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# Combinatorial Decompositions of Rings and Almost Cohen–Macaulay Complexes

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## 0. INTRODUCTION

The concept of a combinatorial decomposition of a graded $K$ algebra was introduced by Baclawski–Garsia [4], and they showed that every (finitely-generated) graded $K$ algebra has such a decomposition. The purpose of this paper is to prove some general properties of combinatorial decompositions, which are useful for finding such decompositions. We then show how to compute combinatorial decompositions for a class of rings based on simplicial complexes. This class of rings is utilized in the theory of lexicographic rings. ([3]). Another interesting consequence of our investigation (Section 4) is a ring-theoretic interpretation of the homology groups of a triangulated compact manifold.

Throughout the paper we use $\mathbb{N}$ for the semigroup of nonnegative integers and $K$ to denote a field; and, unless specified otherwise, cohomology will always be computed with coefficients in $K$. Moreover, most rings will be finitely generated $\mathbb{N}^l$-graded $K$ algebras for some $l$. For such a ring $R$, we write $R_S$ or $\mathscr{H}_S R$ for the graded part of multidegree $S \in \mathbb{N}^l$. We will think of $S$ as a multisubset of $[l] = \{1,..., l\}$. The *Hilbert series* of $R$ is the power series $H(R; t) = H(R; t_1,..., t_l) = \sum_{S \in \mathbb{N}^l} \dim_K(\mathscr{H}_S R)\, t^S$, where $t^S$ is the (multiset) product $\prod_{i \in S} t_i$. The *Krull dimension* of $R$ is the order of the pole at $t = 1$ of the power series $H(R; t,..., t)$. Given two power series $F(t)$ and $G(t)$ in the same variables $t_1,..., t_l$, we write $F(t) \leqslant G(t)$ to mean that $a_S \leqslant b_S$ for every $S \in \mathbb{N}^l$, where $F(t) = \sum a_S t^S$ and $G(t) = \sum b_S t^S$.

Given two homogeneous ideals $I$ and $J$ of $R$, the *ideal quotient* is the homogeneous ideal $(I{:}\,J) = \{f \in R \mid fI \subseteq J\}$. A related concept is the ***annihilator*** of a homogeneous ideal $I$ in a graded $R$-module $N$, given by $\mathrm{ann}_N(I) = \{g \in N \mid Ig = (0)\}$ ([1]).

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It is convenient to use the following notation: The *(reduced) Betti numbers* of a simplicial complex $\Delta$ are the dimensions $\tilde{h}_i(\Delta) = \dim_K \tilde{H}^i(\Delta, K)$. The *(reduced) Euler characteristic of* $\Delta$ is the alternating sum $\mu(\Delta) = \sum_{i=-1}^{\infty} (-1)^i \tilde{h}_i(\Delta)$. For a statement $\mathscr{A}$ we write $\chi(\mathscr{A})$ for the *indicator* of $\mathscr{A}$, i.e., $\chi(\mathscr{A}) = 0$, if $\mathscr{A}$ is false, $=1$, if $\mathscr{A}$ is true.

## 1. Combinatorial Decompositions

The key concept of this paper is the following.

Definition 1.1. Let $R$ be a finitely generated graded $K$ algebra. *A combinatorial decomposition* of $R$ is a triple $(\theta, \mathscr{S}, k)$, where

(a) $\theta = (\theta_1, ..., \theta_r)$ is a homogeneous system of parameters (or *frame*) of $R$, the elements of which are called *quasigenerators*;

(b) $\mathscr{S}$ is a set of homogeneous elements called *separators* of $R$; and

(c) $k: \mathscr{S} \to \{0, 1, ..., r\}$ is a function called the *level function*;

such that the following conditions hold:

(1) the image of $\mathscr{S}$ is a basis of $R/(\theta_1, ..., \theta_r)$;

(2) $\mathscr{E} = \{\eta \prod_{i=1}^{k(\eta)} \theta_i^{n_i} \mid \eta \in \mathscr{S}$ and $n_i \in \mathbb{N}\}$ is a basis of $R$; and

(3) for every $\eta \in \mathscr{S}$, $\eta(\theta_{k(\eta)+1}, ..., \theta_r) \subseteq (\theta_1, ..., \theta_{k(\eta)})$.

We write $\mathscr{S}_l = \{\eta \in \mathscr{S} \mid k(\eta) \leqslant l\}$ for the separators having level at most $l$.

The proof of existence of combinatorial decompositions used in [4, Theorem 2.1] constructed the frame and the separator set $\mathscr{S}_l$ simultaneously and inductively on $l$. In practice it is more convenient to begin with a frame that (we hope) will be the set of quasigenerators for a combinatorial decomposition. The following result shows where to look for the separators in this case:

Let $(\theta_1, ..., \theta_r)$ be a frame for the graded $K$ algebra $R$. Write $A(l)$ for the $R$ module $R/(\theta_1, ..., \theta_l)$, and let $M(l, m) = \{f \in A(l) \mid f g_1 g_2 \cdots g_m = 0$ for every sequence $g_1, g_2, ..., g_m \in A(l)_+\}$. The modules $M(l, m)$ form an ascending sequence of submodules of $A(l)$ and hence the sequence eventually stabilizes; define $M(l)$ to be the limiting module of this sequence.

