leqslant t\leqslant 1, \omega\in\Omega)$, then $c\colon \hat{\Omega}\to K$ defines an element of $\check{H}^1(\hat{\Omega}, \mathbb{R})$, and $h(c)=2\pi$. Thus $2\pi\mathbb{Z}\subset N$.

(2) It can be checked directly that $\rho(\lambda+2\pi)=\rho(\lambda)-\pi$. It is thus natural to regard 2ρ as a map from K into $N_0=N/2\pi\mathbb{Z}$. One can think of N_0 as "the portion of N determined by Ω."

The next theorem gives a simple criterion for determining the *absence* of the absolutely continuous component $\Sigma_w^{ac}\subset\Sigma_\omega$ of the orthogonality measure $d\sigma_\omega$. We call it a Pastur–Ishii theorem because it is analogous to a theorem proved by those authors (Pastur [24], Ishii [14]) for the random Schrödinger operator.

THEOREM 5.8 (Pastur, Ishii). *If $\gamma(z)>0$ on a Borel subset $B\subset K$, then $B\cap\Sigma_w^{ac}=\varnothing$.*

Proof. The first problem is to give a precise definition of Σ_w^{ac} for $\omega\in\Omega$. We do this by referring to Eq. (5.2). Let

$$\Sigma_w^{ac}=\{z=e^{i\psi}\in K \mid \lim_{r\to 1^-} \operatorname{Im} \underline{m}(\hat{\omega}, rz)>0\}. \tag{5.3}$$

Then up to a set of Lebesgue measure zero, Σ_w^{ac} equals the set of points in K where the Radon–Nikodym derivative $d\sigma_w^{ac}/dv$ of the absolutely continuous component $d\sigma_w^{ac}$ of $d\sigma_w$ with respect to Lebesgue measure dv on K exists and is not zero.

Next consider $V=\{(\omega, z)\in \Omega\times K \mid \lim_{r\to 1^-} \underline{m}(\omega, rz) \text{ exists}\}$. For each $\omega\in\Omega$ this set is of full Lebesgue measure, so by Fubini's theorem the set V has full $\mu\times v$-measure. Thus for v-a.a. $z\in B$, there is a set $\Omega_z\subset\Omega$ of full μ-measure for which $\lim_{r\to 1^-} \underline{m}(\omega, rz)$ exists.

On the other hand, fix $z\in B$. Since $\gamma(z)>0$ and since $\mathrm{Tr}(\beta+\lambda\Gamma)$ is pure imaginary, the Oseledec theory [19, 23] implies that the solutions of Eqs. $(4.10)_{\hat{\omega}}$ (with $\lambda=-i\log z$) define a splitting of $\hat{\Omega}\times\mathbb{C}^2$ into a sum of two measurable line bundles:

$$\hat{\Omega}\times\mathbb{C}^2=\hat{W}^+\oplus\hat{W}^-.$$

The fibers $l^{\pm}(\hat{\omega})=\{x\in\mathbb{C}^2 \mid (\hat{\omega}, x)\in\hat{W}^1\}$ are defined for $\hat{\mu}$-a.a. $\hat{\omega}$, say for $\hat{\omega}\in\hat{\Omega}_1\subset\hat{\Omega}$. For each $\hat{\omega}\in\hat{\Omega}_1$, the vector $0\neq x=\binom{a}{b}\in l^{\pm}(\hat{\omega})$ has the property that $b/a\in K$. This follows from the fact that $\beta+\lambda\Gamma\in U(1,1)$. Finally, there are no invariant measurable line bundles in $\hat{\Omega}\times\mathbb{C}^2$ other than $\hat{W}^+$ and $\hat{W}^-$.

Now set $\lim_{r\to 1^-}[(\underline{m}(\omega, rz)+i)/(\underline{m}(\omega, rz)-i)]=m_1(\omega)$ for $\omega\in\Omega_z$. Further set $\Omega_1=\{\omega\in\Omega \mid [\omega, 0]\in\hat{\Omega}_1\}$; by Remark 5.3 the set Ω_1 has full μ-measure. The set $\{(\omega, m_1(\omega)\, c) \mid \in\mathbb{C}, \omega\in\Omega_z\}$ defines an invariant line bundle in $\Omega\times\mathbb{C}^2$, and it follows easily that it must coincide with $W^+=\{(\omega, x)\in\Omega\times\mathbb{C}^2 \mid ([\omega, 0], x)\in\hat{W}^+\}$. Thus $\lim_{r\to 1^-} \underline{m}(\omega, rz)\in\mathbb{R}\cup\{\infty\}$. So if $\omega\in\Omega_z\cap\Omega_1$, then $\lim_{r\to 1^-} \mathrm{Im}\,\underline{m}(\omega, rz)$ is not positive. Thus $B\cap\sum_{\omega}^{ac}=\varnothing$ and the theorem is proved.

