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Note

A characterization of $(2\gamma, \gamma_p)$-trees[☆]

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Abstract

Let $G=(V,E)$ be a graph. A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent with some vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a paired-dominating set of G if S dominates V and $\langle S\rangle$ contains at least one perfect matching. The paired-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set of G. In this paper, we provide a constructive characterization of those trees for which the paired-domination number is twice the domination number.
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1. Introduction

Let $G=(V,E)$ be a graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N(v)=\{u \in V | uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. For $S \subseteq V$, the open neighborhood of S is defined by $N(S)=\cup_{v\in S} N(v)$, and the closed neighborhood of S by $N[S]=N(S)\cup S$. The private neighborhood $\mathrm{PN}(v,S)$ of $v \in S$ is defined by $\mathrm{PN}(v,S)=N(v)-N[S-\{v\}]$. The private neighborhood $\mathrm{PN}(S',S)$ of $S' \subset S$ is defined by $\mathrm{PN}(S',S)=N(S')-N[S-S']$. The subgraph of G induced by the vertices in S is denoted by $\langle S\rangle$. For $X, Y \subseteq V$, if X dominates Y, we write $X \succ Y$, or $X \succ G$ if $Y=V$, or $X \succ y$ if $Y=\{y\}$.

A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent to some vertex in S. (That is, $N[S]=V$.) The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G (similar notation is used for the other domination parameters).

Let $G=(V,E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a paired-dominating set of G if S dominates V and $\langle S\rangle$ contains at least one perfect matching M. If an edge $uv \in M$, we say that u and v are paired in S. The paired-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set of G. Paired-domination in graphs was introduced by Haynes and Slater [7]. Recall that a dominating set $S \subseteq V$ of G is a total dominating set if $\langle S\rangle$ contains no isolated vertices. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Clearly, $\gamma_t(G)\leqslant\gamma_p(G)$ for every connected graph with order at least two. Total domination in graphs was introduced by Cockayne et al. [1]. The concept of domination in graphs, with its many variations, is well studied in graph theory (see [4,5]).

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An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters. For any two graph theoretic parameters λ and μ, G is called a (λ, μ)-graph if $\lambda(G) = \mu(G)$. The class of (γ, i)-trees, that is, trees with equal domination and independent domination numbers was characterized in [2]. In [3], the authors provided a constructive characterization of trees with equal independent domination and restrained domination numbers, and a constructive characterization of trees with equal independent domination and weak domination numbers is also given. In [9], the authors characterized those trees with equal domination and paired-domination numbers. In [8], those trees with equal domination and total domination numbers were characterized. In [6], the authors provided a constructive characterization of the trees T for which (1) $\gamma(T) \equiv i(T)$; (2) $\gamma(T) \equiv \gamma_t(T)$; and (3) $\gamma(T) \equiv \gamma_p(T)$.

Clearly, if G has a paired-dominating set, then $\gamma_p(G)$ is even. For the domination and paired-domination numbers, we have

Fact 1 (*Haynes and Slater [7]*). *Let G be a graph without isolated vertices. Then, G has a paired-dominating set, and $\gamma_p(G) \leqslant 2\gamma(G)$.*

In this paper, we give a constructive characterization of trees for which the paired-domination number is twice the domination number.

2. Main result

Let $T = (V, E)$ be a tree with vertex set V and edge set E. A vertex of T is said to be remote if it is adjacent to a leaf. The set of leaves of T is denoted by $L(T)$. In this paper, we use T_v to denote the subtree of $T - uv$ containing v for $uv \in E(T)$. P_ℓ represents a path with l vertices. $|T|$ denotes the order of a tree T.

We begin with a proposition about the paired-dominating set of a tree T.

Proposition 2. *If S is a γ_p-set of a tree T, then $\langle S\rangle$ contains a unique perfect matching.*

Proof. Let H be a component of $\langle S\rangle$, then H has a perfect matching. So $|H|$ is even. To prove that $\langle S\rangle$ contains a unique perfect matching, it is enough to show that H has a unique perfect matching. We prove H has a unique perfect matching by induction on $2n$, the order of H. If $n = 1$, then $H \cong K_2$, the result is clearly true. Let $n > 1$ and assume that the result is true for every tree H' of order $< 2n$, where H' is a tree containing a perfect matching. Let H be a tree of order $2n$ containing a perfect matching M. Let u be a leaf of H and v be the remote vertex such that $uv \in E(H)$. Then $uv \in M$ and u is the unique leaf adjacent to v in H. Let $H_1, H_2, \ldots, H_k$ be the components of $H - \{u, v\}$. Then every H_i has a perfect matching and $|H_i| < 2n$. By inductive hypothesis, H_i has a unique perfect matching M_i. So, $M = (\cup_{i=1}^{k} M_k) \cup \{u, v\}$ is the unique perfect matching of H. The result follows. □

Let S be a paired-dominating set of a tree T. By Proposition 2, S has a unique perfect matching M. So, for any vertex $v \in S$, the paired vertex of v is unique. We denote the unique paired vertex of $v \in S$ by $\bar{v}$.

To state the characterization of $(2\gamma, \gamma_p)$-trees, we introduce three types of operations.

Type-1 *operation*: Attach a path P_1 to a vertex v of a tree T, where v is in a γ-set of T and $v \notin L(T)$. (As shown in Fig. 1(a).)