Theorem 1.2. *Let $(\theta, \mathscr{S}, k)$ be a combinatorial decomposition of $R$. If $l \leqslant r = K\dim(R)$, then the image of $\mathscr{S}_l$ in $A(l)$ forms a basis of $M(l)$.*

*Proof.* By Definition 1.1(3), the image $\bar{\eta}$ of $\eta \in \mathscr{S}_l$ in $A(l)$ is in $\operatorname{ann}_{A(l)}(\theta_i \mid i > l)$. Let $m = \max\{\deg(\eta) \mid \eta \in \mathscr{S}\}$, and let $\bar{g} \in A(l)$ be homogeneous of degree at least $m+1$. Then we may write $\bar{g}$ uniquely in the form

$$\bar{g} = \sum_{\eta \in \mathscr{S}} \bar{\eta} p_\eta(\theta_{l+1}, \ldots, \theta_{k(\eta)}), \tag{$*$}$$

where each polynomial $p_\eta$ has no constant term since $\deg(\bar{g}) > \deg(\eta)$. Since every $\gamma \in \mathscr{S}_l$ satisfies $\bar{\gamma} \in \operatorname{ann}_{A(l)}(\theta_i \mid i > l)$, it follows that $\bar{\gamma}\bar{g} = 0$ for every $\bar{g} \in A(l)$ of sufficiently high degree. Hence $\mathscr{S}_l \subseteq M(l, m+1) \subseteq M(l)$.

To prove that $\mathscr{S}_l$ is a basis of $M(l)$ we need

LEMMA 1.3. *If $h \in (\theta_1, \ldots, \theta_t)$ is written as a linear combination of elements of $\mathscr{E}$, then every term $\eta \prod_{j=1}^{k(\eta)} \theta_j^{n_j}$ that appears in $h$ satisfies $n_i \neq 0$ for some $i \in [t]$.*

*Proof.* We use induction on $t$, the case $t = 0$ being trivial. Write $h = \sum_{i=1}^{t} h_i \theta_i$ and expand each $h_i$ as a linear combination of elements of $\mathscr{E}$. Thus

$$h = \sum_{i=1}^{t} \sum_{\eta \in \mathscr{S}} \eta p_{i,\eta}(\theta_1, \ldots, \theta_{k(n)})\, \theta_i,$$

where $p_{i,\eta}$ are polynomials in the indicated variables. Those terms in this expression which have $k(\eta) \geqslant i$ are already linear combinations of elements of $\mathscr{E}$ with no more expansion required. Those terms involving a product $\eta\theta_i$, where $k(\eta) < i$, must again be expanded. However, by Definition 1.1(3), we have $\eta\theta_i \in (\theta_1, \ldots, \theta_{k(\eta)})$, and hence by induction on $t$ we know that if $k(\eta) < i$, then $\eta p_{i,\eta}(\theta_1, \ldots, \theta_{k(\eta)})\, \theta_i$ is a linear combination of elements of $\mathscr{E}$ all of which have a factor from $\{\theta_1, \ldots, \theta_{k(\eta)}\} \subseteq \{\theta_1, \ldots, \theta_t\}$. The lemma then follows.

Returning to the proof of the theorem, let $\bar{g} \in M(l)$. By Lemma 1.3, $\bar{g}$ is a unique linear combination of the elements of $\mathscr{E}_l = \{\bar{\eta} \prod_{j=l+1}^{k(\eta)} \theta_j^{n_j} \mid \eta \in \mathscr{S}, n_j \geqslant 0\}$. We wish to expand $\bar{g}\theta_{l+1}^m$ in terms of $\mathscr{E}_l$ when $m > 0$. We do this by observing that

(1) the terms $\bar{\eta} \prod_{j=l+1}^{k(\eta)} \theta_j^{n_j}$ that appear in $\bar{g}$ and have $k(\eta) > l$ have the property that $\bar{\eta} \prod_{j=1}^{k(\eta)} \theta_j^{n_j} \theta_{l+1}^m \in \mathscr{E}_l$;

(2) the terms $\bar{\eta} \prod_{j=l+1}^{k(\eta)} \theta_j^{n_j}$ for which $k(\eta) \leqslant l$ are simply of the form $\bar{\eta}$ and we know that in this case $\eta\theta_{l+1}^m \in (\theta_1, \ldots, \theta_l)$ by Definition 1.1(3), or equivalently that $\bar{\eta}\theta_{l+1}^m = 0$.

By definition of $M(l)$ we know that $\bar{g}\theta_{l+1}^m = 0$ for $m$ sufficiently large. This contradicts the existence of terms of type (1), and hence $\bar{g}$ is a linear combination of the $\bar{\eta}$ for $\eta \in \mathscr{S}_l$. Thus the image of $\mathscr{S}_l$ spans $M(l)$.

Finally, suppose that $\sum_{\eta \in \mathscr{S}_l} c_\eta \bar{\eta} = 0$, or equivalently that $\sum_{\eta \in \mathscr{S}_l} c_\eta \eta \in (\theta_1,..., \theta_l)$, for some constants $c_\eta$. By Lemma 1.3, we may immediately conclude that the image of $\mathscr{S}_l$ in $A(l)$ is linearly independent and hence a basis of $M(l)$. Q.E.D.

An immediate corollary of the proof of Theorem 1.2 is that Definitions 1.1(2) and (3) imply (1). Another consequence is that although combinatorial decompositions are far from being unique, if we fix the frame $(\theta_1,..., \theta_r)$, then the generating function

$$G(t, u) = \sum_{\eta \in \mathscr{S}} t^{\deg(\eta)} u^{k(\eta)}$$

is the same for any combinatorial decomposition $(\theta, \mathscr{S}, k)$.