Our final result, of Kotani type (Kotani [20]), is of a character exactly opposite to the one just proved.

THEOREM 5.9 (Kotani). *Let $B\subset K$ be a Borel set such that $\gamma(z)=0$ for each $z\in B$. Then for μ-a.a. $\omega\in\Omega$: $B-\Sigma_w^{ac}$ has Lebesgue measure zero.*

Recall that Σ_w^{ac} was defined in (5.3).

Proof. First of all, it is convenient to change variables in the spectral problem $(4.10)_{\hat{\omega}}$. Define

$$u=Av, \qquad A=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}, \qquad v\in\mathbb{C}^2.$$

Then Eqs. $(4.10)_{\hat{\omega}}$ take the form

$$J_0 v' = \left[\begin{pmatrix} a_0 & b_0 \\ b_0 & c_0 \end{pmatrix} + i\lambda \begin{pmatrix} 0 & d_0 \\ -d_0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix} \right] v, \qquad (5.4)_{\hat{\omega}}$$

where now $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the functions (of $\hat{\omega}$) a_0, b_0, c_0, d_0, γ_1, γ_2, γ_3 are all real and continuous, and $d_0 = f'(t)/2$ for $\omega \in \Omega$ and $0 \leqslant t \leqslant 1$. We can define m-functions $\underline{m}_{\pm}(\hat{\omega}, \lambda)$ for Eqs. $(5.4)_{\hat{\omega}}$ just as they were defined for Eqs. $(4.10)_{\hat{\omega}}$: we find that

$$\underline{m}_{\pm}(\hat{\omega}, z) = i\, \frac{1 + m_{\pm}(\hat{\omega}, \lambda)}{1 - m_{\pm}(\hat{\omega}, \lambda)} \qquad (5.5)$$

(compare with (5.1)). It follows that

$$\operatorname{sgn}(\operatorname{Im} m_{\pm}(\hat{\omega}, \lambda) \cdot \operatorname{Im} \lambda) = \pm 1$$

if $\operatorname{Im} \lambda \neq 0$. Moreover, $\underline{m}_{\pm}(\hat{\omega}, \cdot)$ is holomorphic in $\operatorname{Im} \lambda \neq 0$, and $\underline{m}_{\pm}(\cdot, \lambda)$ is continuous for fixed λ, $\operatorname{Im} \lambda \neq 0$.

In the rest of the proof, we will use the new m-functions $\underline{m}_{\pm}$ and make no reference to those defined for Eqs. $(4.10)_{\hat{\omega}}$. Accordingly, we drop the bar and write $m_{\pm}(\hat{\omega}, \lambda)$ for $\underline{m}_{\pm}(\hat{\omega}, \lambda)$. This abuse of notation should cause no difficulty.

The m-functions satisfy a Riccati equation, obtained by setting $m = v_2/v_1$ in Eq. $(5.4)_{\hat{\omega}}$. We obtain

$$m' = a_0 + 2b_0 m + c_0 m^2 + \lambda\{\gamma_3 m^2 + 2\gamma_2 m + \gamma_1\}. \qquad (5.6)$$

Note that the d_0-term does not appear in (5.6). Note further that, if $\lambda \in \mathbb{R}$ and $m(0)$ is real, then the solution of Eq. (5.6) is real.

We now follow the arguments in De Concini and Johnson [7], which are motivated by those of Kotani [20]. (See also Sun and Qian [30]).