Type-2 *operation*: Attach a path P_2 to a vertex v of a tree T, where v is a vertex such that for every γ_p-set S of T containing v, $\mathrm{PN}(v, S) = \emptyset$ and $\mathrm{PN}(\{v, \bar{v}\}, S) \neq \emptyset$. (As shown in Fig. 1(b).)

Type-3 *operation*: Attach a path P_3 to a vertex v of a tree T, where either v is a vertex of a γ-set of T such that $v \notin L(T)$ and, for every γ_p-set S of T, $\mathrm{PN}(\{v, \bar{v}\}, S) \neq \emptyset$ if $\bar{v} \notin L(T)$, or v is a vertex such that for every γ_p-set S of T containing v, $\bar{v} \notin N(S - \{v, \bar{v}\})$ if $\mathrm{PN}(\{v, \bar{v}\}, S) = \emptyset$. (As shown in Fig. 1(c).)

Let $\mathscr{J}_p$ be the family of trees for which the paired-domination number is twice the domination number, that is

$$\mathscr{J}_p = \{T : \gamma_p(T) = 2\gamma(T)\}.$$

We define the family $\mathscr{F}_p$ as:

$\mathscr{F}_p = \{T : T$ is obtained from P_3 by a finite sequence of operations of Type-1, Type-2 or Type-3$\}$.

We shall prove that

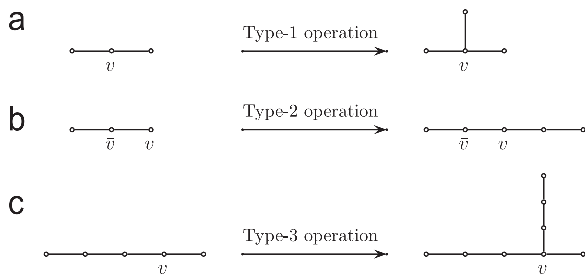

Fig. 1. The k-γ_t-critical graphs.

Theorem 3.

$$\mathscr{J}_p = \mathscr{F}_p \cup \{P_2\}.$$

3. The proof of Theorem 3

We begin with some lemmas.

Lemma 4 (*Haynes and Slater [7]*). *If G is a graph with $\gamma_p(G) = 2\gamma(G)$, then every γ-set of G is an i-set of G, where an i-set of G is an independent set of G with minimum order over all of maximal independent sets of G.*

Lemma 5 (*Haynes and Slater [7], Qiao et al. [9]*). *If v is a remote vertex of a tree T, then for every paired-dominating set S of T, $v \in S$.*

Lemma 6. *Let T be a tree with $\gamma_p(T) = 2\gamma(T)$ and X be a γ-set of T. Then, for every $v \in X$, $\mathrm{PN}(v, X) \neq \emptyset$.*

Proof. By contradiction. Suppose that there is a vertex $v \in X$ such that $\mathrm{PN}(v, S) = \emptyset$. Let $u \in N(v)$. By Lemma 4, X is an i-set of T. So $u \notin X$. Then u must be dominated by another vertex in X, say w. Let $X' = (X - \{v\}) \cup \{u\}$, then X' is a γ-set of T. But u and w are adjacent in X', contradicts the fact that X' is an i-set of T. Therefore, for every $v \in X$, $\mathrm{PN}(v, X) \neq \emptyset$. $\square$

Lemma 7. *If T is a tree with $|T| \geqslant 3$, then T has a γ-set containing no leaves of T.*

Proof. Let X be any γ-set of T. If $X \cap L(T) = \emptyset$, then the result follows. If $X \cap L(T) \neq \emptyset$, let $L'(T) = X \cap L(T)$ and $R'(T)$ be the set of the remote vertices corresponding to $L'(T)$, then $R'(T) \cap X = \emptyset$. Let $X' = (X - L'(T)) \cup R'(T)$. Since $|R'(T)| \leqslant |L'(T)|$ and X is a γ-set of T, X' is a γ-set of T containing no leaves of T. The result follows. $\square$

In the following, a γ-set X of a tree T with $|T| \geqslant 3$ always means that X contains no leaves of T unless otherwise stated.

Lemma 8. *If T is a tree with $\gamma_p(T) = 2\gamma(T)$ and $|T| \geqslant 3$, then T has a unique γ-set containing no leaves of T.*

Proof. By induction on n, the order of the tree T. If $n = 3$, then $T = P_3$ and the result is clearly true. Let $n > 3$ and assume that for all trees $T' \in \mathscr{J}_p$ of order $3 \leqslant n' < n$, T' has a unique γ-set. Let $T \in \mathscr{J}_p$ be a tree of order n and let $v_0 v_1 v_2 \cdots v_l$ be a longest path in T.

Case 1: $d(v_1) \geqslant 3$. Then there exists a leaf $u \neq v_0$ adjacent to v_1. Let $T' = T - \{u\}$. Clearly, every γ-set of T is a dominating set of T'. By Lemma 7, every γ-set X' of T' contains v_1 and so X' is a dominating set of T, too. By Lemma 5, every γ_p-set S' of T' contains v_1 and so S' is a γ_p-set of T. Thus, $\gamma_p(T') = \gamma_p(T) = 2\gamma(T) = 2\gamma(T')$. By inductive hypothesis, T' has a unique γ-set. It follows that T has a unique γ-set.

Case 2: $d(v_1) = 2$.