## 2. ACM Complexes and Hilbert Series

For a ring $R$, the *prime spectrum* of $R$ is the partially ordered set $\mathrm{Spec}(R) = \{P \subseteq R \mid P \text{ is a prime ideal}\}$. If $R$ is also $\mathbb{N}$ graded, then the *projective prime spectrum* is the poset

$$\mathrm{Proj}(R) = \{P \subseteq R \mid P \text{ is a homogeneous prime ideal}, P \subsetneq R_+\}.$$

For $P \in \mathrm{Spec}(R)$, we write $R_P$ for the localization of $R$ at $P$ and $\mathrm{gr}(R_P)$ for the associated graded ring of $R_P$. If $R$ is Noetherian, then so are $R_P$ and $\mathrm{gr}(R_P)$. We say that $R$ is *Cohen–Macaulay at* $P \in \mathrm{Spec}(R)$ if $R_P$ or equivalently $\mathrm{gr}(R_P)$ is CM, and that a subset $Q \subseteq \mathrm{Spec}(R)$ is Cohen–Macaulay if $R$ is CM at every $P \in Q$ ([1, 7]).

Let $\Delta$ be a simplicial complex on the vertex set $V$. The *Stanley–Reisner* or *face ring* [8] of $\Delta$ is the graded $K$ algebra $K[\Delta] = K[X_v \mid v \in V]/(X^S \mid S \subseteq V, S \notin \Delta)$. We say that $\Delta$ is *Cohen–Macaulay* if $K[\Delta]$ is CM, and that $\Delta$ is *almost Cohen–Macaulay* (or simply ACM) if for every $\sigma \in \Delta \backslash \{\varnothing\}$, we have that the subcomplex $\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$ is CM ([2]). The algebraic geometric content of the two definitions above is given by

Proposition 2.1. *Let $\Delta$ be a finite simplicial complex.*

(1) *$\Delta$ is* CM *if and only if* $\mathrm{Spec}(K[\Delta])$ *is* CM.

(2) *$\Delta$ is* ACM *if and only if* $\mathrm{Proj}(K[\Delta])$ *is* CM.

*Proof.* This is an easy consequence of the universal coefficient theorem and the fact that the CM property is preserved by localization. Q.E.D.

The rank $r(\Delta)$ of a simplicial complex $\Delta$ is the largest cardinality of a simplex. We say that $\Delta$ is *pure* if every maximal simplex of $\Delta$ has $r(\Delta)$ vertices. The following result is an improvement of [2, Theorem 6.5]:

THEOREM 2.2. *Let $\Delta$ be a pure* ACM *complex on vertex set $V$. Suppose that we can partition $V$ into two subsets $V_1$ and $V_2$ such that $\Delta \mid V_1$ and $\Delta \mid V_2$ are both pure. Then $\Delta \mid V_1$ is also* ACM *and the inclusion $\Delta \mid V_1 \to \Delta$ induces linear maps*

$$\tilde{H}^i(\Delta) \to \tilde{H}^i(\Delta \mid V_1),$$

*which are isomorphism when $i < r(\Delta \mid V_1) - 1$ and injective when $i = r(\Delta \mid V_1) - 1$.*

When $V$ can be partitioned as in Theorem 2.2, we say that $\Delta$ is *balanced of type* $(r(\Delta \mid V_1), r(\Delta \mid V_2))$. More generally, we say that $\Delta$ is balanced of type $(b_1, ..., b_k)$ when $V$ may be partitioned into subsets $V_1, ..., V_k$ such that $r(\Delta \mid V_i) = b_i$ and $\Delta \mid V_i$ is pure for every $i$. The notation is due to Stanley [9]. If $b_1 = b_2 = \cdots = b_k = 1$, then we say that $\Delta$ is *completely balanced.* In this case, if $S \subseteq [r(\Delta)]$, then we write $\Delta_S$ for $\Delta \mid \bigcup_{i \in S} V_i$.

*Proof.* One may use the method employed in [2, Theorem 6.5], but a more succinct proof may be obtained by using the technique described in the proof of [4, Proposition 3.1]. Q.E.D.

We now use Theorem 2.2 to compute the Hilbert series of the Stanley–Reisner ring of a completely balanced ACM complex. To avoid cumbersome notation we abbreviate $(t_1, ..., t_r)$ to $(t)$ and $(\theta_1, ..., \theta_r)$ to $(\theta)$, and we use the convention that

$$\sum_{U \subseteq [j-1]} t^U t_j \chi(|U| = i) = 1 \qquad \text{when} \quad i = -1.$$

THEOREM 2.3. *Let $\Delta$ be a completely balanced* ACM *complex of rank $r$. Then the Hilbert series of $K[\Delta]$ is*

$$H(K[\Delta]; t) = p_0(t) + \cfrac{p_1(t) + \cfrac{\ddots}{\ddots + \cfrac{p_{r-1}(t) + (p_r(t)/(1-t_r))}{1 - t_{r-1}}}}{1 - t_1},$$

*where $p_j(t) = \sum_{i=-1}^{j-1} \tilde{h}_i(\Delta) \sum_{U \subseteq [j-1]} t^U t_j \chi(|U| = i)$, if $j = 0, ..., r-1$, and $p_r(t) = H(K[\Delta]/(\theta); t) - \sum_{j=0}^{r-1} p_j(t)$.*