First of all, note that

$$\frac{(\operatorname{Im} m)'}{\operatorname{Im} m} = 2 \operatorname{Re}\{b_0 + \lambda\gamma_2 + m(c_0 + \lambda\gamma_3)\}$$

$$+ \frac{\operatorname{Im} \lambda}{\operatorname{Im} m} \{\gamma_1 + 2\gamma_2 \operatorname{Re} m + \gamma_3[(\operatorname{Re} m)^2 + (\operatorname{Im} m)^2]\}.$$

Next, fix $\hat{\omega} \in \hat{\Omega}$ and let $m_+(t) = m_+(\hat{\omega}, t)$. If we linearize (5.6) around $m_+(t)$, we get

$$(\delta m)' = \{2b_0 + 2\lambda\gamma_2 + 2m_+(c_0 + \lambda\gamma_3)\}\, \delta m.$$

Hence

$$2\gamma(\lambda)=\int_{\hat{\Omega}} \operatorname{Re}\{2b_0+2\lambda\gamma_2+2m_+(c_0+\lambda\gamma_3)\}\ d\hat{\mu}(\hat{\omega})$$

(see Johnson [15]), and we have using the Birkhoff ergodic theorem (see (4.6) and (4.7))

$$2\gamma(\lambda)=-\operatorname{Im}\lambda\int_{\hat{\Omega}}\left\{\frac{\gamma_1+2\gamma_2\operatorname{Re}m_++\gamma_3[(\operatorname{Re}m_+)^2+(\operatorname{Im}m_+)^2]}{\operatorname{Im}m_+}\right\}d\hat{\mu}(\hat{\omega}). \tag{5.7}$$

In a similar way, replacing m_+ by m_-, we get

$$2\gamma(\lambda)=\operatorname{Im}\lambda\int_{\hat{\Omega}}\left\{\frac{\gamma_1+2\gamma_2\operatorname{Re}m_-+\gamma_3[(\operatorname{Re}m_-)^2+(\operatorname{Im}m_-)^2]}{\operatorname{Im}m_-}\right\}d\hat{\mu}(\hat{\omega}). \tag{5.8}$$

Now we use the analogue of formula (4.13) for the spectral problem $(5.4)_{\hat{\omega}}$ (with $J_0=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. This gives

$$\begin{aligned}
\operatorname{Im}\frac{d\mathrm{w}}{d\lambda}&=\operatorname{Im}\int_{\hat{\Omega}}\frac{\gamma_1+\gamma_3m_-m_++\gamma_2(m_-+m_+)}{m_--m_+}\,d\hat{\mu}(\hat{\omega})\\
&=\operatorname{Im}\int_{\hat{\Omega}}\frac{\gamma_1+\gamma_3m_-m_++\gamma_2(m_-+m_+)}{|m_--m_+|^2}(\bar{m}_--\bar{m}_+)\,d\hat{\mu}(\hat{\omega})\\
&=\int_{\hat{\Omega}}\frac{\gamma_1(\operatorname{Im}m_+-\operatorname{Im}m_-)+\gamma_3(\operatorname{Im}[m_-m_+(\bar{m}_--\bar{m}_+)]}{|m_--m_+|^2}d\hat{\mu}(\hat{\omega})\\
&\quad+\int_{\hat{\Omega}}\frac{\gamma_2\operatorname{Im}[(m_-+m_+)(\bar{m}_--\bar{m}_+)]}{|m_--m_+|^2}\,d\hat{\mu}(\hat{\omega})\\
&=\int_{\hat{\Omega}}\frac{\gamma_1(\operatorname{Im}m_+-\operatorname{Im}m_-)+\gamma_3[\,|m_-|^2\operatorname{Im}m_+-|m_+|^2\operatorname{Im}m_-]}{|m_--m_+|^2}\,d\hat{\mu}(\hat{\omega})\\
&\quad+\int_{\hat{\Omega}}\frac{\gamma_2[2\operatorname{Re}m_-\ \operatorname{Im}m_+-2\operatorname{Re}m_+\ \operatorname{Im}m_-]}{|m_--m_+|^2}\,d\hat{\mu}(\hat{\omega})\\
&=\int_{\hat{\Omega}}\frac{\operatorname{Im}m_+[\gamma_1+\gamma_3\,|m_-|^2+2\gamma_2\operatorname{Re}m_-]}{|m_--m_+|^2}\,d\hat{\mu}(\hat{\omega})\\
&\quad-\int_{\hat{\Omega}}\frac{\operatorname{Im}m_-[\gamma_1+\gamma_3\,|m_+|^2+2\gamma_2\operatorname{Re}m_+]}{|m_--m_+|^2}\,d\hat{\mu}(\hat{\omega}).
\end{aligned} \tag{5.9}$$