Case 2.1: $d(v_2)\geqslant 3$. We claim that v_2 is not a remote vertex of T. Otherwise, suppose that u is a leaf adjacent to v_2. By Lemma 7, T has a γ-set X such that $\{v_1, v_2\} \subset X$. Contradicts X is an i-set of T. Thus, there exists a remote vertex $u_2 \notin \{v_1, v_3\}$ such that $v_2u_2 \in E(T)$, let u_1 be the leaf adjacent to u_2. Let $T' = T_{v_2}$ be the subtree of $T - v_2u_2$ containing v_2. Let X be any γ-set of T. Then $v_1, u_2 \in X$. So, $v_2 \notin \mathrm{PN}(u_2, X)$, hence $X - \{u_2\} \succ T'$. Thus $\gamma(T')\leqslant\gamma(T)-1$. Clearly, $\gamma(T)\leqslant\gamma(T') + 1$. It follows that $\gamma(T) = \gamma(T') + 1$. Since $\gamma_{\mathrm{p}}(T)\leqslant\gamma_{\mathrm{p}}(T') + 2$ and $\gamma_{\mathrm{p}}(T) = 2\gamma(T)$, we have $2\gamma(T')\geqslant\gamma_{\mathrm{p}}(T')\geqslant\gamma_{\mathrm{p}}(T) - 2 = 2(\gamma(T) - 1) = 2\gamma(T')$. Consequently, $\gamma_{\mathrm{p}}(T') = 2\gamma(T')$ and $T' \in \mathscr{J}_{\mathrm{p}}$. By inductive hypothesis, T' has a unique γ-set X'. It follows that $X = X' \cup \{u_2\}$ is the unique γ-set of T.

Case 2.2: $d(v_2)=2$. We claim that $d(v_3)\geqslant 2$. Suppose to the contrary that $d(v_3)=1$, then $T \cong P_4$. It is easy to check that $\gamma_{\mathrm{p}}(P_4) = \gamma(P_4) = 2$, a contradiction with $\gamma_{\mathrm{p}}(T) = 2\gamma(T)$. Let $T' = T_{v_3}$ be the subtree of $T - v_2v_3$ containing v_3. Let X be any γ-set of T. Then, by Lemma 7, $v_1 \in X$. By Lemma 4, $v_2 \notin X$. So $X - \{v_1\} \succ T'$ and $\gamma(T')\leqslant\gamma(T) - 1$. Clearly, $\gamma(T)\leqslant\gamma(T') + 1$. It follows that $\gamma(T) = \gamma(T') + 1$. Since $\gamma_{\mathrm{p}}(T)\leqslant\gamma_{\mathrm{p}}(T') + 2$ and $\gamma_{\mathrm{p}}(T) = 2\gamma(T)$, we have $2\gamma(T')\geqslant\gamma_{\mathrm{p}}(T')\geqslant\gamma_{\mathrm{p}}(T) - 2 = 2(\gamma(T) - 1) = 2\gamma(T')$. Consequently, $\gamma_{\mathrm{p}}(T') = 2\gamma(T')$ and $T' \in \mathscr{J}_{\mathrm{p}}$. If $|T'| = 2$, then $T = P_5$. The result is clearly true. Henceforth, assume that $|T'|\geqslant 3$. By inductive hypothesis, T' has a unique γ-set X'.

If $v_3 \notin L(T')$, then, clearly, $X' \cup \{v_1\}$ is a unique γ-set of T.

If $v_3 \in L(T')$, then we claim that T' has no minimum dominating set X'' such that $X'' \cap L(T') = \{v_3\}$. Suppose to the contrary that X'' is a minimum dominating set of T' such that $X'' \cap L(T') = \{v_3\}$. By Lemma 4, X'' is an i-set of G. Let M'' be a maximum matching in the bipartite subgraph of T' with partite sets X'' and $N_{T'}(X'')$ and with edge set all edges of T' incident with vertices in X'' (note that this subgraph is not necessarily induced in T' since there may be edges in T' joining vertices of $N_{T'}(X'')$). Let S'' be the set of all vertices saturated by M''. (Note that S'' paired dominates T', and $|S''|\leqslant 2|X''| = 2\gamma(T')$.) Since $\gamma_{\mathrm{p}}(T') = 2\gamma(T')$ and $|X''| = \gamma(T')$, $|S''| = \gamma_{\mathrm{p}}(T')$ and $X'' \subseteq S''$. Since $v_3 \in L(T')$, $v_3v_4 \in M''$. If $v_5 \notin S''$, assume that v_5 is dominated by $v' \in X''$ and $v'v'' \in M''$, then $(S'' - \{v''\}) \cup \{v_5\}$ is a γ_{p}-set of T', too. Hence we can always extend X'' to a γ_{p}-set S' of T' such that v_3, v_4 are paired in S' and $v_5 \in S'$. Let $S = (S' - \{v_3, v_4\}) \cup \{v_1, v_2\}$. Then S is a PDS of T with cardinality $2\gamma - 2$, contradicts $\gamma_{\mathrm{p}}(T) = 2\gamma(T)$. It follows that $X = X' \cup \{v_1\}$ is the unique γ-set of T. $\square$

Lemma 9. *Let $T \in \mathscr{J}_{\mathrm{p}}$ with $|T|\geqslant 3$ and X be the unique γ-set of T containing no leaves. Then X is contained in every γ_{p}-set of T.*

Proof. By induction on n, the order of tree T. If $n = 3$, then $T = P_3 \in \mathscr{J}_{\mathrm{p}}$, the result is clearly true. Let $n > 3$ and assume that the result is true for all trees T' of order $3\leqslant n' < n$. Let $T \in \mathscr{J}_{\mathrm{p}}$ be a tree of order n and let $v_0v_1 \cdots v_l$ be a longest path in T.