*Proof.* By Theorem 2.2, we can compute $\mu(\Delta_S)$ for any $S \subseteq [r]$,

$$\mu(\Delta_S) = \sum_{i=-i}^{|S|-1} (-1)^i \tilde{h}_i(\Delta_S) = \sum_{i=-1}^{|S|-2} (-1)^i \tilde{h}_i(\Delta) + (-1)^{|S|-1} \tilde{h}_{|S|-1}(\Delta_S).$$

If we multiply this equation by $(-1)^{|S|-1} t^S$ and sum over all $S \subseteq [r]$, we obtain

$$\sum_{S \subseteq [r]} (-1)^{|S|-1} \mu(\Delta_S) t^S = \sum_{S \subseteq [r]} (-1)^{|S|-1} \sum_{i=-1}^{|S|-2} (-1)^i \tilde{h}_i(\Delta) t^S$$
$$+ \sum_{S \subseteq [r]} \tilde{h}_{|S|-1}(\Delta_S) t^S.$$

By a computation due to Stanley (see [4, Proposition 3.2]), the left-hand side of Eq. (2.1) is $H(K[\Delta]; t) \prod_{i=1}^r (1-t_i)$. On the other hand, the second term on the right-hand side is $H(K[\Delta]/(\theta); t)$. This follows from [4, Theorem 5.1]. Although the result discussed there was only for order complexes of partially ordered sets, the result easily generalizes to completely balanced complexes. We therefore have

$$H(K[\Delta]; t) \prod_{i=1}^{r} (1-t_i) = \sum_{S \subseteq [r]} (-1)^{|S|-1} \sum_{i=-1}^{|S|-2} (-1)^i \tilde{h}_i(\Delta) t^S$$
$$+ H(K[\Delta]/(\theta); t). \tag{2.2}$$

It remains to find a suitable expression for the first term on the right-hand side of Eq. (2.2).

First we note that $\tilde{h}_{-1}(\Delta) = 0$ unless $\Delta = \{\varnothing\}$. In this case it is easy to verify the theorem directly. Thus we henceforth assume that $\Delta \neq \{\varnothing\}$. Now rewrite $\sum_{i=0}^{|S|-2}$ as a sum over $j \in S'$, where $S'$ is obtained from $S$ by deleting its last element. Let $U$ be the set of elements of $S$ that precede $j$, and let $V$ be the set of elements of $S$ that follow $j$. We recover $i$ as $|U|$, while $S$ is just $u \cup \{j\} \cup V$. As $S$ varies over all subsets of $[r]$ having at least two elements, $j$ goes from 1 to $r-1$, $U$ varies over all subsets of $[j-1]$ and $V$ varies over all nonempty subsets of $[r] \backslash [j]$. Thus

$$\sum_{S \subseteq [r]} (-1)^{|S|-1} \sum_{i=0}^{|S|-2} (-1)^i \tilde{h}_i(\Delta) t^S$$
$$= \sum_{S \subseteq [r]} \chi(|S| \geqslant 2)(-1)^{|S|-1} \sum_{j \in S'} (-1)^{|U|} \tilde{h}_{|U|}(\Delta) t^S$$
$$= \sum_{j=1}^{r-1} \sum_{U} \chi(U \subseteq [j-1]) \sum_{V} \chi(\varnothing \neq V \subseteq [r] \backslash [j])$$
$$\times (-1)^{|U|+|V|} (-1)^{|U|} \tilde{h}_{|U|}(\Delta) t^U t_j t^V$$

$$= \sum_{j=1}^{r-1} \sum_{i=0}^{j-1} \sum_{U} \chi(U \subseteq [j-1])\chi(|U| = i)\, \tilde{h}_i(\Delta)\, t^U t_j$$

$$\times \sum_{V} \chi(\varnothing \neq V \subseteq [r] \backslash [j])(-1)^{|V|}\, t^V$$

$$= \sum_{j=1}^{r-1} \sum_{i=0}^{j-1} \tilde{h}_i(\Delta) \sum_{U \subseteq [j-1]} \chi(|U| = i)\, t^U t_j \left( \prod_{k=j+1}^{r} (1 - t_k) - 1 \right)$$

$$= \sum_{j=1}^{r-1} p_j(t) \left( \prod_{k=j+1}^{r} (1 - t_k) - 1 \right).$$

Combining this with (2.2) gives us

$$H(K[\Delta]; t) \prod_{i=1}^{r} (1 - t_i) = \sum_{j=0}^{r-1} p_j(t) \left( \prod_{k=j+1}^{r} (1 - t_k) - 1 \right) + H(K[\Delta]/(\theta); t). \tag{2.3}$$

The result now follows easily. Q.E.D.

As this paper was being prepared, we learned that Schenzel [5] (see also [6]) had proved Theorem 2.3 without the requirement that $\Delta$ be completely balanced.

## 3. Quasigenerators and Separators

We now give a ring-theoretic interpretation of the peculiar form of the Hilbert series of $K[\Delta]$ given in Theorem 2.3, by exhibiting a combinatorial decomposition of $K[\Delta]$. The calculation of the separators depends on the factorization of the homomorphism in Theorem 2.2,

Lemma 3.1. *Let $\Delta$ be a completely balanced complex of rank $r$. Let $S \subsetneq [r]$ be a proper subset, and write $s$ for $|S|$. Then there are linear maps*

$$\tilde{H}^{s-1}(\Delta) \xrightarrow{\bar{\omega}} \mathscr{H}_S \operatorname{ann}_A(\theta_i \mid i \notin S) \xrightarrow{\bar{\omega}} \tilde{H}^{s-1}(\Delta_S),$$

*where $A = K[\Delta]/(\theta_i \mid i \in S)$. Moreover, the composition of the two maps above is the injective map given by Theorem 2.2.*

*Proof.* There is a natural inclusion $\phi\colon \tilde{C}^*(\Delta, K) \to K[\Delta]$ defined by $\phi(\sigma^*) = \prod_{v \in \sigma} X_v$, where $\sigma^*$ is the cochain dual to the chain $\sigma \in \Delta$. Furthermore, the coboundary map $\delta$ on $\tilde{C}^*(\Delta, K)$ can be computed in $K[\Delta]$.

