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Estimates for solutions of Burgers type equations and some applications

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Abstract

We obtain precise large time asymptotics for the Cauchy problem for Burgers type equations satisfying shock profile condition. The proofs are based on the exact a priori estimates for (local) solutions of these equations and the result of [G.M. Henkin, A.A. Shananin, Asymptotic behavior of solutions of the Cauchy problem for Burgers type equations, J. Math. Pures Appl. 83 (2004) 1457–1500].

Résumé

Nous déterminons des asymptotiques précises, en grand temps, des solutions du problème de Cauchy pour des équations du type Burgers, solutions admettant des profils de chocs. Les démonstrations utilisent des résultats de [G.M. Henkin, A.A. Shananin, Asymptotic behavior of solutions of the Cauchy problem for Burgers type equations, J. Math. Pures Appl. 83 (2004) 1457–1500] et des estimations a priori précises des solutions de ces équations.
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1. Introduction

The Burgers type equations have been introduced for studying different models of fluids [1,3,4,10]. The difference-differential analogues of these equations have been proposed in some models of economic development [5,6].

One of the most useful versions of the Burgers type equations is the following [4,12,14]:

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \varepsilon \frac{\partial^2 f}{\partial x^2}, \tag{1.1}$$

where $\varepsilon > 0$, $(x,t) \in \Omega \subset \mathbb{R}^2$.

One of the most interesting difference-differential analogues of Eq. (1.1) is the following [5,6]:

$$\frac{\partial F}{\partial t} + \varphi(F)\frac{F(x,t) - F(x-\varepsilon,t)}{\varepsilon} = 0, \tag{1.2}$$

where $\varepsilon > 0$, $(x,t) \in \Omega \subset \mathbb{R}^2$.

The interesting and difficult problems, related with Eqs. (1.1), (1.2), are the following:

Problem I [4,12]. Find asymptotic ($t \to \infty$) of the solution $f(x,t)$, $x \in \mathbb{R}$, $t \geqslant t_0$, of Eq. (1.1) with initial condition:

$$\alpha \leqslant f(x,t_0) \leqslant \beta, \quad \int_{-\infty}^{0} \big(f(x,t_0) - \alpha\big)\,\mathrm{d}x + \int_{0}^{\infty} \big(\beta - f(x,t_0)\big)\,\mathrm{d}x < \infty. \tag{1.3}$$

Problem II [6]. Find asymptotic ($t \to \infty$) of the solution $F(n,t)$, $n \in \mathbb{Z}$, $t \geqslant t_0$, of Eq. (1.2) with $\varepsilon = 1$, and initial condition:

$$\alpha \leqslant F(n,t_0) \leqslant \beta, \quad \sum_{-\infty}^{0}\big(F(n,t_0) - \alpha\big) + \sum_{0}^{\infty}\big(\beta - F(n,t_0)\big) < \infty. \tag{1.4}$$

See [7,14] for a review of several recent results on these problems.

In this paper we present a complete solution of these problems for the special case of equations, satisfying the shock profile condition. The detailed study of this special case is highly important for solving these problems (see [6,7]).

Definition. Eq. (1.1) (correspondingly (1.2)) satisfies (α,β)-shock profile condition, if there exist wave-train solutions of this equation of the form $f = \tilde{f}(x - Ct)$ (correspondingly $F = \widetilde{F}(x - Ct)$) such that $\tilde{f}(x) \to \beta$, $x \to +\infty$, $\tilde{f}(x) \to \alpha$, $x \to -\infty$ (correspondingly $\widetilde{F}(x) \to \beta$, $x \to +\infty$, $\widetilde{F}(x) \to \alpha$, $x \to -\infty$).

From the results of [4,13], it follows that Eq. (1.1) with positive φ satisfies $(0,1)$-shock profile condition iff:

$$\frac{1}{u}\int_0^u \varphi(y)\,\mathrm{d}y > C = \int_0^1 \varphi(y)\,\mathrm{d}y, \quad \forall u \in (0,1). \tag{1.5}$$

From the results of [5,2] it follows that Eq. (1.2) with positive φ satisfies $(0,1)$-shock profile condition iff:

$$\frac{1}{u}\int_0^u \frac{\mathrm{d}y}{\varphi(y)} < \frac{1}{C} = \int_0^1 \frac{\mathrm{d}y}{\varphi(y)}, \quad \forall u \in (0,1). \tag{1.6}$$

Let further φ be a positive piecewise twice continuously differential function on the interval $[0,1]$.

Theorem 1. (i) *Let Eq.* (1.1) *satisfy* $(0,1)$*-shock profile condition* (1.5); $\varphi'(0) \neq 0$ *if* $\varphi(0) = C$; $\varphi'(1) \neq 0$ *if* $\varphi(1) = C$. *Let* $f(x,t)$ *be a solution of* (1.1) *with initial condition* (1.3), *where* $\tilde f(x - Ct)$ *is a wave-train solution of* (1.1), (1.3), *where* $\alpha = 0$, $\beta = 1$. *Then there exist constants* γ_0 *and* d_0 *such that*

$$\sup_{x\in\mathbb{R}}\bigl|f(x,t) - \tilde f(x - Ct + \varepsilon\gamma_0 \ln t + d_0)\bigr| \to 0, \quad t \to \infty, \tag{1.7}$$

$$\gamma_0 = \begin{cases} 0, & \text{if } \varphi(0) > C > \varphi(1),\\ \frac{1}{\varphi'(1)}, & \text{if } \varphi(0) > C = \varphi(1),\\ -\frac{1}{\varphi'(0)}, & \text{if } \varphi(0) = C > \varphi(1),\\ \frac{1}{\varphi'(1)} - \frac{1}{\varphi'(0)}, & \text{if } \varphi(0) = C = \varphi(1). \end{cases}$$

(ii) *Let Eq.* (1.2) *satisfy* $(0,1)$*-shock profile condition* (1.6); $\varphi'(0) \neq 0$ *if* $\varphi(0) = C$; $\varphi'(1) \neq 0$ *if* $\varphi(1) = C$. *Let* $F(n,t)$ *be a solution of* (1.2) *with initial condition* (1.4), *where* $\widetilde F(x - Ct)$ *is a wave-train solution of* (1.2), (1.4), *where* $\alpha = 0$, $\beta = 1$; $\varepsilon = 1$. *Let* $\Delta F(n,t_0) \stackrel{\text{def}}{=} F(n,t_0) - F(n-1,t_0) \geqslant 0$. *Then there exist constants* Γ_0 *and* D_0 *such that*

$$\sup_{n\in\mathbb{Z}}\bigl|F(n,t) - \widetilde F(n - Ct + \Gamma_0 \ln t + D_0)\bigr| \to 0, \quad t \to \infty, \tag{1.8}$$

$$\Gamma_0 = \begin{cases} 0, & \text{if } \varphi(0) > C > \varphi(1),\\ \frac{C}{2\varphi'(1)}, & \text{if } \varphi(0) > C = \varphi(1),\\ -\frac{C}{2\varphi'(0)}, & \text{if } \varphi(0) = C > \varphi(1),\\ \frac{C}{2}\bigl(\frac{1}{\varphi'(1)} - \frac{1}{\varphi'(0)}\bigr), & \text{if } \varphi(0) = C = \varphi(1). \end{cases}$$

Remarks. (1) In the case $\varphi(0) > C > \varphi(1)$ the statement (i) of Theorem 1 is the main result of [9] and the statement (ii) of Theorem 1 is the main result of [5].

(2) For the other cases when $\varphi(0) = C$ or $\varphi(1) = C$ or $\varphi(0) = \varphi(1) = C$ in the previous work [7] it was already obtained the existence of the shift-functions $\gamma(t) = \mathrm{O}(\ln t)$ and $\Gamma(t, \{x\}) = \mathrm{O}(\ln t)$ with the properties:

$$\sup_x \bigl| f(x,t) - \tilde{f}\bigl(x - Ct + \varepsilon\gamma(t)\bigr)\bigr| \to 0,$$
$$\sup_x \bigl| F(x,t) - \widetilde{F}\bigl(x - Ct + \varepsilon\Gamma\bigl(t, \{x/\varepsilon\}\bigr)\bigr)\bigr| \to 0, \quad t \to \infty,$$

where f, F are solutions of (1.1), (1.2) under conditions (1.3), (1.4), $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

(3) It is interesting to compare the statements (i), (ii) of Theorem 1 with the L^1-stability results presented in the paper of D. Serre [14]. Results of [14] give in particular the following.

Let $f(x,t)$ and $F(n,t)$ be solutions of Eqs. (1.1) and (1.2) correspondingly with such initial conditions that

$$\int_{-\infty}^{\infty} \bigl| f(x,0) - \tilde{f}(x)\bigr| \,\mathrm{d}x < \infty, \qquad \sum_{-\infty}^{\infty} \bigl| F(n,0) - \widetilde{F}(n)\bigr| < \infty,$$

where $\tilde{f}(x - Ct)$ and $\widetilde{F}(n - Ct)$ are wave-trains solutions of (1.1), (1.2). Then

$$\int_{-\infty}^{\infty} \bigl| f(x,t) - \tilde{f}(x - Ct + d_0)\bigr| \,\mathrm{d}x \to 0,$$
$$\sum_{-\infty}^{\infty} \bigl| F(n,t) - \widetilde{F}(n - Ct + D_0)\bigr| \to 0, \quad t \to \infty,$$

where constants d_0 and D_0 are being calculated from equations:

$$\int_{-\infty}^{\infty} \bigl(f(x,0) - \tilde{f}(x + d_0)\bigr) \,\mathrm{d}x = 0 \quad \text{and} \quad \sum_{-\infty}^{\infty} \int_{\widetilde{F}(n+D_0)}^{F(n,0)} \frac{\mathrm{d}y}{\varphi(y)} = 0.$$

The proof of Theorem 1 is based on the results of [7] and the following crucial a priori estimates of (local) solutions for (1.1) and (1.2).

Without loss of generality we will put further parameter ε equal to 1. Otherwise, we make substitution: $t \to t/\varepsilon$, $x \to x/\varepsilon$.