Case 1: $d(v_1)\geqslant 3$. Then there exists a leaf $u \neq v_0$ such that $uv_1 \in E(T)$. Let $T' = T - \{u\}$. As shown in Lemma 8, $T' \in \mathscr{J}_{\mathrm{p}}$, and T and T' have the same unique γ-set, say X. By inductive hypothesis, every γ_{p}-set S' of T' contains X. Let S be a γ_{p}-set of T. If $u \notin S$, then S is a γ_{p}-set of T', too, hence $X \subset S$. If $u \in S$, then $v_0 \notin S$. So $(S - \{u\}) \cup \{v_0\}$ is a γ_{p}-set of T', hence $X \subset (S - \{u\}) \cup \{v_0\}$. Since $v_0, u \notin X$, $X \subset S$. It follows that every γ_{p}-set of T contains X.

Case 2: $d(v_1) = 2$.

Case 2.1: $d(v_2)\geqslant 3$. As proved in Lemma 8, v_2 is not a remote vertex. Let $u_2 \neq v_1$ be a remote vertex adjacent to v_2 and u_1 be the leaf adjacent to u_2. Let $T' = T_{v_2}$ be the subtree of $T - u_2v_2$ containing v_2. As discussed in Lemma 8, $\gamma_{\mathrm{p}}(T') = 2\gamma(T') = 2(\gamma(T) - 1) = \gamma_{\mathrm{p}}(T) - 2$, and $T' \in \mathscr{J}_{\mathrm{p}}$. Let X' be the unique γ-set of T'. Then $X = X' \cup \{u_2\}$ is the unique γ-set of T. By inductive hypothesis, for every γ_{p}-set S' of T', $X' \subset S'$. Let S be any γ_{p}-set of T. By Lemma 5, $v_1 \in S$ and $u_2 \in S$. If $\bar{u}_2 = u_1$, then $S - \{u_1, u_2\}$ is a γ_{p}-set of T'. So $X' \subset S - \{u_1, u_2\}$, and $X = X' \cup \{u_2\} \subset S$. If $\bar{u}_2 = v_2$, then $\bar{v}_1 = v_0$ and $S - \{u_2, v_0\}$ is a γ_{p}-set of T'. So $X' \subset S - \{u_2, v_0\}$, and $X = X' \cup \{u_2\} \subset S$.

Case 2.2: $d(v_2)=2$. As discussed in Lemma 8, we have $d(v_3)\geqslant 2$. Let $T' = T_{v_3}$ be the subtree of $T - v_2v_3$ containing v_3. As shown in Lemma 8, $\gamma_{\mathrm{p}}(T') = 2\gamma(T') = 2(\gamma(T) - 1) = \gamma_{\mathrm{p}}(T) - 2$, and $T' \in \mathscr{J}_{\mathrm{p}}$. Let X' be the unique γ-set of T'. Then $X = X' \cup \{v_1\}$ is the unique γ-set of T. By inductive hypothesis, for every γ_{p}-set S' of T', $X' \subset S'$. Let S be any γ_{p}-set of T, by Lemma 5, $v_1 \in S$. In the following, we prove that $X \subset S$.

If $\bar{v}_1 = v_0$, then $S - \{v_0, v_1\}$ is a γ_{p}-set of T'. So $X' \subset S - \{v_0, v_1\}$ and $X = X' \cup \{v_1\} \subset S$.

If $\bar{v}_1 = v_2$ and $v_3 \notin \mathrm{PN}(v_2, S)$, then $S - \{v_1, v_2\}$ is a γ_{p}-set of T'. So $X' \subset S - \{v_1, v_2\}$ and $X = X' \cup \{v_1\} \subset S$.

If $\bar{v}_1 = v_2$ and $v_3 \in \mathrm{PN}(v_2, S)$, then none of $N[v_3] - \{v_2\}$ are contained in S. By Lemma 5 and $v_3 \in \mathrm{PN}(v_2, S)$, neither v_3 is adjacent to a leaf nor v_3 is adjacent to a remote vertex.

Claim A. $d(v_3)\geqslant 3$.