Let $f_T \in K[\Delta]$ be homogeneous of multidegree $T \subseteq [r]$. Then $f_T$ is the image of an element $\alpha \in \tilde{C}^*(\Delta, K)$ and

$$\phi(\delta(\alpha)) = \sum_{i \notin T} (-1)^{n(i,T)} \theta_i f_T, \tag{3.1}$$

where $n(i, T)$ is the number of elements of $T$ that precede $i$. To avoid cumbersome notation we will henceforth regard $\phi$ as an inclusion.

Now let $f \in \tilde{C}^s(\Delta, K) \subseteq K[\Delta]$ be homogeneous of degree $s$. Write $f_T$ for the homogeneous part of $f$ of multidegree $T$, where $T \subseteq [r]$ and $|T| = s$. By formula (3.1), the homogeneous part of $\delta(f)$ having multidegree $U$ is easily seen to be

$$\sum_{j \in U} (-1)^{n(j, U \setminus \{j\})} \theta_j f_{U \setminus \{j\}} \tag{3.2}$$

Therefore if $f \in \operatorname{Ker}(\delta)$ then all the above polynomials vanish. In particular, if $U$ has the form $S \cup \{i\}$, for a fixed $i \notin S$, then we have that

$$(-1)^{n(i,S)} \theta_i f_S = -\sum_{j \in S} (-1)^{n(j, U \setminus \{j\})} \theta_i f_{U \setminus \{j\}},$$

and hence that $\theta_i f_S \in (\theta_j \mid j \in S)$. Since this holds for any $i \notin S$, we can define a map

$$\psi \colon \operatorname{Ker}(\delta^{s-1}) \to ((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S))$$

by $\psi(f) = f_S$.

We now show that $\psi(\operatorname{Im}(\delta^{s-2})) \subseteq (\theta_j \mid j \in S)$. Let $f \in \operatorname{Im}(\delta^{s-2})$. Choose $g$ so that $\delta(g) = f$. By formula (3.2),

$$\psi(f) = f_S = \delta(g)_S = \sum_{j \in S} (-1)^{n(j, S \setminus \{j\})} \theta_j g_{S \setminus \{j\}}.$$

Therefore $\psi(f) \in (\theta_j \mid j \in S)$ as desired, and hence $\psi$ induces a map

$$\bar{\psi} \colon \tilde{H}^{s-1}(\Delta) \to ((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S))/(\theta_j \mid j \in S) = \operatorname{ann}_A(\theta_i \mid i \notin S).$$

It remains to define a map

$$\mathscr{H}_S \operatorname{ann}_A(\theta_i \mid i \notin S) \to \tilde{H}^{s-1}(\Delta_S).$$

We begin with the inclusion

$$\omega \colon \mathscr{H}_S((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S)) \to \tilde{C}^{s-1}(\Delta_S, K).$$

Now suppose that $f \in \mathscr{H}_S(\theta_j \mid j \in S) \cap ((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S))$. Then we may write $f$ in the form $\sum_{j \in S} \theta_j f_j$, where $f_j$ is homogeneous of multidegree $S \setminus \{j\}$. Set $g_j = (-1)^j f_j$. Let $\delta_S$ be the coboundary map of the complex $\tilde{C}^*(\Delta_S, K)$. Then $\delta_S(\sum_{j \in S} g_j) = \sum_{j \in S} (-1)^j \theta_j g_j = \sum_{j \in S} \theta_j f_j = f$. Hence the map $\omega$ induces a map

$$\mathscr{H}_S \operatorname{ann}_A(\theta_i \mid i \notin S) = \mathscr{H}_S((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S))/(\theta_j \mid j \in S) \xrightarrow{\bar{\omega}} \tilde{H}^{s-1}(\Delta_S).$$

Finally the composition $\bar{\omega} \circ \bar{\psi}$ is induced from the map $\alpha \colon \tilde{C}^{s-1}(\Delta, K) \to \tilde{C}^{s-1}(\Delta_S, K)$ given by

$$\begin{aligned} \alpha(\sigma^*) &= \sigma^*, \qquad && \text{if} \quad \sigma \in \Delta_S, \\ &= 0, && \text{if} \quad \sigma \notin \Delta_S. \end{aligned}$$

This is precisely the map in Theorem 2.2. Q.E.D.

We are now ready to construct the separators for the frame $(\theta_1, \ldots, \theta_r)$ of $K[\Delta]$.

DEFINITION 3.4. Let $\Delta$ be as in Lemma 3.1.

(a) For each $S \subsetneq [r]$ choose sets $\mathscr{A}(S)$ and $\mathscr{B}(S)$ of elements of $\mathscr{H}_S K[\Delta]$ so that

(1) $\mathscr{A}(S) \subseteq \mathscr{H}_S((\theta_i \mid i \notin S) \colon (\theta_j \mid j \in S))$;

(2) $\bar{\omega}(\mathscr{A}(S))$ is a basis of $\operatorname{Im}(\bar{\omega})$;

(3) the image of $\mathscr{A}(S) \cup \mathscr{B}(S)$ is a basis of $\tilde{H}^{s-1}(\Delta_S)$; and

(4) $\mathscr{A}(S) \cap \mathscr{B}(S) = \varnothing$.