Theorem 2. *Let in* (1.1), (1.2) *parameter* $\varepsilon = 1$. *Let* $C = \varphi(0) > 0$, $\gamma_0 > |\varphi'(0)|$, $\bar{x} \stackrel{\text{def}}{=} \frac{x - Ct}{\sqrt{Ct}}$,

$$\Omega_\sigma = \left\{(x,t)\colon\ a_1 < \bar{x} < a_2 + \sigma\sqrt{Ct}\right\}, \quad 0 < a_1 < a_2 < \infty,\ \sigma \geqslant 0.$$

(i) *If function* $f(x,t)$ *defined in the domain* Ω_0 *satisfies Eq.* (1.1), *and*

$$\left|f(x,t)\right| \leqslant \frac{\gamma}{\sqrt{Ct}}, \quad (x,t) \in \Omega_0,\ t \geqslant t_0, \tag{1.9}$$

then the following estimate holds:

$$\left|\frac{\partial f}{\partial x}(x,t)\right| \leqslant \frac{b\gamma}{Ct}, \quad (x,t) \in \Omega_0,\ t \geqslant t_0, \tag{1.10}$$

where

$$b = \frac{b_0}{C}\left(\gamma\gamma_0 + \frac{1}{\delta}\right)\left(1 + \ln_+ \frac{\gamma\gamma_0 + 1/\delta}{\sqrt{C}}\right),$$

$d = \min(\bar{x} - a_1, a_2 - \bar{x}, a_2/2)$, $\delta = \min(1, d)$, b_0 *is absolute constant.*

(ii) *If function* $F(x,t)$ *defined in the domain* Ω_σ, $\sigma > 0$, *satisfies Eq.* (1.2), $\Delta F(x,t) \stackrel{\text{def}}{=} F(x,t) - F(x-1,t) \geqslant 0$, $t \geqslant t_0$, *and*

$$\left|F(x,t)\right| \leqslant \frac{\Gamma \cdot \bar{x}}{\sqrt{Ct}}, \quad \textit{where } \bar{x} \in \left(a_1, a_2 + \sigma\sqrt{Ct}\right),\ t \geqslant t_0, \tag{1.11}$$

then the following estimate holds:

$$0 \leqslant \Delta F(x,t) \leqslant \frac{B\Gamma \cdot \bar{x}}{Ct}, \quad \textit{where } \bar{x} \in \left(a_1, a_2 + \sigma_0\sqrt{Ct}\right),\ \sigma > \sigma_0,\ t \geqslant t_0 \geqslant a_1^2,$$

$$B = B_0\left[\frac{\sqrt{1+\sigma}}{\sqrt{\sigma - \sigma_0}} + \frac{\gamma_0\Gamma}{C} + \frac{1}{d} + \frac{\gamma_0\Gamma \cdot a_1}{C}\right], \quad d = \bar{x} - a_1, \tag{1.12}$$

B_0 *is absolute constant.*

(ii)′ *If function* $F(x,t)$ *defined in the domain* Ω_0, *satisfies Eq.* (1.2), $\varphi'(0) \geqslant 0$, $\Delta F(x,t) \stackrel{\text{def}}{=} F(x,t) - F(x-1,t) \geqslant 0$, $t \geqslant t_0$, *and*

$$\left|F(x,t)\right| \leqslant \frac{\Gamma}{\sqrt{Ct}}, \quad (x,t) \in \Omega_0,\ t \geqslant t_0, \tag{1.11$'$}$$

then the following estimate holds:

$$0 \leqslant \Delta F(x,t) \leqslant \frac{B\Gamma}{Ct}, \quad (x,t) \in \Omega_0,\ t \geqslant t_0, \tag{1.12$'$}$$

where

$$B = B_0\left[a_2 + \left(\frac{1}{d} + \frac{\gamma_0 \Gamma}{C}\right)\bigl(1 + \ln(1 + a_2)\bigr)\right], \quad d = \min(\bar{x} - a_1, a_2 - \bar{x}),$$

B_0 is absolute constant.

Remarks. (1) Theorem 2(i) in the weak form (condition $|f(x,t)| = \mathrm{O}(1/\sqrt{t})$ for $\bar{x} \in [a_1, a_2]$ implies the estimate $|\frac{\partial f}{\partial x}(x,t)| = \mathrm{O}(1/t)$ for $\bar{x} \in [\tilde{a}_1, \tilde{a}_2] \subset (a_1, a_2)$) can be deduced from the general theory of quasilinear parabolic equations developed in [16].

Theorem 2(ii)′ in the weak form (condition $0 \leqslant F(x,t) \leqslant \mathrm{O}(1/\sqrt{t})$ for $\bar{x} \in [a_1, a_2]$ implies the estimate $0 \leqslant \Delta F(x,t) \leqslant \mathrm{O}(1/t)$ for $\bar{x} \in [\tilde{a}_1, \tilde{a}_2] \subset (a_1, a_2)$) was formulated in [7] (with the reference to the present paper) and was essentially used in [7].

(2) Theorem 2(ii) is used for the proof of Theorem 1(ii) of this paper. Theorem 2(i) is needed for the proof of Theorem 1(i).

(3) Theorem 2 can be applied to the study of Problems I, II (see [7]), because the necessary conditions (1.9), (1.11) are satisfied due to [15,6].

(4) Theorem 2 can be applied also to the study of Problems I, II in other cases. For example, in the important case $\alpha = \beta = 0$, $\varphi'(\alpha) \neq 0$ the necessary conditions (1.9), (1.11) are valid globally: $|f(x,t)| = \mathrm{O}(1/\sqrt{t})$, $|F(x,t)| = \mathrm{O}(1/\sqrt{t})$, $x \in \mathbb{R}$, $t > 0$ (see [8,11,6]).

Theorem 1(ii) is proved in Section 2. The proof of Theorem 2(ii) and sketch of the proof of Theorem 2(ii)′ are given in Section 3. Theorems 1(i) and 2(i) will be proved in the another paper.

2. Asymptotics for solutions of Burgers type equations with shock profile conditions

The detailed proof of Theorem 1(ii) will be given below, only in the principal case: $\alpha = 0$, $\beta = 1$, $\varepsilon = 1$, $\varphi(0) > C = \varphi(1)$, $x = n \in \mathbb{Z}$. Other cases can be proved by very similar arguments.

Let $F(n,t)$, $n \in \mathbb{Z}$, $t \in \mathbb{R}_+$, be a solution of the equation:

$$\frac{\mathrm{d}F(n,t)}{\mathrm{d}t} = \varphi\bigl(F(n,t)\bigr)\bigl(F(n-1,t) - F(n,t)\bigr), \tag{2.1}$$

under initial conditions: $F(n-1,t_0) \leqslant F(n,t_0)$, $n \in \mathbb{Z}$,

$$\sum_{-\infty}^{0} F(n,t_0) + \sum_{0}^{\infty}\bigl(1 - F(n,t_0)\bigr) < \infty. \tag{2.2}$$

By the shock profile condition there exists a wave-train solution $\widetilde{F}(n - Ct)$ for (2.1) with overfall $(0,1)$.

Let $\Phi(F) = \int_F^1 \mathrm{d}y/\varphi(y)$. Let $d_A(t)$, $A > 0$, be such function that

$$\sum_{k=-\infty}^{[Ct+A\sqrt{t}]} \big(\Phi\big(F(k,t) - \Phi\big(\widetilde{F}\big(k - Ct + d_A(t)\big)\big) + \big(Ct + A\sqrt{t} - \big[Ct + A\sqrt{t}\,\big]\big)\big)$$
$$\times \big(\Phi\big(F\big(\big[Ct + A\sqrt{t}\,\big] + 1, t\big)\big) - \Phi\big(\widetilde{F}\big(\big[Ct + A\sqrt{t}\,\big] + 1 - Ct + d_A(t)\big)\big)\big) = 0. \tag{2.3}$$

By Theorem 1 from [7] for any $A > 2\sqrt{C}$, we have:

$$\frac{\Gamma_-}{t} < d'_A(t) \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} d_A(t) \leqslant \frac{\Gamma_+}{t}, \tag{2.4}$$

where $0 < \Gamma_- \leqslant \Gamma_+ < \infty$, $t > t_0 > 0$, and

$$\sup_n \big|F(n,t) - \widetilde{F}\big(n - Ct + d_A(t)\big)\big| \to 0, \quad t \to \infty. \tag{2.5}$$

To prove Theorem 1(ii) we use statement (2.5) and the following crucial improvement of the statement (2.4).

Proposition 1. *Let $A > 2\sqrt{C}$. Then the shift-function, defined by* (2.3)*, has the following asymptotic behavior*:

$$d_A(t) = \frac{1}{2}\frac{C}{\varphi'(1)} \ln t + const + \mathrm{o}(1), \quad t \to \infty. \tag{2.6}$$

The proof of Proposition 1 is based on the appropriate comparison of statements for Burgers type equations and on Theorem 2(ii) proved in Section 3. Besides known comparison results [5,7] we need also the following new one.

Lemma 1. *Let*

$$\psi(z) = \frac{C}{\varphi'(1)} \exp\left(-\frac{z^2}{2}\right)\left(\int_{-\infty}^{z/2} \exp\big(-2y^2\big)\,\mathrm{d}y\right)^{-1}.$$

For any solution $F(n,t)$ of the Cauchy problem (2.1), (2.2) *and for any $0 < \delta_0 < \delta < 1$ and $A > 2\sqrt{C}$, there exist $t_0 > 0$, $T > 0$, such that*

$$F(n, t - T) > 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{n - Ct - 2\sqrt{Ct} - \delta\sqrt{Ct}}{\sqrt{Ct}}\right), \tag{2.7}$$

if $Ct + 2\sqrt{Ct} + (\delta - \delta_0)\sqrt{Ct} < n < Ct + A\sqrt{Ct}$, $t > t_0$.

Remark. The function $u(\xi,t) = 1 - \frac{1}{\sqrt{t}}\psi(\frac{\xi}{\sqrt{t}})$ is one of the most important (in fluid mechanics) solutions of the classical Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} = \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}$ (see [10]).

For the proof of Lemma 1 we need additional lemmas about subsolutions for Eq. (2.1) and about patching of these subsolutions.

The next lemma shows that the function $1-\frac{1}{\sqrt{t}}\psi(\frac{x-Ct}{\sqrt{Ct}})$, being a solution of classical Burgers equation, is also the subsolution for Eq. (2.1) in the domains:

$$\left\{(x,t)\colon\ B<\frac{x-Ct}{\sqrt{Ct}}<A,\ t>t_0\right\},\quad t_0=t_0(A,B).$$

This subsolution will be called asymptotic subsolution.