Proof of Claim A. If not, then $d(v_3)=2$. Then $v_4 \notin S$ since $v_3 \in \mathrm{PN}(v_2, S)$. Let $T''=T'-\{v_3\}$, then $S-\{v_1, v_2\}$ is a paired-dominating set of T''. So $\gamma_{\mathrm{p}}(T'') \leqslant \gamma_{\mathrm{p}}(T)-2$. However, any γ_{p}-set of T'' can be extended to a paired-dominating set of T by adding the vertices v_1 and v_2. So $\gamma_{\mathrm{p}}(T) \leqslant \gamma_{\mathrm{p}}(T'')+2$. Consequently, $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T'')+2$. Clearly, $\gamma(T)-2 \leqslant \gamma(T'') \leqslant \gamma(T)$. If $\gamma(T'')=\gamma(T)-1=\gamma(T')$, then $\gamma_{\mathrm{p}}(T)=2\gamma(T)=2(\gamma(T'')+1)=2\gamma(T'')+2 \geqslant \gamma_{\mathrm{p}}(T'')+2=\gamma_{\mathrm{p}}(T)$. So, $\gamma_{\mathrm{p}}(T'')=2\gamma(T'')$. By Lemma 8, T'' has a unique γ-set. Let X' be the unique γ-set of T'. Then $v_4 \in X'$, and X' is a γ-set of T''. If v_4 is not a leaf of T'', then X' is the unique γ-set of T''. Applying the inductive hypothesis to T'', for any γ_{p}-set S'' of T'', we have $X' \subset S''$.So $X' \subset S-\{v_1, v_2\}$ and $v_4 \in X' \subset S$, contradicts $v_4 \notin S$. If v_4 is a leaf of T'', let $T'''=T''-\{v_4\}$, then $\gamma(T) \leqslant \gamma(T''')+2$. However, for any γ-set X of T, $|X \cap \{v_0, v_1, v_2, v_3, v_4\}| \geqslant 2$, so $\gamma(T''') \leqslant \gamma(T)-2$. Consequently, $\gamma(T''')=\gamma(T)-2$. Clearly, $\gamma_{\mathrm{p}}(T) \leqslant \gamma_{\mathrm{p}}(T''')+4$. So, $2\gamma(T''') \geqslant \gamma_{\mathrm{p}}(T''') \geqslant \gamma_{\mathrm{p}}(T)-4=2\gamma(T)-4=2(\gamma(T''')+2)-4=2\gamma(T''')$. Thus, $\gamma_{\mathrm{p}}(T''')=2\gamma(T''')=\gamma_{\mathrm{p}}(T)-4$. By Lemma 8, T''' has a unique γ-set X'''. Applying inductive hypothesis to T''', for any γ_{p}-set S''' of T''', we have $X''' \subset S'''$. We claim that for any γ_{p}-set S''' of T''', $v_5 \notin S'''$. If not, $S''' \cup \{v_1, v_2\}$ is a paired-dominating set of T, so $\gamma_{\mathrm{p}}(T) \leqslant \gamma_{\mathrm{p}}(T''')+2$, contradicts $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T''')+4$. Consequently, $v_5 \notin X'''$. Since v_3 is not adjacent to a remote vertex, v_5 is not a leaf of T. Then $d(v_5) \geqslant 2$. Let u be a vertex such that $u \in X'''$ and $u \succ v_5$, then $u \in S'''$. So $(S'''-\{\bar{u}\}) \cup \{v_5\}$ is a γ_{p}-set of T''' containing v_5, a contradiction. If$\gamma(T'')=\gamma(T)-2$, then $\gamma_{\mathrm{p}}(T)=2\gamma(T)=2(\gamma(T'')+2)=2\gamma(T'')+4=\gamma_{\mathrm{p}}(T'')+4$, contradicts $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T'')+2$. The Claim follows. $\square$

By Claim A and since neither v_3 is adjacent to a leaf nor v_3 is adjacent to a remote vertex, there exists a P_3 with vertex set $\{u_1, u_2, u_3\}$ such that $u_3 v_3 \in E(T)$. By Lemma 5, $u_2 \in S$. Since $u_3 \in N[v_3]-\{v_2\}$, $u_3 \notin S$. So $\bar{u}_2=u_1$. Let $T''=T_{v_3}$ be the subtree of $T-u_3v_3$ containing v_3. Clearly, $T'' \cong T'$. So, $\gamma_{\mathrm{p}}(T'')=2\gamma(T'')=2(\gamma(T)-1)=\gamma_{\mathrm{p}}(T)-2$, and $T'' \in \mathscr{I}_{\mathrm{p}}$. Let X'' be the unique γ-set of T''. Then $X=X'' \cup \{u_2\}$ is the unique γ-set of T. By inductive hypothesis, for every γ_{p}-set S'' of T'', $X'' \subset S''$. Since $S-\{u_1, u_2\}$ is a γ_{p}-set of T'', $X'' \subset S-\{u_1, u_2\}$. So, $X=X'' \cup \{u_2\} \subset S$. $\square$

Let $T \in \mathscr{I}_{\mathrm{p}}$, X be the unique γ-set of T and S be a γ_{p}-set of T. Then, by Lemma 9, for any $u \in S-X$, $\mathrm{PN}(u, S)=\emptyset$.

Lemma 10. *If $T' \in \mathscr{I}_{\mathrm{p}}$ with $|T'| \geqslant 3$ and T is obtained from T' by a Type-i operation, $i=1,2,3$, then $T \in \mathscr{I}_{\mathrm{p}}$.*

Proof. Let $P_i=u_1u_2\cdots u_i$ be a path with vertex set $\{u_1, u_2, \ldots, u_i\}$, $T' \in \mathscr{I}_{\mathrm{p}}$ with $|T'| \geqslant 3$. T is obtained from T' by attaching u_i to a vertex v of T', where v satisfies the conditions required by Type-i operation, $i=1,2,3$. By Lemma 8, T' has a unique γ-set X'. By Lemma 9, for every γ_{p}-set S' of T', $X' \subset S'$.

Case 1: $i=1$. Then T is obtained from T' by attaching vertex u_1 to v of T', where $v \in X'$. Clearly, X' is a γ-set of T, too. So, $\gamma(T)=\gamma(T')$. Since $v \in X' \subset S'$, S' is a γ_{p}-set of T, too. Thus, $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')=2\gamma(T')=2\gamma(T)$. Hence, $T \in \mathscr{I}_{\mathrm{p}}$.