Using the identifications in the proof of Lemma 3.1, one chooses $\mathscr{A}(S) \subseteq \tilde{C}^{s-1}(\Delta_S, K)$ to form a basis of $\operatorname{Im}(\bar{\omega})$, and then this set is extended to form a basis $\mathscr{A}(S) \cup \mathscr{B}(S)$ of $\tilde{H}^{s-1}(\Delta_S)$.

(b) For $S = [r]$, we define $\mathscr{A}(S)$ to be $\varnothing$ and $\mathscr{B}(S)$ to be any set of elements of $\mathscr{H}_S K[\Delta]$ whose image in $\tilde{H}^{r-1}(\Delta)$ forms a basis.

(c) The complete set of separators is the union

$$\mathscr{S} = \bigcup_{S \subseteq [r]} (\mathscr{A}(S) \cup \mathscr{B}(S)).$$

(d) The level map $k \colon \mathscr{S} \to \mathbb{N}$ is given by

(1) if $f \in \mathscr{A}(S)$, then $k(f) = \max\{j \mid j \in S\}$;

(2) if $f \in \mathscr{B}(S)$, then $k(f) = r$.

We will later see (Theorem 4.1) that when $\Delta$ is ACM the map $\bar{\psi}$ is an isomorphism and hence the set $\mathscr{A}(S)$ may be regarded as a basis of $\tilde{H}^{|S|-1}(\Delta)$.

THEOREM 3.5. *Let $\Delta$ be a completely balanced complex of rank r. The set of homogeneous polynomials*

$$\mathscr{E} = \left\{ \gamma \prod_{i=1}^{k(\gamma)} \theta_i^{n_i} \mid \gamma \in \mathscr{S} \text{ and } n_i \geqslant 0 \right\}$$

*spans $K[\Delta]$, where $(\theta, \mathscr{S}, k)$ is defined in Definition* 3.4.

*Proof.* By [4, Theorem 5.1] it follows that the set $\{\gamma \prod_{i=1}^{r} \theta_i^{n_i} \mid \gamma \in \mathscr{S}, n_i \geqslant 0\}$ spans $K[\Delta]$ because $\mathscr{S}$ defines a basis of $\oplus \sum_{S \subseteq [r]} \tilde{H}^{|S|-1}(\Delta_S)$. For an expression $\gamma \prod_{i=1}^{r} \theta_i^{n_i}$ let $l(n_1,..., n_r)$ be the largest index $l$ such that $n_l > 0$. We wish to show that if $l(n_1,..., n_r) > k(\gamma)$, then $\gamma \prod_{i=1}^{r} \theta_i^{n_i}$ is a linear combination of elements of $\mathscr{E}$. We do this by induction on degree, and for a fixed degree $n$ we use induction on $l(n_1,..., n_r)$.

The case $n = 0$ is trivial; and for fixed $n > 0$, the case $l(n_1,..., n_r) = 1$ is vacuous because $k(\gamma) \geqslant 1$ if $\gamma \in \mathscr{S}$ and $\Delta \neq \{\varnothing\}$. Next let $\gamma \prod_{i=1}^{l} \theta_i^{n_i}$ have degree $n$ and satisfy $n_l \neq 0$. If $\gamma \in \mathscr{B}(S)$ for some $S$ there is nothing to show, so we may assume that $\gamma \in \mathscr{A}(S)$ for some $S \subsetneq [r]$ and that $l > k(\gamma)$. By the choice of $\gamma$, we know that $\gamma \in ((\theta_i \mid i \notin S) : (\theta_j \mid j \in S))$. Now $k(\gamma)$ is the largest element of $S$ and hence $l \notin S$. Since $n_l \neq 0$, we have that $\prod_{i=1}^{l} \theta_i^{n_i} \in (\theta_i \mid i \notin S)$. Therefore $\gamma \prod_{i=1}^{l} \theta_i^{n_i} \in (\theta_j \mid j \in S)$, say $\gamma \prod_{i=1}^{l} \theta_i^{n_i} = \sum_{j \in S} \theta_j g_j$. Now each $g_j$ has degree $n-1$ and so by induction is in the span of $\mathscr{E}$. It follows immediately that $\gamma \prod_{i=1}^{l} \theta_i^{n_i}$ is in the span of $\bigcup_{j \in S} \theta_j \mathscr{E}$. Let $\beta \prod_{i=1}^{k(\beta)} \theta_i^{m_i} \in \mathscr{E}$, and consider $\beta \theta_j \prod_{i=1}^{k(\beta)} \theta_i^{m_i}$. If $j \leqslant k(\beta)$, then $\beta \theta_j \prod_{i=1}^{k(\beta)} \theta_i^{m_i} \in \mathscr{E}$. If $j > k(\beta)$, then since $j < l$ we also have $k(\beta) < l$; hence $\beta \theta_j \prod_{i=1}^{k(\beta)} \theta_i^{m_i}$ has the form $\beta \prod_{i=1}^{r} \theta_i^{m'_i}$, where $l(m'_1,..., m'_r) < l$. By induction on $l$, $\beta \theta_j \prod_{i=1}^{k(\beta)} \theta_i^{m_i}$ is in the span of $\mathscr{E}$. Thus every element of $\bigcup_{j \in S} \theta_j \mathscr{E}$ is in the span of $\mathscr{E}$. Therefore, $\gamma \prod_{i=1}^{l} \theta_i$ is in the span of $\mathscr{E}$ and the result follows. Q.E.D.