Lemma 2. *For any $B<A$ and increasing function $D(t)=\mathrm{O}(\sqrt{t})$ there exists $t_0>0$ such that for $t\geqslant t_0$ and $x\in(Ct+B\sqrt{Ct},Ct+A\sqrt{Ct})$ the function $\widehat{F}(x,t)=1-\frac{1}{\sqrt{t}}\psi(\frac{x-Ct-D(t)}{\sqrt{Ct}})$ satisfies inequality*:

$$\frac{\partial\widehat{F}}{\partial t}(x,t)\leqslant\varphi\big(\widehat{F}(x,t)\big)\big(\widehat{F}(x-1,t)-\widehat{F}(x,t)\big).\tag{2.8}$$

Remark. For the proof of Lemma 1 we will use Lemma 2 in the domain:

$$\left\{(x,t)\colon\ 2-\delta<\frac{x-Ct}{\sqrt{Ct}}<A\right\}\quad\text{for } D(t)=(2+\delta_0)\sqrt{Ct},\ \delta_0<\delta<1.$$

In other domains $\{(x,t)\colon\ 1<\frac{x-Ct}{\sqrt{Ct}}\leqslant 2-\delta\}$ and $\{(x,t)\colon\ \frac{x-Ct}{\sqrt{Ct}}\leqslant 1\}$ we will need other subsolutions for (2.1): so called diffusion subsolution $\widehat{F}(x,t)=\varphi^{(-1)}(\frac{x-2\sqrt{Ct}}{t})$, and wave-train subsolution $\widetilde{F}_\sigma(x-C_\sigma t)$ with overfall $[-\sigma,1]$, $\sigma>0$ (see the properties of these subsolutions in [5,6]).

Proof of Lemma 2. We will use the equality:

$$\frac{\partial\widehat{F}(x,t)}{\partial t}=\frac{1}{2t^{3/2}}\hat{\psi}\left(\frac{x-Ct}{2\sqrt{t}}\right)-\frac{1}{\sqrt{t}}\frac{\mathrm{d}\hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{\mathrm{d}\bar{x}}\cdot\left(-\frac{x}{4t^{3/2}}-\frac{C}{4\sqrt{t}}\right),$$

where

$$\hat{\psi}(\bar{x})=\frac{C}{\varphi'(1)}\exp\left(-\frac{2}{C}\bar{x}^2\right)\left(\int_{-\infty}^{\bar{x}}\exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y\right)^{-1},\quad \bar{x}=\frac{x-Ct}{2\sqrt{t}}.$$

Let us fix $\beta>0$. Then for $\bar{x}=\frac{x-Ct}{2\sqrt{t}}\geqslant-\beta$ and $t\to+\infty$, we have:

$$\varphi\big(\widehat{F}(x,t)\big)=C-\frac{\varphi'(1)}{\sqrt{t}}\hat{\psi}\left(\frac{x-Ct}{2\sqrt{t}}\right)+\mathrm{O}\left(\frac{\psi^2(-\beta)}{t}\right),$$

$$\widehat{F}(x-1,t)-\widehat{F}(x,t)=-\frac{\partial\widehat{F}(x,t)}{\partial x}+\frac{1}{2}\frac{\partial^2\widehat{F}}{\partial x^2}(x,t)+\cdots$$
$$=2\left(\frac{1}{2\sqrt{t}}\right)^2\frac{\mathrm{d}\hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{\mathrm{d}\bar{x}}-\left(\frac{1}{2\sqrt{t}}\right)^3\frac{\mathrm{d}^2\hat{\psi}(\frac{x-Ct}{2\sqrt{t}})}{\mathrm{d}\bar{x}^2}+\mathrm{O}\big(1/t^2\big).$$

Hence, for $\bar{x}\geqslant-\beta$, we obtain:

$$\frac{\partial\widehat{F}(x,t)}{\partial t}-\varphi\big(\widehat{F}(x,t)\big)\big(\widehat{F}(x-1,t)-\widehat{F}(x,t)\big)$$
$$=\frac{1}{2t^{3/2}}\hat{\psi}(\bar{x})+\frac{1}{t^2}\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}\left(\frac{2Ct+2\bar{x}\sqrt{t}}{4}\right)$$
$$-\left(C-\frac{\varphi'(1)}{\sqrt{t}}\hat{\psi}(\bar{x})\right)\left(\frac{1}{2t}\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}-\frac{1}{8t^{3/2}}\frac{\mathrm{d}^2\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}^2}\right)+\mathrm{O}\big(1/t^2\big)$$
$$=\frac{1}{2t^{3/2}}\left(\hat{\psi}(\bar{x})+\bar{x}\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}+\varphi'(1)\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}+\frac{C}{4}\frac{\mathrm{d}^2\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}^2}\right)+\mathrm{O}\big(1/t^2\big). \quad (2.9)$$

By direct differentiation with respect to $\bar{x}$, we obtain:

$$\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}\bar{x}^2}+\left(\frac{4}{C}\bar{x}+\frac{2\varphi'(1)}{C}\hat{\psi}(\bar{x})\right)\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}+\frac{4}{C}\hat{\psi}(\bar{x})=0.$$

Hence,

$$\hat{\psi}(\bar{x})+\bar{x}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\bar{x}}+\varphi'(1)\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}+\frac{C}{4}\frac{\mathrm{d}^2\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}^2}$$
$$=\hat{\psi}(\bar{x})+\bar{x}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\bar{x}}+\varphi'(1)\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}-\bar{x}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\bar{x}}-\frac{\varphi'(1)}{2}\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}-\hat{\psi}(\bar{x})$$
$$=\frac{\varphi'(1)}{2}\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}.$$

Let us check the inequality:

$$\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}<0,\quad \forall\bar{x}\in\mathbb{R}. \quad (2.10)$$

By direct differentiation, we have:

$$\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}}=-\frac{4}{C}\bar{x}\hat{\psi}(\bar{x})-\frac{\varphi'(1)}{C}\hat{\psi}^2(\bar{x}).$$

This implies the equality:

$$\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}} = -\hat{\psi}(\bar{x})\left(\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y\right)^{-1} \times \left(\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y + \frac{C\exp(-2\bar{x}^2/C)}{4\bar{x}}\right)\frac{4\bar{x}}{C}.$$

Hence, (2.8) is equivalent to the inequality:

$$\frac{4}{C}\bar{x}\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y + \exp\left(-\frac{2}{C}\bar{x}^2\right) > 0.$$

For $\bar{x} \geqslant 0$ this inequality is obvious. For $\bar{x} < 0$ this inequality follows from the relations:

$$\lim_{\bar{x}\to-\infty}\left(\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y + \frac{C\exp(-2\bar{x}^2/C)}{4\bar{x}}\right) = 0 \quad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\left(\int_{-\infty}^{\bar{x}} \exp\left(-\frac{2}{C}y^2\right)\mathrm{d}y + \frac{C\exp(-2\bar{x}^2/C)}{4\bar{x}}\right) = -\frac{C\exp(-2\bar{x}^2/C)}{4\bar{x}^2} < 0.$$

From (2.9), (2.10) it follows that there exists $\sigma > 0$ such that

$$\sup\left\{\hat{\psi}(\bar{x})\frac{\mathrm{d}\hat{\psi}(\bar{x})}{\mathrm{d}\bar{x}} \;\middle|\; -\beta \leqslant \bar{x} \leqslant \alpha\right\} < -\sigma.$$

Hence, for $x \in [Ct - \beta\sqrt{t}, Ct + \alpha\sqrt{t}\,]$ we obtain the estimate:

$$\frac{\partial \widehat{F}(x,t)}{\partial t} - \varphi\big(\widehat{F}(x,t)\big)\big(\widehat{F}(x-1,t) - \widehat{F}(x,t)\big) \leqslant -\frac{\varphi'(1)}{4t^{3/2}}\sigma + \mathrm{O}\big(1/t^2\big).$$

It means that there exists $t_0 > 0$ such that for $t \geqslant t_0$ and $x \in [Ct - \beta\sqrt{t}, Ct + \alpha\sqrt{t}\,]$ the inequality (2.8) is valid if

$$\widehat{F}(x,t) = 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{x - Ct}{\sqrt{Ct}}\right).$$

This inequality is also valid if

$$\widehat{F}(x,t) = 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{x - Ct - D(t)}{\sqrt{Ct}}\right) \quad \text{for } x \in \big[Ct + D(t) - \beta\sqrt{t}, Ct + D(t) + \alpha\sqrt{t}\,\big]$$

because $t \mapsto D(t)$ is increasing function. □

The next lemma gives conditions for patching diffusion subsolutions and asymptotic subsolutions.

Lemma 3. *For any $\delta \in (0,1)$ and constant $\Gamma > 0$ there exists $t_0 > 0$ such that for $t \geqslant t_0$ and $n \in [Ct + (2-\delta)\sqrt{Ct} - \Gamma, Ct + (2-\delta)\sqrt{Ct} + \Gamma]$ the following inequality is valid*:

$$\varphi^{(-1)}\left(\frac{n - 2\sqrt{Ct}}{t}\right) > 1 - \frac{1}{\sqrt{t}}\psi\left(\frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}}\right). \tag{2.11}$$

Proof. We have equalities:

$$\lim_{\bar{x}\to-\infty} \frac{1}{\bar{x}} \exp(-2\bar{x}^2)\left(\int_{-\infty}^{\bar{x}} \exp(-2y^2)\,\mathrm{d}y\right)^{-1} = -4,$$

$$\lim_{\bar{x}\to-0} \frac{1}{\bar{x}} \exp(-2\bar{x}^2)\left(\int_{-\infty}^{\bar{x}} \exp(-2y^2)\,\mathrm{d}y\right)^{-1} = -\infty.$$

Hence, for any $\varepsilon \in (0,1)$ there exists $\bar{x}^*(\varepsilon) < 0$ such that

$$\exp(-2(\bar{x}^*)^2)\left(\int_{-\infty}^{\bar{x}^*} \exp(-2y^2)\,\mathrm{d}y\right)^{-1} = -\frac{4\bar{x}^*(\varepsilon)}{1-\varepsilon}. \tag{2.12}$$

Besides, $\bar{x}^*(\varepsilon) \to 0$ when $\varepsilon \to 1$. Let us take $n \in [Ct + (2+2\bar{x}^*)\sqrt{Ct} - \Gamma, Ct + (2+2\bar{x}^*)\sqrt{Ct} + \Gamma]$. Then $\frac{n-Ct-2\sqrt{Ct}}{\sqrt{Ct}} = 2\bar{x}^* + \mathrm{O}(1/\sqrt{t})$.