Case 2: $i=2$. Let S' be a γ_{p}-set of T' such that $v \in S'$. We claim that $v \notin X'$. If not, then $\bar{v} \notin X'$. So $\mathrm{PN}(\bar{v}, S')=\emptyset$. But the Type-2 operation requires that $\mathrm{PN}(v, S')=\emptyset$ and $\mathrm{PN}(\{v, \bar{v}\}, S') \neq \emptyset$, so we have $\mathrm{PN}(\bar{v}, S') \neq \emptyset$, a contradiction. Thus, $v \notin X'$ and $\bar{v} \in X'$. So $X' \cup \{u_2\} \succ T$, hence $\gamma(T) \leqslant \gamma(T')+1$.

Claim B. $\gamma(T)=\gamma(T')+1$.

Proof of Claim B. If not, then $\gamma(T)=\gamma(T')$. Let X be a γ-set of T. Then $u_2 \in X$ since $L(T) \cap X=\emptyset$. If $v \notin \mathrm{PN}(u_2, X)$, then $X-\{u_2\}$ is a γ-set of T', contradicts $\gamma(T')=\gamma(T)$. So, $v \in \mathrm{PN}(u_2, X)$. Then $(X-\{u_2\}) \cup \{v\}$ is a γ-set of T'. If $v \notin L(T')$, then $(X-\{u_2\}) \cup \{v\}$ is the unique γ-set of T'. Hence $X'=(X-\{u_2\}) \cup \{v\}$ and $v \in X'$, a contradiction. If $v \in L(T')$, let $X^*=(X-\{u_2\}) \cup \{v\}$, then $\mathrm{PN}(v, X^*)=\emptyset$ in T' since $X-\{u_2\} \succ T'-\{v\}$. By Lemma 6, $\mathrm{PN}(v, X^*) \neq \emptyset$, a contradiction. Claim B follows. $\square$

Claim C. $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')+2$.

Proof of Claim C. Clearly, $\gamma_{\mathrm{p}}(T) \leqslant \gamma_{\mathrm{p}}(T')+2$. If $\gamma_{\mathrm{p}}(T) \neq \gamma_{\mathrm{p}}(T')+2$, then $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')$. Let S be a γ_{p}-set of T with $u_2 \in S$.

If $v \notin S$, then $\bar{u}_2=u_1$. If $v \notin \mathrm{PN}(u_2, S)$, then $S-\{u_1, u_2\}$ is a paired-dominating set of T', contradicts $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')$. If $v \in \mathrm{PN}(u_2, S)$, then there exists a vertex $w \neq u_2$ such that w is adjacent to v and $w \notin S$. So, $S'=(S-\{u_1, u_2\}) \cup \{v, w\}$

is a γ_{p}-set of T'. Since $S-\{u_1,u_2\} \succ T'-\{v\}$, $\mathrm{PN}(\{v,w\},S')=\emptyset$, contradicts the conditions required by Type-2 operation.

If $v \in S$ and $\bar{v}=u_2$, then there exists a vertex $w \neq u_2$ such that w is adjacent to v and $w \notin S$ (otherwise, $S-\{u_2,v\}$ is a paired-dominating set of T', contradicts $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')$). So, $S'=(S-\{u_2\})\cup\{w\}$ is a γ_{p}-set of T'. By Lemma 9, $X' \subset S'$. Since $v \notin X'$, $\mathrm{PN}(v,S')=\emptyset$. Since $S-\{u_2\} \succ T'$, $\mathrm{PN}(w,S')=\emptyset$. Hence $\mathrm{PN}(\{v,w\},S')=\emptyset$, contradicts the conditions required by Type-2 operation.

If $v \in S$ and $\bar{v} \neq u_2$, then $\bar{u}_2=u_1$. So $S-\{u_1,u_2\}$ is a paired-dominating set of T', contradicts $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')$. Claim C is true. □

Therefore, $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')+2=2\gamma(T')+2=2(\gamma(T')+1)=2\gamma(T)$. Hence $T \in \mathscr{J}_{\mathrm{p}}$.

Case 3: $i=3$. Clearly, $\gamma(T) \leqslant \gamma(T')+1$. Let X be any γ-set of T. Then $u_2 \in X$. If $u_3 \notin X$, then $X-\{u_2\} \succ T'$. So $\gamma(T') \leqslant \gamma(T)-1$. If $u_3 \in X$, then $(X-\{u_2,u_3\})\cup\{v\} \succ T'$. So, $\gamma(T') \leqslant \gamma(T)-2+1=\gamma(T)-1$. Consequently, $\gamma(T)=\gamma(T')+1$.

Clearly, $\gamma_{\mathrm{p}}(T) \leqslant \gamma_{\mathrm{p}}(T')+2$. Let S be any γ_{p}-set of T. By Lemma 5, $u_2 \in S$. If $u_3 \notin S$, then $\bar{u}_2=u_1$. Hence $S-\{u_1,u_2\}$ is a paired-dominating set of T'. So $\gamma_{\mathrm{p}}(T') \leqslant \gamma_{\mathrm{p}}(T)-2$. Now assume that $u_3 \in S$. If $\bar{u}_2=u_1$, then $\bar{u}_3=v$. Hence, $\mathrm{PN}(v,S) \neq \emptyset$ (otherwise, $S-\{u_1,v\}$ is a paired-dominating set of T, contradicts S is a γ_{p}-set of T). Let $w \in \mathrm{PN}(v,S)$, then $(S-\{u_1,u_2,u_3\})\cup\{w\}$ is a paired-dominating set of T'. So, $\gamma_{\mathrm{p}}(T') \leqslant \gamma_{\mathrm{p}}(T)-3+1=\gamma_{\mathrm{p}}(T)-2$. If $\bar{u}_2=u_3$ and $v \notin \mathrm{PN}(u_3,S)$,then $S-\{u_2,u_3\}$ is a paired-dominating set of T'. So $\gamma_{\mathrm{p}}(T') \leqslant \gamma_{\mathrm{p}}(T)-2$. Consequently, $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')+2$.