Next write

$$Q(m) = \gamma_1 + 2\gamma_2 \operatorname{Re} m + \gamma_3 |m|^2 \qquad (m \in \mathbb{C}),$$

and note that

$$Q(m) = [\gamma_1 + 2\gamma_2 \operatorname{Re} m + \gamma_3 (\operatorname{Re} m)^2] + \gamma_3 (\operatorname{Im} m)^2 \qquad (m \in \mathbb{C}).$$

Since the $\left(\begin{smallmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{smallmatrix}\right)$matrix is negative semi-definite (and negative definite if $f(t)\ f'(t) \neq 0$), we have

$$[\gamma_1 + 2\gamma_2 \operatorname{Re} m + \gamma_3 (\operatorname{Re} m)^2] \leqslant 0, \qquad Q(m) \leqslant 0. \tag{5.10}$$

We now prove:

Lemma 5.10. *If* $\operatorname{Im} \lambda > 0$ *there holds*

$$\begin{aligned} -4 &\left(\frac{\operatorname{Re} \mathrm{w}}{\operatorname{Im} \lambda} + \operatorname{Im} \frac{d\mathrm{w}}{d\lambda} \right) \\ &= \int_{\hat{\Omega}} \left\{ \frac{Q(m_+)}{\operatorname{Im} m_+} - \frac{Q(m_-)}{\operatorname{Im} m_-} \right\} \\ &\quad \times \left\{ \frac{(\operatorname{Re} m_- - \operatorname{Re} m_+)^2 + (\operatorname{Im} m_- + \operatorname{Im} m_+)^2}{|m_- - m_+|^2} \right\} d\hat{\mu}(\hat{\omega}). \end{aligned}$$

Proof. From Observation 4.6 and Eqs. (5.7)–(5.9) we have

$$\begin{aligned} -4 &\left(\frac{\operatorname{Re} \mathrm{w}}{\operatorname{Im} \lambda} + \operatorname{Im} \frac{d\mathrm{w}}{d\lambda} \right) \\ &= \int_{\hat{\Omega}} \left\{ \frac{Q(m_+)}{\operatorname{Im} m_+} - \frac{Q(m_-)}{\operatorname{Im} m_-} \right\} d\hat{\mu}(\hat{\omega}) \\ &\quad - 4 \int_{\hat{\Omega}} \left\{ \frac{\operatorname{Im} m_+ \cdot Q(m_-)}{|m_- - m_+|^2} - \frac{\operatorname{Im} m_- \cdot Q(m_+)}{|m_- - m_+|^2} \right\} d\hat{\mu}(\hat{\omega}) \\ &= \int_{\hat{\Omega}} \{ \operatorname{Im} m_- \cdot Q(m_+) - \operatorname{Im} m_+ \cdot Q(m_-) \} \\ &\quad \times \left\{ \frac{1}{\operatorname{Im} m_+ \operatorname{Im} m_-} + \frac{4}{|m_- - m_+|^2} \right\} d\hat{\mu}(\hat{\omega}) \end{aligned}$$

$$= \int_{\hat{\Omega}} [\operatorname{Im} m_- Q(m_+) - \operatorname{Im} m_+ Q(m_-)]$$

$$\times \left\{ \frac{|m_- - m_+|^2 + 4 \operatorname{Im} m_- \operatorname{Im} m_+}{\operatorname{Im} m_+ \operatorname{Im} m_- |m_- - m_+|^2} \right\} d\hat{\mu}(\hat{\omega})$$

$$= \int_{\hat{\Omega}} \left\{ \frac{Q(m_+)}{\operatorname{Im} m_+} - \frac{Q(m_-)}{\operatorname{Im} m_-} \right\}$$

$$\times \left\{ \frac{(\operatorname{Re} m_- - \operatorname{Re} m_+)^2 + (\operatorname{Im} m_- + \operatorname{Im} m_+)^2}{|m_- - m_+|^2} \right\} d\hat{\mu}(\hat{\omega}).$$

This completes the proof of the lemma.