We have now from one side,

$$1 - \varphi^{(-1)}\left(\frac{n-2\sqrt{Ct}}{t}\right) = \frac{C}{\varphi'(1)} \frac{(-2\bar{x}^*)}{\sqrt{Ct}} + \mathrm{O}(1/t);$$

from the other side, we obtain using (2.12):

$$\begin{aligned}
\frac{1}{\sqrt{t}}\psi\left(\frac{n-Ct-2\sqrt{Ct}}{\sqrt{Ct}}\right) &= \frac{1}{\sqrt{t}}\psi\big(2\big(\bar{x}^* + \mathrm{O}\big(1/\sqrt{t}\big)\big)\big) \\
&= \frac{C}{\varphi'(1)\sqrt{t}} \exp\big(-2\big(\bar{x}^* + \mathrm{O}\big(1/\sqrt{t}\big)\big)^2\big)\left(\int_{-\infty}^{\bar{x}^*+\mathrm{O}(1/\sqrt{t})} \exp(-2y^2)\,\mathrm{d}y\right)^{-1} \\
&= \frac{C}{\varphi'(1)\sqrt{t}}\left(\frac{-4\bar{x}^*}{1-\varepsilon}\right) + \mathrm{O}(1/t).
\end{aligned}$$

If $\frac{2}{\sqrt{C}} < \frac{4}{1-\varepsilon}$, then there exists $t_0 > 0$ such that

$$1 - \varphi^{(-1)}\left(\frac{n - 2\sqrt{Ct}}{t}\right) < \frac{1}{\sqrt{t}}\psi\left(\frac{n - Ct - 2\sqrt{Ct}}{\sqrt{Ct}}\right).$$

Besides, if $1-\varepsilon$ small enough we have $-2\bar{x}^* \in [0,1]$. So, we can finish the proof by putting $\delta = -2\bar{x}^*$. □

Proof of Lemma 1. Let the function $\varphi(F)$ be extended for negative values of F as a smooth strictly decreasing function. Then there exists a wave-train solution $\widetilde{F}_\sigma(n - C_\sigma t)$ for (2.1) with overfall $(-\sigma, 1)$, $\sigma > 0$. Put $\sigma = \sigma(t) = \exp(-t^{1/3})$. Proposition 1, Lemmas 5, 6 from [7] together with Lemmas 2, 3 above imply the following statement.

For any $\delta \in (0,1)$, $l > 1$, $A > 2\sqrt{C}$ there exist $t_0 > 0$ and increasing functions $\gamma_1(t) = \mathrm{O}(t^{1/3})$, $\gamma_2(t) = 2\sqrt{Clt} + a(l)$:

$$F^-(n,t) = \begin{cases} \widetilde{F}_{\sigma(t)}(n - Ct - \gamma_1), & n \leqslant Ct + \sqrt{Clt} + a(l), \\ \varphi^{(-1)}(\frac{n-\gamma_1-\gamma_2}{t}), & Ct + \sqrt{Clt} + a(l) < n < Ct + \gamma_1 + \gamma_2 - \delta\sqrt{Ct}, \\ 1 - \frac{1}{\sqrt{t}}\psi(\frac{n - Ct - \gamma_1 - \gamma_2}{\sqrt{Ct}}), & Ct + \gamma_1 + \gamma_2 - \delta\sqrt{Ct} \leqslant n < Ct + A\sqrt{Ct}, \\ 1 - \delta, & n \geqslant Ct + A\sqrt{t}, \end{cases} \tag{2.13}$$

is a subsolution for (2.1), if $t \geqslant t_0$.

This statement and comparison principle from [6] imply that for any solution $F(n,t)$ of the Cauchy problem (2.1), (2.2) there exists $T > 0$ such that

$$F(n,t) > F^-(n, t+T), \tag{2.14}$$

if $n \in \mathbb{Z}$, $t \geqslant -T + t_0$.

Lemma 1 follows from (2.13) and (2.14). □

Proof of Proposition 1. Put $\kappa(t) = \{Ct + A\sqrt{t}\}$, $0 \leqslant \kappa(t) < 1$, $N(t) = [Ct + A\sqrt{t}]$, $F = F(N(t), t)$, $F_1 = F(N(t)+1, t)$, $\widetilde{F} = \widetilde{F}(N(t) - Ct + d_A(t))$, $\widetilde{F}_1 = \widetilde{F}(N(t) + 1 - Ct + d_A(t))$. Proposition 3 from [7] implies the following asymptotic formula for:

$$d'_A(t) \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\big(d_A(t)\big),$$

$$\begin{aligned} d'_A(t)\big(1 + \mathrm{O}\big(1/\sqrt{t}\big)\big) &= C(1-\kappa)(F_1 - F) - C(1-\kappa)(\widetilde{F}_1 - \widetilde{F}) \\ &\quad + \frac{A(1-\widetilde{F}_1)}{2\sqrt{t}} + \frac{1}{2}\varphi'(1)(1-\widetilde{F}_1)^2 - \left(\frac{A(1-F_1)}{2\sqrt{t}} + \frac{1}{2}\varphi'(1)(1-F_1)^2\right). \end{aligned} \tag{2.15}$$

Let us estimate now all terms of (2.15). The assumption $\Delta F(n,t_0)\geqslant 0$ implies (by Theorem 1 in [5]) that $\Delta F(n,t)\geqslant 0$ $\forall t\geqslant t_0$. From this and from inequality (2.7) it follows for $n\geqslant N(t)+1$ and $t\geqslant t_0$:

$$\begin{aligned}
0\leqslant 1-F(n,t)\leqslant 1-F_1\leqslant 1-F\bigl(N(t)+1,t+T\bigr)\leqslant \frac{1}{\sqrt{t}}\psi(A-\delta_0-2)\\
\leqslant \frac{C}{\varphi'(1)\sqrt{t}}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\left(\int_{-\infty}^{0}\exp\bigl(-2y^2\bigr)\,\mathrm{d}y\right)^{-1}\\
\leqslant \mathrm{O}\left(\frac{1}{\sqrt{t}}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\right). \qquad (2.16)
\end{aligned}$$

From (2.16) and from inequality (1.12) of Theorem 2(ii) for $t\geqslant t_0\geqslant A^2$, $n\geqslant N(t)$ we obtain the crucial inequality:

$$F_1-F=\mathrm{O}\left(\frac{A}{t}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\right). \qquad (2.17)$$

From [5, Theorems 2, 2′] and [6, Theorems 6.1, 6.2] it follows asymptotic formula:

$$\widetilde{F}_1=1-\frac{C}{\varphi'(1)(A\sqrt{t}+d_A(t))}+\mathrm{O}\left(\frac{1}{(A\sqrt{t}+d_A(t))^2}\right).$$

This formula and estimate $d_A(t)\geqslant 0$ (see (2.4)) gives inequalities:

$$0<1-\widetilde{F}_1\leqslant \mathrm{O}\left(\frac{1}{A\sqrt{t}}\right),\qquad \widetilde{F}_1-\widetilde{F}=\mathrm{O}\left(\frac{1}{A^2t}\right). \qquad (2.18)$$

Let us put estimates (2.16)–(2.18) into formula (2.15). We obtain:

$$\begin{aligned}
d_A'(t)\bigl(1+\mathrm{O}\bigl(1/\sqrt{t}\bigr)\bigr)&=C(1-\kappa)(F_1-F)-C(1-\kappa)(\widetilde{F}_1-\widetilde{F})\\
&\quad+\frac{C}{2\varphi'(1)t}+\frac{1}{2}\frac{C^2}{\varphi'(1)A^2t}-\frac{(1-F_1)}{2}\left(\frac{A}{\sqrt{t}}+\varphi'(1)(1-F_1)\right)\\
&=(1-\kappa)\mathrm{O}\left(\frac{A}{t}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\right)-(1-\kappa)\mathrm{O}\left(\frac{1}{A^2t}\right)\\
&\quad+\frac{C}{2\varphi'(1)t}+\frac{1}{2}\frac{C^2}{\varphi'(1)A^2t}+\mathrm{O}\left(\frac{A}{t}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\right)\\
&=\frac{C}{2\varphi'(1)t}+\mathrm{O}\left(\frac{1}{A^2t}\right)+\mathrm{O}\left(\frac{A}{t}\exp\bigl(-(A-\delta_0-2)^2/2\bigr)\right). \qquad (2.19)
\end{aligned}$$

Estimate (2.9) implies asymptotic formula:

$$d_A(t) = \frac{C}{2\varphi'(1)} \ln t + \mathrm{O}\big(1/A^2\big) \ln t + const.$$

From result (2.5) it follows that for any $A_1 > 2\sqrt{C}$ and $A_2 > 2\sqrt{C}$ we have $d_{A_1}(t) - d_{A_2}(t) \to 0$, $t \to \infty$.

Hence,

$$d_A(t) = \frac{C}{2\varphi'(1)} \ln t + const + \mathrm{o}(1). \qquad \square$$

3. A priori estimates for local solutions of Burgers type equations

Without loss of generality we will put further $C = 1$ and $\varepsilon = 1$. Otherwise we make substitutions: $t \to Ct/\varepsilon$, $x \to x/\varepsilon$ for Eq. (1.2) and $t \to C^2 t/\varepsilon$, $x \to Cx/\varepsilon$ for Eq. (1.1). We will give here a complete proof of Theorem 2(ii) which is sufficient for all current applications and a sketch of the proof of Theorem 2(ii)$'$. Theorem 2(i) will be proved in a separate paper.

The first step in the proof of Theorem 2(ii) is the Green–Poisson type representation formula (for function u in Ω_σ) associated with operator $u \mapsto u'_t + \Delta u$, where $\Delta u \stackrel{\mathrm{def}}{=} u(x,t) - u(x-1,t)$, $u'_t = \frac{\partial u(x,t)}{\partial t}$.