If $\bar{u}_2=u_3$ and $v \in \mathrm{PN}(u_3,S)$, we claim that $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')+2$. If not, then $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')$. Then there exists a vertex $w \in N(v)$ such that $w \neq u_3$ and $w \notin S$ (otherwise, $v \notin \mathrm{PN}(u_3,S)$, a contradiction). Thus, $S'=(S-\{u_2,u_3\})\cup\{v,w\}$ is a γ_{p}-set of T'. By Lemma 9, $X' \subset S'$. Since $S'-\{v,w\}=S-\{u_2,u_3\} \succ T'-\{v\}$, $\mathrm{PN}(\{v,w\},S')=\emptyset$ and $w \in N(S'-\{v,w\})$. So we must be in the first case of the Type-3 operation, that is $v \in X'$ and $w \notin X'$. Since $v \in X'$ and $v \notin S$, neither v is a leaf nor v is a remote vertex in T'. So $w \notin L(T')$. Then the Type-3 operation requires that $\mathrm{PN}(\{v,w\},S') \neq \emptyset$, a contradiction.

Therefore, $\gamma_{\mathrm{p}}(T)=\gamma_{\mathrm{p}}(T')+2=2\gamma(T')+2=2\gamma(T)$. Hence $T \in \mathscr{J}_{\mathrm{p}}$. □

Lemma 11.

$$\mathscr{F}_{\mathrm{p}} \subseteq \mathscr{J}_{\mathrm{p}}.$$

Proof. Note that $P_3 \in \mathscr{J}_{\mathrm{p}}$. Let $T \in \mathscr{F}_{\mathrm{p}}$ be a tree obtained from P_3 by a number of operations of Type-1, Type-2, or Type-3. By Lemma 10, we can easily prove that $T \in \mathscr{J}_{\mathrm{p}}$ by induction on the number of operations required to construct the tree T. □

Lemma 12.

$$\mathscr{J}_{\mathrm{p}}-\{P_2\} \subseteq \mathscr{F}_{\mathrm{p}}.$$

Proof. Let $T \in \mathscr{J}_{\mathrm{p}}$. If $|T|<3$, then $T=P_2$. If $|T| \geqslant 3$, we prove that $T \in \mathscr{F}_{\mathrm{p}}$ by induction on n, the order of the tree T. If $n=3$, then $T=P_3 \in \mathscr{F}_{\mathrm{p}}$. Assume that the result is true for all trees $T' \in \mathscr{J}_{\mathrm{p}}$ of order $3 \leqslant n'<n$. Let $T \in \mathscr{J}_{\mathrm{p}}$ be a tree of order n and let $v_0v_1v_2\cdots v_l$ be a longest path in T. By Lemma 8, T has a unique γ-set X containing no leaves of T. By Lemma 7, $v_1 \in X$.

Case 1: $d(v_1) \geqslant 3$. Then there exists a leaf u such that $u \neq v_0$ and $uv_1 \in E(T)$. Let $T'=T-\{u\}$. As shown in Lemma 8, $\gamma_{\mathrm{p}}(T')=\gamma_{\mathrm{p}}(T)=2\gamma(T)=2\gamma(T')$. Hence, $T' \in \mathscr{J}_{\mathrm{p}}$. By the inductive hypothesis, $T' \in \mathscr{F}_{\mathrm{p}}$. Hence T is obtained from T' by a Type-1 operation. By Lemma 10, $T \in \mathscr{F}_{\mathrm{p}}$.

Case 2: $d(v_1)=2$.

Case 2.1: $d(v_2) \geqslant 3$. As shown in Lemma 8, v_2 is not a remote vertex of T, and v_2 is adjacent to a remote vertex $u_2 \neq v_1, v_3$. Let $T'=T_{v_2}$ be the subtree of $T-v_2u_2$ containing v_2. As discussed in Lemma 8, $\gamma_{\mathrm{p}}(T')=\gamma_{\mathrm{p}}(T)-2=2\gamma(T)-2=2\gamma(T')$, and $T' \in \mathscr{J}_{\mathrm{p}}$. By inductive hypothesis, $T' \in \mathscr{F}_{\mathrm{p}}$. We claim that v_2 is a vertex of T' satisfying the conditions required by Type-2 operation. Let S' be a γ_{p}-set of T' with $v_2 \in S'$. By Lemma 8, T' has a unique γ-set X'. By Lemma 9, $X' \subset S'$. By Lemma 7, $v_1 \in X' \subset S'$. By Lemma 4, $v_2 \notin X'$. If $\bar{v}_2=v_1$, then $v_0 \in \mathrm{PN}(\{v_1,v_2\},S')$. So $\mathrm{PN}(\{v_1,v_2\},S') \neq \emptyset$. If $\bar{v}_2=v_3$, then $\bar{v}_1=v_0$. So, $\mathrm{PN}(\{v_2,v_3\},S') \neq \emptyset$ (otherwise, $S'-\{v_0,v_3\}$ is a paired-dominating set of T', contradicts S' is a $\gamma_{\rm p}$-set of T'). Since $X' \subset S'$ and $v_2 \notin X'$, $\mathrm{PN}(v_2, S') = \emptyset$. The claim is true. Hence T is obtained from T' by a Type-2 operation. By Lemma 10, $T \in \mathscr{F}_{\rm p}$.