THEOREM 3.6. *Let $\Delta$ be a completely balanced* ACM *complex of rank r. Then $(\theta, \mathscr{S}, k)$, as defined in Definition* 3.4, *is a combinatorial decomposition of $K[\Delta]$.*

*Proof.* The (fine-graded) Hilbert series of $\mathscr{E}$ is easily seen to be

$$F(t) = \sum_{\gamma \in \mathscr{S}} t^{\deg(\gamma)} \prod_{i=1}^{k(\gamma)} (1 - t_i)^{-1},$$

where $\deg(\gamma)$ is the multidegree of $\gamma$. By the construction of $\mathscr{E}$ we have that

$$F(t)=\sum_{S\subsetneq[r]}\left(\sum_{\gamma\in\mathscr{A}(S)} t^S\prod_{i=1}^{k(S)}(1-t_i)^{-1}+\sum_{\gamma\in\mathscr{B}(S)} t^S\prod_{i=1}^{r}(1-t_i)^{-1}\right)$$
$$+\sum_{\gamma\in\mathscr{B}([r])}\prod_{i=1}^{r} t_i(1-t_i)^{-1}.$$
$$=\sum_{S\subsetneq[r]}\left(d_S t^S\prod_{i=1}^{k(S)}(1-t_i)^{-1}+(\tilde{h}_{|S|-1}(\Delta_S)-d_S)\,t^S\prod_{i=1}^{r}(1-t_i)^{-1}\right)$$
$$+\tilde{h}_{r-1}(\Delta)\,t^{[r]}\prod_{i=1}^{r}(1-t_i)^{-1},$$

where $d_S=\dim\operatorname{Im}(\bar{\omega})$ and $\bar{\omega}$ is the map given in Lemma 3.1. Finally, we rearrange the sum above to give

$$F(t)=\sum_{S\subseteq[r]} d_S t^S\prod_{i=1}^{k(S)}(1-t_i)^{-1}+\sum_{S\subseteq[r]}(\tilde{h}_{|S|-1}(\Delta_S)-d_S)\,t^S\prod_{i=1}^{r}(1-t_i)^{-1},$$

where $d_{[r]}=0$. By Theorems 2.1 and 3.1, it follows that $d_S\geqslant\tilde{h}_{|S|-1}(\Delta)$.

We now use the computation in Theorem 2.3 of the Hilbert series of $K[\Delta]$. The form most useful to us is (2.3)

$$H(K[\Delta];t)\prod_{i=1}^{r}(1-t_i)=\sum_{j=0}^{r-1}p_j(t)\left(\prod_{k>j}(1-t_k)-1\right)+H(K[\Delta]/(\theta);t)$$
$$=\sum_{j=0}^{r}p_j(t)\left(\prod_{k>j}(1-t_k)-1\right)+\sum_{S\subseteq[r]}\tilde{h}_{|S|-1}(\Delta_S)\,t^S,$$

where $p_j(t)=\sum_{i=0}^{j-1}\tilde{h}_i(\Delta)\sum_{U\subseteq[j-1]}\chi(|U|=i)\,t^U t_j$. Next we rewrite $F(t)$ in a form that begins to resemble $H(K[\Delta];t)$, by making the change of variables $S=U\cup\{k\}$, where $k=k(S)$

$$F(t)=\sum_{k=1}^{r}\sum_{U\subseteq[k-1]} d_{U\cup\{k\}}t^U t_k\prod_{i=1}^{k}(1-t_i)^{-1}$$
$$+\sum_{S\subseteq[r]}\tilde{h}_{|S|-1}(\Delta_S)\,t^S\prod_{i=1}^{r}(1-t_i)^{-1}$$
$$-\sum_{k=1}^{r}\sum_{U\subseteq[k-1]} d_{U\cup\{k\}}t^U t_k\prod_{i=1}^{r}(1-t_i)^{-1}.$$

Therefore,

$$F(t)\prod_{i=1}^{r}(1-t_i)=\sum_{k=1}^{r}\sum_{U\subseteq[k-1]}d_{U\cup\{k\}}t^U t_k\left(\prod_{i=k+1}^{r}(1-t_i)-1\right)$$
$$+\sum_{S\subseteq[r]}\tilde{h}_{|S|-1}(\Delta_S)\,t^S.$$

Now form the difference

$$(H(K[\Delta];t)-F(t))\prod_{i=1}^{r}(1-t_i)$$
$$=\sum_{k=1}^{r}\sum_{U\subseteq[k-1]}(\tilde{h}_{|U|}(\Delta)-d_{U\cup\{k\}})\,t^U t_k\left(\prod_{i=k+1}^{r}(1-t_i)-1\right).$$

Hence,

$$H(K[\Delta];t)-F(t)=\sum_{k=1}^{r}\sum_{U\subseteq[k-1]}(d_{U\cup\{k\}}-\tilde{h}_{|U|}(\Delta))$$
$$\times\, t^U t_k\left(\prod_{i=1}^{r}(1-t_i)^{-1}-\prod_{i=1}^{k}(1-t_i)^{-1}\right).$$

Now since $\prod_{i=1}^{r}(1-t_i)^{-1}-\prod_{i=1}^{k}(1-t_i)^{-1}$ has nonnegative coefficients (and vanishes when $k=r$) and since $d_{U\cup\{k\}}\geqslant\tilde{h}_{|U|}(\Delta)$ when $k\neq r$, we conclude that $F(t)\leqslant H(K[\Delta];t)$. On the the other hand, since $F(t)$ is the generating function of the set $\mathscr{E}$ which spans $K[\Delta]$, we also know that $H(K[\Delta];t)\leqslant F(t)$ and hence that $F(t)=H(K[\Delta];t)$. We conclude that $\mathscr{E}$ is a basis for $K[\Delta]$ and as a bonus we also have that $d_S=\tilde{h}_{|S|-1}(\Delta)$ for every $S\subsetneq[r]$. This gives us Definition 1.1(2). Since Definition 1.1(3) holds by definition of $\mathscr{S}$, Theorem 3.6 follows. Q.E.D.