Let $\chi_0 : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that

$$\begin{aligned} &0 \leqslant \chi_0 \leqslant 1, \quad \chi_0|_{(-\infty,a_1)} \equiv 0, \quad \chi_0|_{[\tilde a_1,+\infty]} \equiv 1, \quad 0 < a_1 < \tilde a_1 < \infty, \\ &|\chi'_{0t}| \leqslant \frac{A_0}{\delta} \quad \text{and} \quad |\chi''_{0t}| \leqslant \frac{A_0}{\delta^2}, \end{aligned} \tag{3.1}$$

where $\delta = \tilde a_1 - a_1$. Put $\chi(x,t) = \chi_0(\frac{x-t}{\sqrt{t}})$.

Proposition 2. *Let function $u(x,t)$ be defined in the domain*

$$\Omega_\sigma = \left\{ (x,t)\colon\, a_1 < \bar x \stackrel{\mathrm{def}}{=} \frac{x-t}{\sqrt{t}} < a_2 + \sigma\sqrt{t} \right\}, \quad \sigma > 0,$$

and $\tilde u(x,t) = u(x,t) \cdot \chi(x,t)$. Let $0 < \sigma_0 < \sigma$ and $\alpha \in (\frac{1+\sigma_0}{1+\sigma}, 1)$. Then function $\tilde u$ can be represented in Ω_{σ_0} by the following formula of the Green–Poisson type:

$$\begin{aligned} \tilde u(x,t) = {}& \int_{-\infty}^{\infty} G(x-\xi, t-\alpha t) \tilde u(\xi, \alpha t)\, \mathrm{d}\xi \\ &+ \int_{\alpha t}^{t} \mathrm{d}\tau \int_{-\infty}^{\infty} G(x-\xi, t-\tau)(\tilde u'_\tau + \Delta \tilde u)(\xi,\tau)\, \mathrm{d}\xi, \end{aligned} \tag{3.2}$$

where

$$G(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\mathrm{i}\xi x)\exp\bigl([\mathrm{e}^{\mathrm{i}\xi}-1]t\bigr)\,\mathrm{d}\xi.$$

Besides,

$$G(x,t)=\sum_{n=-\infty}^{\infty}G_n(t)\delta(n-x), \tag{3.3}$$

where

$$\begin{cases} G_n(t)=0, & \text{if } n<0,\\ G_n(t)=\dfrac{t^n}{n!}\mathrm{e}^{-t}, & \text{if } n\geqslant 0,\end{cases}$$

is Poisson-distribution.

This statement is certainly classical but we did not find the precise reference. So, we will indicate the abridge proof.

The operator $\frac{\partial}{\partial t}+\Delta$ can be considered as a parabolic operator of infinite order in x and it can be represented by the following formula:

$$\frac{\partial}{\partial t}+\Delta=\frac{\partial}{\partial t}+\left(1-\exp\left(-\frac{\partial}{\partial x}\right)\right).$$

We will apply further to the Cauchy problem for this operator the same Fourier method as for parabolic operator of finite order and we will obtain (3.2). The formula (3.3) is the Fourier inversion formula for the classical Poisson distribution through its characteristic function.

It is important to remark that the function $\tilde{u}(\xi,\tau)$ is well defined for (ξ,τ): $\xi<\tau+a_2\sqrt{\tau}+\sigma\tau$, $\tilde{u}(\xi,\tau)\equiv 0$ for $\xi\leqslant\tau+a_1\sqrt{\tau}$ and function $\xi\mapsto G(x-\xi,t-\tau)$ is equal to zero for $\xi>x=t+\bar{x}\sqrt{t}$. So, the function $\xi\mapsto\tilde{u}(\xi,\tau)\cdot G(x-\xi,t-\tau)$ can be naturally interpreted in the formula (3.2) as a function with compact support in $\mathbb{R}$ if the following inequality is satisfied:

$$\tau+a_2\sqrt{\tau}+\sigma\tau\geqslant x=t+\bar{x}\sqrt{t}\quad\text{for}$$
$$\bar{x}\in\bigl(a_1,a_2+\sigma_0\sqrt{t}\bigr),\ \sigma_0<\sigma\ \text{and}\ \tau\geqslant\alpha t\geqslant t_0(\sigma,\sigma_0).$$

In order to satisfy these inequalities we choose $\alpha\in(0,1)$ such that for $t>t_0(\sigma,\sigma_0)$ the following inequality is valid:

$$\alpha t+a_2\sqrt{\alpha t}+\sigma\alpha t>t+a_2\sqrt{t}+\sigma_0 t,$$

i.e., we must take $\alpha>\frac{1+\sigma_0}{1+\sigma}$.

Corollary (Integral representation for $\Delta u(x,t)$). *Let function $u(x,t)$ satisfy* (1.2) *in Ω_σ with $\varphi(0)=C=1$ and $\varepsilon=1$. Put $\varphi_0=\varphi-C$. Then in assumption of Proposition* 2 *for,*

$$(x,t)\in\widetilde{\Omega}_{\sigma_0}=\left\{(x,t)\in\Omega_{\sigma_0}\colon\ x\geqslant t+\tilde a_1\sqrt{t}\right\},$$
$$\sigma_0<\sigma,\ t>t^*=\alpha t\geqslant t_0,\ \alpha\in\left(\frac{1+\sigma_0}{1+\sigma},1\right),$$

we have the equality:

$$\Delta u(x,t)=I_0u+I_1u+I_2u+I_3u+I_4u, \tag{3.4}$$

where

$$I_0u(x,t)=-\int_{t^*}^{t}\mathrm{d}\tau\int_{\bar\xi>\tilde a_1}\Delta_x G(x-\xi,t-\tau)\varphi_0(u)\Delta u(\xi,\tau)\,\mathrm{d}\xi,$$

$$I_1u(x,t)=-\int_{t^*}^{t}\mathrm{d}\tau\int_{\bar\xi\in[a_1,\tilde a_1]}\Delta_x G(x-\xi,t-\tau)\varphi_0(u)\Delta u(\xi,\tau)\chi(\xi,\tau)\,\mathrm{d}\xi,$$

$$I_2u(x,t)=\int_{\bar\xi\geqslant a_1}\Delta_x G(x-\xi,t-t^*)u(\xi,t^*)\chi(\xi,t^*)\,\mathrm{d}\xi,$$

$$I_3u(x,t)=\int_{t^*}^{t}\mathrm{d}\tau\int_{\bar\xi\in[a_1,\tilde a_1]}\Delta_x G(x-\xi,t-\tau)(u\chi'_\tau+u\Delta\chi)(\xi,\tau)\,\mathrm{d}\xi,$$

$$I_4u(x,t)=-\int_{t^*}^{t}\mathrm{d}\tau\int_{\bar\xi\in[a_1,\tilde a_1]}\Delta_x G(x-\xi,t-\tau)\Delta u(\xi,\tau)\Delta\chi(\xi,\tau)\,\mathrm{d}\xi.$$

Remark. We will use below several times the following simple relation: let $u=u(x)$, $v=v(x)$, then $\Delta(u\cdot v)=u\cdot\Delta v+v(x-1)\Delta u$, where $\Delta u\overset{\text{def}}{=}u(x)-u(x-1)$.

Proof of Corollary. We have relations:

$$\tilde u(\xi,\tau)=u(\xi,\tau)\cdot\chi(\xi,\tau),$$
$$\tilde u'_\tau=(u\cdot\chi)'_\tau=u'_\tau\cdot\chi+u\chi'_\tau,$$
$$\Delta\tilde u=\Delta(u\cdot\chi)=\Delta u\cdot\chi(\xi-1,t)+u\cdot\Delta\chi=\Delta u\cdot\chi+u(\xi-1,t)\Delta\chi.$$

Using (1.2), we obtain:

$$(u'_\tau+\Delta u)\cdot\chi=-\varphi_0(u)\Delta u\chi=-\varphi_0(u)\bigl(\Delta\tilde u-u(\xi-1,t)\cdot\Delta\chi\bigr),$$

$$(\tilde{u}'_\tau + \Delta\tilde{u}) = -\varphi_0(u)\Delta\tilde{u} + \varphi_0(u)\cdot u(\xi-1,\tau)\Delta\chi + u(\chi'_\tau + \Delta\chi) - \Delta u\cdot\Delta\chi$$
$$= -\varphi_0(u)\Delta u\cdot\chi + u\cdot(\chi'_\tau + \Delta\chi) - \Delta u\cdot\Delta\chi.$$

Plugging these relations into (3.2) and using the equality $\tilde{u}(\xi,\tau) = u(\xi,\tau)$ for $\bar{\xi} > \tilde{a}_1$ we obtain (3.4).

For the estimates of terms $I_1u,\ I_2u,\ I_3u,\ I_4u$ in formula (3.4) we will use elementary estimates for cut-off function $\chi(x,t)$ and rather precise estimates for Green–Poisson function $G(x,t)$. □

Lemma 4. *Let $\chi(x,t)$ be cut-off function defined by* (3.1). *Then the following estimates for derivatives of χ are valid*:

$$\left|\Delta\chi(x,t)\right| \leqslant \frac{A_0}{\delta\sqrt{t}}, \qquad \left|\Delta^2\chi(x,t)\right| \leqslant \frac{A_0}{\delta^2 t},$$
$$\left|(\chi'_t + \Delta\chi)(x,t)\right| \leqslant \frac{A_0}{t}\left(\frac{1}{\delta^2} + \frac{\tilde{a}_1}{2\delta}\right),$$

where $(x,t)\in\Omega_\sigma$, $\delta = \tilde{a}_1 - a_1$.