Case 2.2: $d(v_2) = 2$. Let $T' = T_{v_3}$ be the subtree of $T - v_2v_3$ containing v_3. As discussed in Lemma 8, $\gamma_{\rm p}(T') = \gamma_{\rm p}(T) - 2$, $\gamma(T') = \gamma(T) - 1$, and $T' \in \mathscr{J}_{\rm p}$. By inductive hypothesis, $T' \in \mathscr{F}_{\rm p}$. We claim that v_3 is a vertex of T' satisfying the conditions required by Type-3 operation. Let S' be a $\gamma_{\rm p}$-set of T' with $v_3 \in S'$. By Lemma 8, T' has a unique γ-set X'. By Lemma 9, $X' \subset S'$.

If $v_3 \in X'$, then v_3 is not a leaf of T'. So $d(v_3) \geqslant 3$. Since $T' \in \mathscr{J}_{\rm p}$, by Lemma 4, every neighbor of v_3 in T' is not contained in X'. Let u be the paired vertex of v_3 in S'. Then u is not a remote vertex of T'. If $u \neq v_4$, then either u is a leaf or u is adjacent to a remote vertex. If u is not a leaf, we prove that $\mathrm{PN}(\{v_3, u\}, S') \neq \emptyset$. Let u_1 be the remote vertex which is adjacent to u and u_2 be the leaf which is adjacent to u_1 in T'. By Lemma 5, $u_1 \in S'$ and $\bar{u}_1 = u_2$ since $u = \bar{v}_3$. If $\mathrm{PN}(\{v_3, u\}, S') = \emptyset$, then $S' - \{v_3, u_2\}$ is a paired-dominating set of T', contradicts S' is a $\gamma_{\rm p}$-set of T'. If $u = v_4$ and v_4 is not a leaf of T, we claim that $\mathrm{PN}(\{v_3, v_4\}, S') \neq \emptyset$. If there is a leaf $w \in N(v_3) - \{v_2, v_4\}$, then $w \in \mathrm{PN}(\{v_3, v_4\}, S')$. So $\mathrm{PN}(\{v_3, v_4\}, S') \neq \emptyset$. If there are no such leaves, let $w \in N(v_3) - \{v_2, v_4\}$ and T_w be the subtree of $T - wv_3$ containing w, then $T_w = P_3$ since $w \notin X'$. Let $T_w = ww_2w_1$, then $w_2 \in X' \subset S'$. Hence $w \notin \mathrm{PN}(\{v_3, v_4\}, S')$. If $\mathrm{PN}(\{v_3, v_4\}, S') = \emptyset$ and there is a vertex $x \in N(v_4) - \{v_3\}$ such that $x \in S'$, then $S' - \{v_3, v_4\}$ is a paired-dominating set of T' if $\bar{w}_2 = w$, or $S' - \{v_3, v_4, w_1\} \cup \{w\}$ is a paired-dominating set of T' if $\bar{w}_2 = w_1$. Contradicts S' is a $\gamma_{\rm p}$-set of T'. If $\mathrm{PN}(\{v_3, v_4\}, S') = \emptyset$ and there is no vertex $x \in N(v_4) - \{v_3\}$ such that $x \in S'$, then, for every $x \in N(v_4) - \{v_3\}$, $x \notin X'$. Let $y \in X' \subset S'$ such that $y \succ x$. Then $(S' - \{v_3, v_4, \bar{y}\}) \cup \{x\}$ is a paired-dominating set of T' if $\bar{w}_2 = w$, or $(S' - \{v_3, v_4, \bar{y}, w_1\}) \cup \{x, w\}$ is a paired-dominating set of T' if $\bar{w}_2 = w_1$. Contradicts S' is a $\gamma_{\rm p}$-set of T'. Therefore, $\mathrm{PN}(\{v_3, v_4\}, S') \neq \emptyset$.

If $v_3 \notin X'$, then $\bar{v}_3 \in X'$. If $\bar{v}_3 \in N(S' - \{v_3, \bar{v}_3\})$ and $\mathrm{PN}(\{v_3, \bar{v}_3\}, S') = \emptyset$, then $(S' - \{v_3, \bar{v}_3\}) \cup \{v_1, v_2\}$ is a paired-dominating set of T. So, $\gamma_{\rm p}(T) \leqslant \gamma_{\rm p}(T')$, contradicts $\gamma_{\rm p}(T) = \gamma_{\rm p}(T') + 2$. Hence, $\bar{v}_3 \notin N(S' - \{v_3, \bar{v}_3\})$ or $\mathrm{PN}(\{v_3, \bar{v}_3, S'\}) \neq \emptyset$.

The claim is true. Thus, T is obtained from T' by a Type-3 operation. By Lemma 10, $T \in \mathscr{F}_{\rm p}$. The proof is completed. □

Theorem 3 follows as an immediate consequence of Lemmas 11 and 12.

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