Next we repeat the proof of Lemma 4.1 of Kotani [20] to get:

LEMMA 5.11 (Kotani [20]). *Let $d\alpha_{\mathrm{ac}}(\lambda)$ be the absolutely continuous part of the measure $d\alpha$, and let $B_* \subset \mathbb{R}$ be a compact set such that $\gamma(\lambda) = 0$ for a.a. $\lambda \in B_*$. Then*

$$\lim_{\varepsilon \to 0} \int_{B_*} \frac{\operatorname{Re} \mathrm{w}(\lambda + i\varepsilon)}{\varepsilon} \, d\lambda = -\lim_{\varepsilon \to 0} \int_{B_*} \operatorname{Im} \mathrm{w}'(\lambda + i\varepsilon) \, d\lambda = -\int_{B_*} d\alpha_{\mathrm{ac}}(\lambda).$$

The proof of Lemma 5.11 uses the inequality $\operatorname{Re} \mathrm{w}/\operatorname{Im} \lambda + \operatorname{Im}(d\mathrm{w}/d\lambda) \leqslant 0$ for $\operatorname{Im} \lambda > 0$, and this in turn follows from Lemma 5.10.

We can now finish the proof of Theorem 5.9. First, a calculation shows that $\gamma_1(\hat{\omega}) < 0$ on a set of positive $\hat{\mu}$-measure (in fact

$$\gamma_1([\omega, t]) = -\frac{c_0^2(t)}{2} [f'(t) + 2f(t)\, f'(t) \operatorname{Re} \alpha_1(\omega) + f^2(t)\, f'(t)\, |\alpha_1(\omega)|^2]$$

and the statement follows). We have

$$\frac{2 \operatorname{Re} \mathrm{w}(\lambda + i\varepsilon)}{\varepsilon} = \int_{\hat{\Omega}} \frac{\gamma_1 + 2\gamma_2 \operatorname{Re} m_+ + \gamma_3 |m_+|^2}{\operatorname{Im} m_+} \, d\hat{\mu}(\hat{\omega}).$$

By definiteness of $\left(\begin{smallmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{smallmatrix}\right)$ and continuity of the entries, there is a set $F \subset \hat{\Omega}$ of positive $\hat{\mu}$-measure and a positive constant δ such that

$$\gamma_1 + 2\gamma_2 \operatorname{Re} m_+ + \gamma_3 |m_+|^2 \leqslant -\delta \qquad (\hat{\omega} \in F).$$

Using Fatou's lemma, we find that

$$\int_F \int_{B_*} \frac{\delta}{\operatorname{Im} m_+(\hat{\omega}, \lambda + i0)} \, d\lambda \leqslant -\lim_{\varepsilon \to 0} \int_{B_*} \int_{\hat{\Omega}} \frac{\gamma_1 + 2\gamma_2 \operatorname{Re} m_+ + \gamma_3 |m_+|^2}{\operatorname{Im} m_+} \, d\hat{\mu}(\hat{\omega})$$

$$= -2 \int_{B_*} d\alpha_{ac}(\lambda),$$

and hence $\operatorname{Im} m_+(\hat{\omega}, \lambda + i0) > 0$ for Lebesgue-a.a. $\lambda \in B_*$, for $\hat{\mu}$-a.a. $\hat{\omega} \in \hat{\Omega}$. The set of such $\hat{\omega} \in \hat{\Omega}$ is invariant, hence (by ergodicity of $\hat{\mu}$) has full $\hat{\mu}$-measure, hence by Remark 5.3 we obtain that, for μ-a.a. $\omega \in \Omega$, the set $B = \{z = -i \log \lambda \mid \lambda \in B_*\}$ has the property that $B - \sum_{\omega}^{ac}$ has zero Lebesgue measure. This completes the proof of Theorem 5.9.

We can strengthen the conclusion of Theorem 5.9 if $\gamma(z) = 0$ for a.a. z in an *interval* $I \subset K$. In this case, γ extends harmonically through I, and hence w extends holomorphically through I. Using Lemma 5.10, one finds easily that for μ-a.a. $\hat{\omega} \in \hat{\Omega}$:

$$\operatorname{Im} m_-(\hat{\omega}, \lambda + i0) = -\operatorname{Im} m_+(\hat{\omega}, \lambda + i0)$$

$$\operatorname{Re} m_-(\hat{\omega}, \lambda + i0) = \operatorname{Re} m_+(\hat{\omega}, \lambda + i0)$$

for a.a. $\lambda \in I$. By Schwarz reflection, $m_\pm$ extend holomorphically through I. In particular we have:

COROLLARY 5.12. *If $\gamma(z) = 0$ a.e. on an interval $I \subset K$, then $\sigma_\omega = \sigma_\omega^{ac}$ on I and σ_ω^{ac} has an analytic density function on I. Moreover the functions $m_\pm(\cdot, \lambda)$ are continuous in $\hat{\omega}$ for $\lambda \in I$.*

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