Proof. We have:

$$\chi'(x,t) = -\left(\frac{1}{\sqrt{t}} + \frac{x-t}{2t^{3/2}}\right)\chi'_0\left(\frac{x-t}{\sqrt{t}}\right),$$
$$\Delta\chi(x,t) = \frac{1}{\sqrt{t}}\int_{x-1}^{x}\chi'_0\left(\frac{y-t}{\sqrt{t}}\right)\mathrm{d}y,$$

$$(\chi' + \Delta\chi)(x,t) = -\frac{1}{\sqrt{t}}\int_{x-1}^{x}\left(\chi'_0\left(\frac{x-t}{\sqrt{t}}\right) - \chi'_0\left(\frac{y-t}{\sqrt{t}}\right)\right)\mathrm{d}y - \chi'_0\left(\frac{x-t}{\sqrt{t}}\right)\frac{x-t}{2t^{3/2}}$$
$$= -\frac{1}{t}\int_{x-1}^{x}\int_{y}^{x}\chi''_0\left(\frac{z-t}{\sqrt{t}}\right)\mathrm{d}z\,\mathrm{d}y - \chi'_0\left(\frac{x-t}{\sqrt{t}}\right)\frac{x-t}{2t^{3/2}}.$$

From these relations and from estimates (3.1) for χ_0 we obtain necessary estimates for $\chi(x,t)$. □

Lemma 5 (Estimates for Green–Poisson distribution $G(x,t)$). *Let*

$$G(x,t) = \sum_{n=0}^{\infty} G_n(t)\delta(n-x)$$

be the Poisson distribution (3.3). *The following estimates for $\{G_n(t)\}$ are valid*:

(i) *if* $p = n - t \geqslant 0$, *then*

$$G_n(t) \leqslant \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-p^2/(2n)};$$

(ii) *if* $q = t - n > 0$, $q \leqslant t$, *then*

$$G_n(t) \leqslant \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-q^2/(2t)};$$

(iii) *if* $n = t + a\sqrt{t}$, *then*

$$G_n(t) = \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(n-t)^2}{2t}\right)\left(1 + \mathrm{O}\left(\frac{(n-t)^3}{t^2}\right)\right)$$
$$= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2}\right)\left(1 + \mathrm{O}\left(\frac{a+a^3}{\sqrt{t}}\right)\right).$$

Proof. By Stirling's formula we have:

$$n! = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right).$$

Then

$$G_n(t) = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{n \ln t - n \ln n + n - t}\left(1 - \mathrm{O}\left(\frac{1}{n}\right)\right).$$

If $p = n - t > 0$, then

$$\ln \frac{t}{n} = \ln\left(1 - \frac{p}{n}\right) = -\frac{p}{n} - \frac{p^2}{2n^2} - \cdots.$$

If $q = t - n > 0$, then

$$\ln \frac{n}{t} = \ln\left(1 - \frac{q}{t}\right) = -\frac{q}{t} - \frac{q^2}{2t^2} - \cdots.$$

Hence,

$$G_n(t) = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-p^2/(2n)}\left(1 - \mathrm{O}\left(\frac{1}{n}\right)\right), \quad \text{if } p = n - t > 0,$$

$$G_n(t) = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-q^2/(2t) - (1/2 - 1/3)(q^3/t^2) - \cdots}\left(1 - \mathrm{O}\left(\frac{1}{n}\right)\right),$$

if $q = t - n > 0,\ q < t$.

These relations give (i)–(iii). □

Lemma 6 (Estimates for $\Delta G(x,t)$). *Let* $G(x,t)=\sum_{n=0}^{\infty} G_n(t)\delta(n-x)$ *be the Poisson distribution. We put*:

$$\Delta G_n(t) = G_n(t) - G_{n-1}(t),$$
$$\Delta_x G(x-\xi, t-\tau) = G(x-\xi, t-\tau) - G(x-1-\xi, t-\tau),$$
$$\bar{\xi} = \frac{\xi-\tau}{\sqrt{\tau}}, \quad \bar{x} = \frac{x-t}{\sqrt{t}}.$$

Then the following estimates are valid:

(i)
$$\Delta G_n(t) = G_n(t)\frac{(t-n)}{t}, \quad \textit{and as a consequence},$$
$$\Delta G_n(t) > 0, \quad \textit{if } n < t, \qquad \Delta G_n(t) < 0, \quad \textit{if } n > t;$$
$$\Delta^2 G_n(t) = G_n(t)\left(1 - \frac{2n}{t} + \frac{n(n-1)}{t^2}\right), \quad \textit{and as a consequence},$$
$$\Delta^2 G_n(t) < 0, \quad \textit{if } n - t - \frac{1}{2} \in \left(-\sqrt{t+1/4}, +\sqrt{t+1/4}\right),$$
$$\Delta^2 G_n(t) \geqslant 0, \quad \textit{if } n - t - \frac{1}{2} \notin \left(-\sqrt{t+1/4}, +\sqrt{t+1/4}\right);$$

(ii) $\forall s \geqslant 0$ *and* $p \geqslant 0$ *we have inequalities*:

$$-\Delta G_{p+s}(s) \leqslant A_1 s^{-3/2} p \exp\left(-\frac{p^2}{4s}\right), \quad \textit{if } p < s,$$
$$-\Delta G_{p+s}(s) \leqslant A_1 p^{-1/2} \mathrm{e}^{-p/4}, \quad \textit{if } p > s;$$

(iii) $\forall s \geqslant 0$ *and* $q \in (0,s)$, *we have inequalities*:

$$\Delta G_{s-q}(s) \leqslant A_1 \frac{q}{s\sqrt{s-q}} \exp\left(-\frac{q^2}{2s}\right);$$

(iv)
$$\sum_{n=-1}^{\infty} \left|\Delta G_n(t)\right| = \min\left\{2, \frac{2}{\sqrt{2\pi t}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right)\right\},$$
$$\sum_{n=-2}^{\infty} \left|\Delta^2 G_n(t)\right| = \min\left\{4, \frac{4}{\sqrt{2\pi \mathrm{e}}}\frac{1}{t}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right)\right\};$$

(v) $\forall \bar{x} > \tilde{a}_1$ *and* $t > \tau > \alpha t$, *we have inequality*:

$$I = \int\limits_{\bar\xi > \tilde a_1} \left|\Delta_x G(x-\xi, t-\tau)\right| \left(1 + \ln_+ \frac{1}{\bar\xi - \tilde a_1}\right)(1+\bar\xi)\,\mathrm{d}\xi$$
$$\leqslant \frac{A_1}{\sqrt{t-\tau}}\left(1 + \sqrt{(1-\alpha)/\alpha}\right)\left(1 + \ln_+ \frac{1}{\bar x - \tilde a_1}\right)\left(1 + \bar x/\sqrt{\alpha}\right),$$

where A_1 is absolute constant.

Remark. We will use further several times the differential and integral relations:

$$-\Delta_\xi\big(G(x-\xi-1, t-\tau)u(\xi)\big)$$
$$= G(x-\xi, t-\tau)\Delta u(\xi) + \Delta_\xi G(x-\xi-1, t-\tau)\cdot u(\xi),$$
$$-\Delta_\xi G(x-\xi-1, t-\tau) = \Delta_x G(x-\xi, t-\tau);$$

if $G(x-\xi-1, t-\tau)\cdot u(\xi)$ has compact support with respect to ξ, then

$$-\int\limits_{\xi\in\mathbb{R}} \Delta_\xi\big(G(x-\xi-1, t-\tau)\cdot u(\xi)\big)\,\mathrm{d}\xi = 0, \quad \text{and hence}$$
$$-\int\limits_{\xi\in\mathbb{R}} \Delta_x G(x-\xi, t-\tau)\cdot u(\xi)\,\mathrm{d}\xi = \int\limits_{\xi\in\mathbb{R}} G(x-\xi, t-\tau)\Delta u(\xi)\,\mathrm{d}\xi.$$

Proof of Lemma 6. (i) We have, from (3.3):

$$\Delta G_n(t) = \left(\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}\right)\mathrm{e}^{-t} = G_n(t)\frac{(t-n)}{t},$$
$$\Delta^2 G_n(t) = \left(\frac{t^n}{n!} - 2\frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!}\right)\mathrm{e}^{-t} = G_n(t)\left(1 - \frac{2n}{t} + \frac{n(n-1)}{t^2}\right);$$

(ii) follows from (i) and Lemma 5(i);
(iii) follows from (i) and Lemma 5(ii);
(iv) putting in (i) $p = n - t = a\sqrt{t}$ and using Lemma 5(iii), we obtain:

$$\Delta G_n(t) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-a^2/2}\left(-\frac{a}{\sqrt t}\right)\left(1 - \mathrm{O}\left(\frac{a^3}{\sqrt t}\right)\right), \quad \text{and}$$
$$\text{as a consequence} \quad \Delta G_n(t) = 0 \quad \text{if } a = 0.$$

So,

$$\sum_{n=-1}^{\infty}\left|\Delta G_n(t)\right| = \left(\sum_{n\geqslant t}\Delta G_n(t) - \sum_{n\leqslant t}\Delta G_n(t)\right).$$

Then

$$\sum_{n=-1}^{\infty}\left|\Delta G_n(t)\right| = \left[G_{[t]}(t) - G_{-\infty}(t)\right] - \left[G_{+\infty}(t) - G_{[t]}(t)\right]$$

$$= 2G_{[t]}(t) = \frac{2}{\sqrt{2\pi t}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right), \quad t \geqslant t_0.$$

For all $t > t_0$, we have:

$$\sum_{n=-1}^{\infty}\left|\Delta G_n(t)\right| = \min\left\{2, \frac{2}{\sqrt{2\pi t}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right)\right\}.$$

By similar arguments we have:

$$\sum_{n=-2}^{\infty}\left|\Delta^2 G_n(t)\right| = \sum_{-\infty}^{[t-\sqrt{t}\,]}\Delta^2 G_n(t) - \sum_{[t-\sqrt{t}\,]+1}^{[t+\sqrt{t}\,]}\Delta^2 G_n(t) + \sum_{[t+\sqrt{t}\,]+1}^{\infty}\Delta^2 G_n(t)$$

$$= 2\Delta G_{[t-\sqrt{t}\,]}(t) + 2\left|\Delta G_{[t+\sqrt{t}\,]}(t)\right|$$

$$= \frac{4}{t\sqrt{2\pi \mathrm{e}}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right), \quad t \geqslant t_0.$$

For all $t > 0$, we have:

$$\sum_{n=-2}^{\infty}\left|\Delta^2 G_n(t)\right| = \min\left\{4, \frac{4}{t\sqrt{2\pi \mathrm{e}}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right)\right\}.$$

(v) Put $t - \tau = s$, $x - \xi = y$. We have $p = y - s = \bar{x}\sqrt{t} - \bar{\xi}\sqrt{\tau}$. Put $I = I_+ + I_-$, where

$$I_\pm = \int\limits_{\pm\Delta_x G<0}\left|\Delta_x G(x-\xi, t-\tau)\right|\left(1 + \ln_+\frac{1}{(\bar{\xi} - \tilde{a}_1)}\right)(1 + \bar{\xi})\,\mathrm{d}\xi.$$

By part (i) $\Delta_x G(x-\xi, t-\tau) < 0$ iff $p = (x-\xi) - (t-\tau) > 0$.