## 4. Homology and Annihilators

For a completely balanced CM complex $\Delta$, the only nonzero cohomology appears in the top dimension. For this cohomology we can give a ring-theoretic interpretation

$$\tilde{H}^{r-1}(\Delta)\cong\mathscr{H}_{[r]}K[\Delta]/(\theta_1,\dots,\theta_r).$$

For a completely balanced ACM complex, we can have nonzero cohomology in every dimension. Our purpose in this section is to give a ring-theoretic expression for every (reduced) cohomology module

$$\tilde{H}^{l-1}(\Delta) \cong \mathscr{H}_S \operatorname{ann}_{A(S)}(\theta_i \mid i \notin S),$$

where $S$ is any subset of $[r]$ having $l$ elements and $A(S) = K[\Delta]/(\theta_i \mid i \in S)$. More precisely, we have

THEOREM 4.1. *Let $\Delta$ be a completely balanced* ACM *complex. Then the map $\bar{\psi}$ in Theorem* 2.2 *is an isomorphism*

$$\bar{\psi}\colon \tilde{H}^{|S|-1}(\Delta) \simeq \mathscr{H}_S \operatorname{ann}_{A(S)}(\theta_i \mid i \notin S),$$

*for any* $S \subseteq [\operatorname{rank}(\Delta)]$.

*Proof.* We may assume that $S$ is a proper subset of $[\operatorname{rank}(\Delta)]$, since the case $S = \varnothing$ is trivial while the case $S = [r]$ follows from [4, Theorem 5.1]. By Theorem 2.2 we know that $\bar{\psi}$ is injective; and we also know that if $\mathscr{A}(S)$ is any subset of $((\theta_j \mid j \in S)\colon (\theta_i \mid i \notin S))$ whose image (under $\omega$) is a basis of $\operatorname{Im}(\bar{\omega})$, then $|\mathscr{A}(S)| = \tilde{h}_{|S|-1}(\Delta)$. The result will follow if we can show that $\mathscr{A}(S)$ spans $\operatorname{ann}_{A(S)}(\theta_i \mid i \notin S)$.

Let $\bar{f} \in \mathscr{H}_S \operatorname{ann}_{A(S)}(\theta_i \mid i \notin S)$. Then $\bar{f}\theta_i = 0$ in $A(S)$ for any $i \notin S$. Let $l$ be the largest element of $S$. Then $\bar{f}\theta_i \in (\theta_j \mid j \leqslant l)$ for every $i > l$. As in the proof of Theorem 1.2, we have that $\bar{\bar{f}} \in M(l)$, where $\bar{\bar{f}}$ is the image of $\bar{f}$ in $A(l)$. By Theorem 1.2, we know that $\bar{\bar{f}}$ can be written as $\bar{\bar{f}} = \sum_{\eta \in S_l} c_\eta \bar{\bar{\eta}}$, where $c_\eta \in K$ for every $\eta$. However, since $\bar{f}$ has multidegree $S$ we must have $c_\eta = 0$ for $\eta \notin \mathscr{A}(S)$ if $l \neq r$. For the time being we assume that $l < r$. Thus

$$\bar{\bar{f}} = \sum_{\eta \in \mathscr{A}(S)} c_\eta \bar{\bar{\eta}}.$$

Returning to $A(S)$, the equations above imply that

$$\bar{f} - \sum_{\eta \in \mathscr{A}(S)} c_\eta \bar{\eta} \subseteq (\theta_1, \ldots, \theta_l).$$

However, the expression above has multidegree $S$ while any homogeneous element of $(\theta_1, \ldots, \theta_l)$ having this multidegree is necessarily of the form $\sum_{j \in S} h_j \theta_j$ and hence is in $(\theta_j \mid j \in S)$. Therefore we have that $\bar{f} = \sum_{\eta \in \mathscr{A}(S)} c_\eta \bar{\eta}$, and the result follows in the case $l < r$.

To obtain the general case we note that the choice of $\mathscr{A}(S)$ and of $\mathscr{B}(S)$ does not depend on the ordering of elements of $[r]$; only the level function requires this order. Thus any other total order on $[r]$ will produce a new combinatorial decomposition $(\theta', \mathscr{S}, k')$, having the same set of separators. Since $S$ is a proper subset of $[r]$, we can choose a total order on $[r]$ so that the largest elements of $S$ and of $[r]$ are different, and this is all we need for the proof above to apply. Q.E.D.

The requirement in Theorem 4.1 that $\Delta$ be completely balanced may always be arranged by passing to the barycentric subdivision. It is not clear whether we can omit the requirement that $\Delta$ be ACM, but experimental evidence suggests that one can weaken this hypothesis or possibly even omit it entirely.

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