Hence, $I_+ = I'_+ + I''_+$, where

$$I'_+ = -\int\limits_{\bar{\xi}>\tilde{a}_1:\ 0<p<s}\Delta_x G(x-\xi, t-\tau)\left(1 + \ln_+\frac{1}{(\bar{\xi} - \tilde{a}_1)}\right)(1 + \bar{\xi})\,\mathrm{d}\xi,$$

$$I''_+ = -\int\limits_{\bar{\xi}>\tilde{a}_1:\ p>s}\Delta_x G(x-\xi, t-\tau)\left(1 + \ln_+\frac{1}{(\bar{\xi} - \tilde{a}_1)}\right)(1 + \bar{\xi})\,\mathrm{d}\xi.$$

Put $p_1 = \bar{x}\sqrt{t} - \tilde{a}_1\sqrt{\tau}$. We have $\tilde{a}_1 - \bar{\xi} = \frac{p-p_1}{\sqrt{\tau}} < 0$ and $p = p_1$ iff $\bar{\xi} = \tilde{a}_1$.

For I'_+, when $p \in (0, s)$, we use (ii) and obtain:

$$I'_+ \leqslant As^{-3/2} \int_0^{p_1} \mathrm{e}^{-p^2/(4s)} p \left(1 + \ln_+ \frac{\sqrt{\tau}}{p_1 - p}\right)\left(1 + \tilde{a}_1 + \frac{p_1 - p}{\sqrt{\tau}}\right) \mathrm{d}p$$

(putting $p = \rho\sqrt{s}$)

$$\leqslant As^{-1/2} \int_0^{p_1/\sqrt{s}} \mathrm{e}^{-\rho^2/4} \rho \left(1 + \ln_+ \frac{\sqrt{\tau/s}}{(p_1/\sqrt{s} - \rho)}\right)\left(1 + \tilde{a}_1 + \frac{p_1}{\sqrt{\tau}} - \rho\right) \mathrm{d}\rho$$

(by Lemma A.1 of Appendix A)

$$\leqslant As^{-1/2}\left(1 + \ln_+ \frac{\sqrt{\tau}}{p_1}\right)\left(1 + \tilde{a}_1 + \frac{p_1}{\sqrt{\tau}}\right)$$

$$\leqslant As^{-1/2}\left(1 + \ln_+ \frac{1}{\bar{x}\sqrt{t/\tau} - \tilde{a}_1}\right)\left(1 + \bar{x}\sqrt{t/\tau}\,\right)$$

$$\leqslant A_1 s^{-1/2}\left(1 + \ln_+ \frac{1}{(\bar{x} - \tilde{a}_1)}\right)\left(1 + \bar{x}/\sqrt{\alpha}\,\right).$$

For I''_+, when $p > s$ we use (ii), and obtain

$$I''_+ \leqslant As^{-1/2} \int_0^{p_1} \mathrm{e}^{-p/4} \left(1 + \ln_+ \frac{\sqrt{\tau}}{p_1 - p}\right)\left(1 + \tilde{a}_1 + \frac{p_1}{\sqrt{\tau}} - \rho\right) \mathrm{d}p$$

$$\leqslant As^{-1/2}\left(1 + \ln_+ \frac{\sqrt{\tau}}{p_1}\right)\left(1 + \bar{x}\sqrt{t/\tau}\,\right)$$

$$\leqslant A_1 s^{-1/2}\left(1 + \ln_+ \frac{1}{(\bar{x} - \tilde{a}_1)}\right)\left(1 + \bar{x}/\sqrt{\alpha}\,\right).$$

Let us estimate now integral I_-. Put $q = (t - \tau) - (x - \xi)$. By part (i) $\Delta_x G(x - \xi, t - \tau) > 0$ iff $q \in (0, s)$. We use now part (iii) and obtain:

$$I_- \leqslant \frac{A}{s} \int_0^{s} \mathrm{e}^{-q^2/(2s)} \frac{q}{\sqrt{s - q}} \left(1 + \ln_+ \frac{\sqrt{\tau}}{p_1 + q}\right)\left(1 + \tilde{a}_1 + \frac{p_1 + q}{\sqrt{\tau}}\right) \mathrm{d}q$$

$$\leqslant \frac{A}{s}\left(1 + \ln_+ \frac{1}{\bar{x} - \tilde{a}_1}\right)\left(\left(1 + \bar{x}/\sqrt{\alpha}\,\right) \int_0^{s} \frac{\mathrm{e}^{-q^2/(2s)} q}{\sqrt{s - q}} \,\mathrm{d}q + \frac{1}{\sqrt{\tau}} \int_0^{s} \frac{\mathrm{e}^{-q^2/(2s)} q^2}{\sqrt{s - q}} \,\mathrm{d}q\right),$$

where

$$\int_0^s \frac{e^{-q^2/(2s)}q}{\sqrt{s-q}}\,dq \leqslant \int_0^{s/2} \frac{e^{-q^2/(2s)}q}{\sqrt{s/2}}\,dq + \int_{s/2}^{s} \frac{e^{-s/8}s}{\sqrt{s-q}}\,dq$$
$$\leqslant \sqrt{2s}\int_0^{s/8} e^{-y}\,dy + 2se^{-s/8}\sqrt{s/2}$$
$$= \sqrt{2s}\bigl(1 - e^{-s/2} + se^{-s/8}\bigr) = O\bigl(\sqrt{s}\,\bigr),$$

$$\int_0^s \frac{e^{-q^2/(2s)}q^2}{\sqrt{s-q}}\,dq \leqslant \int_0^{s/2} \frac{e^{-q^2/(2s)}q^2}{\sqrt{s/2}}\,dq + \int_{s/2}^{s} \frac{e^{-s/8}s^2}{\sqrt{s-q}}\,dq$$
$$\leqslant 2s\int_0^{s/8} \sqrt{y}e^{-y}\,dy + 2s^2e^{-s/8}\sqrt{s/2} = O(s).$$

Hence,

$$I_- \leqslant \frac{A_2}{\sqrt{s}}\left(1 + \ln_+ \frac{1}{\bar{x} - \tilde{a}_1}\right)\bigl(1 + \sqrt{(1-\alpha)/\alpha}\,\bigr)\bigl(1 + \bar{x}/\sqrt{\alpha}\,\bigr).$$

Lemma 6 is proved. □

Now we are ready to estimate terms I_2u and I_3u of formula (3.4).

Lemma 7. *Let function $F = u$ satisfy the conditions of Theorem* 2(ii) *and Δu is represented in Ω_σ by formula* (3.4), *$\alpha \geqslant \sup\{1/2, \frac{1+\sigma_0}{1+\sigma}\}$, $\sigma_0 < \sigma$. Then terms I_2u and I_3u of formula* (3.4) *admit the following estimates*:

$$\bigl|I_2u(x,t)\bigr| \leqslant A_2\frac{\Gamma \cdot \bar{x}}{\sqrt{(1-\alpha)t}}, \tag{3.5}$$

$$\bigl|I_3u(x,t)\bigr| \leqslant A_0\frac{\Gamma \cdot \tilde{a}_1}{t^{3/2}}\left(\frac{1}{\delta^2} + \frac{\tilde{a}_1}{2\delta}\right)K^+,$$
$$\text{where } K^+ = \int_{\alpha t}^{t} d\tau \int_{\bar{\xi}\in[a_1,\tilde{a}_1]} \bigl|\Delta_x G(x-\xi, t-\tau)\bigr|\,d\xi, \tag{3.6}$$

A_2 is absolute constant, $\bar{x} \in (\tilde{a}_1, a_2 + \sigma_0\sqrt{t}\,)$, $t > t_0(\sigma_0, \sigma)$.

Remark. I_2u is the only term in representation (3.4), where $1 - \alpha$ is in the denominator.

Proof. The definitions of I_2u and I_3u, condition (1.11) and Lemma 4 imply estimates:

$$\left|I_2u(x,t)\right| \leqslant \Gamma \int\limits_{\bar{\xi}>a_1} \left|\Delta_x G(x-\xi, t-\alpha t)\right| \frac{\bar{\xi}\,\mathrm{d}\xi}{\sqrt{\alpha t}}, \tag{3.7}$$

where $\bar{\xi} = \frac{\xi - \alpha t}{\sqrt{\alpha t}}$,

$$\left|I_3u(x,t)\right| \leqslant \Gamma A_0\left(\frac{1}{\delta^2} + \frac{\tilde{a}_1}{2\delta}\right) \int\limits_{\alpha t}^{t} \mathrm{d}\tau \int\limits_{\bar{\xi}\in(a_1,\tilde{a}_1)} \frac{|\Delta_x G(x-\xi, t-\tau)|}{\tau} \frac{\bar{\xi}\,\mathrm{d}\xi}{\sqrt{\tau}}, \tag{3.8}$$

where $\bar{\xi} = \frac{\xi - \tau}{\sqrt{\tau}}$.

Using Lemmas 6(i), 5(i), 6(iv) (see also (3.14)), we obtain further from (3.7):

$$\begin{aligned}
\left|I_2u(x,t)\right| &\leqslant \frac{\Gamma}{\sqrt{\alpha t}}\Bigg[\int\limits_{\bar{\xi}<\bar{x}\sqrt{t/(\alpha t)}} \Delta_\xi G(x-\xi, t-\alpha t)\bar{\xi}\,\mathrm{d}\xi \\
&\qquad - \int\limits_{\bar{\xi}>\bar{x}\sqrt{t/(\alpha t)}} \Delta_\xi G(x-\xi, t-\alpha t)\bar{\xi}\,\mathrm{d}\xi\Bigg] \\
&\leqslant \frac{\Gamma}{\sqrt{\alpha t}}\Bigg[-\int\limits_{\bar{\xi}<\bar{x}/\sqrt{\alpha}} G\cdot\Delta_\xi\bar{\xi}\,\mathrm{d}\xi + \int\limits_{\bar{\xi}>\bar{x}/\sqrt{\alpha}} G\cdot\Delta_\xi\bar{\xi}\,\mathrm{d}\xi + G\bar{\xi}\big|_{a_1}^{\bar{x}/\sqrt{\alpha}} \\
&\qquad - G\bar{\xi}\big|_{\bar{x}/\sqrt{\alpha}}^{\bar{x}/\sqrt{\alpha}+(1-\alpha)\sqrt{t}/\sqrt{\alpha}}\Bigg] \\
&\leqslant \frac{\Gamma}{\sqrt{\alpha t}}\Bigg[\int\limits_{\bar{\xi}>\bar{x}/\sqrt{\alpha}} \frac{1}{\sqrt{\alpha t}} G(x-\xi, t-\alpha t)\,\mathrm{d}\xi + 2G(t-\alpha t, t-\alpha t)\frac{\bar{x}}{\sqrt{\alpha}}\Bigg] \\
&\leqslant \frac{\Gamma}{\sqrt{\alpha t}}\left(\frac{1}{\sqrt{\alpha t}} + \frac{2}{\sqrt{2\pi(t-\alpha t)}}\left(\frac{\bar{x}}{\sqrt{\alpha}}\right)\right) \\
&\leqslant \frac{A_2\Gamma}{\sqrt{(1-\alpha)\alpha}}\frac{1}{t}\left(\frac{\bar{x}}{\sqrt{\alpha}}\right), \quad \text{if } t \geqslant t_0.
\end{aligned}$$

From (3.8) we deduce:

$$\left|I_3u(x,t)\right| \leqslant \frac{\Gamma A_0}{t^{3/2}}\left(\frac{1}{\delta^2} + \frac{\tilde{a}_1}{2\delta}\right)\tilde{a}_1 \int\limits_{\alpha t}^{t} \mathrm{d}\tau \int\limits_{\bar{\xi}\in(a_1,\tilde{a}_1)} \left|\Delta_x G(x-\xi, t-\tau)\right|\mathrm{d}\xi.$$

We have proved (3.5), (3.6). □

We will estimate now the terms I_1u and I_4u of (3.4).

Lemma 8. *Let function u satisfy the conditions of Theorem* 2(ii) *and Δu be represented in Ω_σ by formula* (3.4)*. Then terms I_1u and I_4u of formula* (3.4) *admit the following* (*preliminary*) *estimates for $\bar{x} \geqslant \tilde{a}_1$ and $t \geqslant t_0$*:

$$|I_1u| \leqslant \frac{4\gamma_0\Gamma^2 \cdot \tilde{a}_1^2}{\alpha t}(K^- + K_1), \tag{3.9}$$

$$|I_4u| \leqslant \frac{2A_0\Gamma \cdot \tilde{a}_1}{\delta\alpha t}(K^- + K_1), \tag{3.10}$$

where

$$K^- = \int_{\alpha t}^{t} \left|\Delta_x G(x-\xi, t-\tau)\right|_{\bar{\xi}=a_-,\ \bar{\xi}<\tilde{a}_1} \mathrm{d}\tau,$$

$$K_1 = \int_{\alpha t}^{t} \left|\Delta_x G(x-\xi, t-\tau)\right|_{\bar{\xi}=\tilde{a}_1} \mathrm{d}\tau,$$

$$a_- = \bar{x}\sqrt{t/\tau} - \frac{1}{2\sqrt{\tau}} - \sqrt{(t-\tau)/\tau + 1/(4\tau)}.$$

Proof. If t_0 is large enough and $\tau \geqslant t_0$ we have, using (1.11) inequalities:

$$\left|u(\xi,\tau)\right| \leqslant \frac{\Gamma \cdot \bar{\xi}}{\sqrt{\tau}}, \quad \left|\varphi_0(u)\right| \leqslant 2\gamma_0|u|, \quad \left|\Delta_\xi \chi(\xi,\tau)\right| \leqslant \frac{A_0}{\delta\sqrt{\tau}}.$$

From these relations and from definitions of I_1u, I_4u it follows (using also that $\bar{\xi} \leqslant \tilde{a}_1 \leqslant \bar{x}$):

$$|I_1u| \leqslant \frac{2\gamma_0\Gamma \cdot \tilde{a}_1}{\sqrt{\alpha t}} I_5u \quad \text{and} \quad |I_4u| \leqslant \frac{A_0}{\delta\sqrt{\alpha t}} I_5u, \tag{3.11}$$

where

$$I_5u = \int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi}\in[a_1,\tilde{a}_1]} \left|\Delta_x G(x-\xi, t-\tau)\right| \cdot \left|\Delta_\xi u(\xi,\tau)\right| \mathrm{d}\xi. \tag{3.12}$$

The assumption of Theorem 2(ii) implies that

$$\Delta_\xi u(\xi,\tau) \geqslant 0 \quad \forall \tau \geqslant \tau_0. \tag{3.13}$$

By Lemma 6 we have also inequalities:

$$\begin{aligned}
&\Delta_x G(x-\xi, t-\tau) < 0 \quad \text{iff} \quad \bar{\xi} < \bar{x}\sqrt{t/\tau},\\
&\Delta_x G(x-\xi, t-\tau) > 0 \quad \text{iff} \quad \bar{\xi} > \bar{x}\sqrt{t/\tau}.
\end{aligned} \tag{3.14}$$

From (3.12)–(3.14), we deduce:

$$\begin{aligned}
I_5 u &= -\int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi}\in[a_1,\tilde{a}_1]} \Delta_x G \Delta_\xi u \, \mathrm{d}\xi \\
&= -\int_{\alpha t}^{t} \mathrm{d}\tau \left(\int_{\bar{\xi}\in[a_1,\tilde{a}_1]} \Delta_x^2 G \cdot u \, \mathrm{d}\xi + \Delta_x G \cdot u|_{\bar{\xi}=\tilde{a}_1} - \Delta_x G \cdot u|_{\bar{\xi}=a_1} \right).
\end{aligned}$$

Using inequality $|u(\xi,\tau)| \leqslant \frac{\Gamma\cdot\bar{\xi}}{\sqrt{\tau}}$, we obtain:

$$|I_5 u| \leqslant \frac{\Gamma\cdot\tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} \mathrm{d}\tau \left[\int_{\bar{\xi}\in[a_1,\tilde{a}_1]} |\Delta_x^2 G| \, \mathrm{d}\xi + |\Delta_x G|_{\bar{\xi}=a_1} + |\Delta_x G|_{\bar{\xi}=\tilde{a}_1} \right]. \tag{3.15}$$

From Lemma 6, we have:

$$\begin{aligned}
&\Delta_x^2 G(x-\xi, t-\tau) < 0, \quad \text{iff} \quad \bar{\xi} \in (a_-, a_+),\\
&\quad \text{where } a_\pm = \bar{x}\sqrt{t/\tau} - \frac{1}{2\sqrt{\tau}} \pm \sqrt{(t-\tau)/\tau + 1/(4\tau)}.
\end{aligned} \tag{3.16}$$

If t_0 is large enough and $\tilde{a}_1 > a_1\sqrt{\alpha} + \sqrt{1-\alpha}$ we have inequality: $a_- > a_1$.

Put $\bar{\xi}_- = \inf\{\tilde{a}_1, a_-\}$.

From (3.14)–(3.16), we deduce:

$$\begin{aligned}
|I_5 u| &\leqslant \frac{\Gamma\cdot\tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} \mathrm{d}\tau \left[\int_{\bar{\xi}\in[a_1,\bar{\xi}_-]} \Delta_x^2 G \, \mathrm{d}\xi - \int_{\bar{\xi}\in[\bar{\xi}_-,\tilde{a}_1]} \Delta_x^2 G \, \mathrm{d}\xi - \Delta_x G|_{\bar{\xi}=a_1} - \Delta_x G|_{\bar{\xi}=\tilde{a}_1} \right] \\
&\leqslant \frac{\Gamma\cdot\tilde{a}_1}{\sqrt{\alpha t}} \int_{\alpha t}^{t} \mathrm{d}\tau [\Delta_x G|_{\bar{\xi}=a_1} - \Delta_x G|_{\bar{\xi}=\bar{\xi}_-} + \Delta_x G|_{\bar{\xi}=\tilde{a}_1} \\
&\quad - \Delta_x G|_{\bar{\xi}=\bar{\xi}_-} - \Delta_x G|_{\bar{\xi}=a_1} - \Delta_x G|_{\bar{\xi}=\tilde{a}_1}] \\
&\leqslant \frac{\Gamma\cdot\tilde{a}_1}{\sqrt{\alpha t}} (-2\Delta_x G|_{\bar{\xi}=\bar{\xi}_-}) \leqslant \frac{2\Gamma\cdot\tilde{a}_1}{\sqrt{\alpha t}} (K^- + K_1).
\end{aligned}$$

The last estimate together with estimates (3.12) imply (3.9), (3.10). □

The following lemma gives more precise estimates for terms I_1u, I_3u, I_4u.

Lemma 9. *In conditions and notations of Lemmas* 7, 8 *we have estimates*:

$$|I_3u| \leqslant A_2 \frac{A_0\Gamma \cdot \tilde{a}_1}{t}\left(\frac{\sqrt{1-\alpha}}{\delta^2} + \frac{1}{\delta} + \frac{\tilde{a}_1}{\delta\sqrt{t}}\right), \tag{3.17}$$

$$|I_1u| \leqslant A_2 \frac{\gamma_0\Gamma^2 \cdot \tilde{a}_1^2}{t}\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right), \tag{3.18}$$

$$|I_4u| \leqslant A_2 \frac{A_0\Gamma \cdot \tilde{a}_1}{\delta t}\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right), \tag{3.19}$$

where A_2 is absolute constant, α is sufficiently close to 1.

Proof. In order to prove (3.17)–(3.19), it is sufficient to prove estimates:

$$K^- \leqslant A\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right), \tag{3.20}$$

$$K_1 \leqslant A, \tag{3.21}$$

$$K^+ \leqslant A\sqrt{t}\,\inf\left\{\sqrt{1-\alpha}, \frac{1}{\tilde{a}_1} + \frac{1}{\sqrt{t}}\right\}, \tag{3.22}$$

where K^+, K^-, K_1 are integrals from (3.6), (3.9), (3.10).

Let us prove firstly (3.21). Put $\varepsilon = \bar{x} - \tilde{a}_1$, indicating that it can be arbitrary small, $y = x - \xi$, $s = t - \tau$. We have:

$$\begin{aligned} p = y - s &= \bar{x}\sqrt{t} - \tilde{a}_1\sqrt{\tau} \\ &= \varepsilon\sqrt{t} + \tilde{a}_1(\sqrt{t} - \sqrt{t-s}) = \varepsilon\sqrt{t} + \frac{\tilde{a}_1 s}{2\theta\sqrt{t}} > 0, \\ &\text{where } \theta(s) = \frac{\sqrt{t} + \sqrt{t-s}}{2\sqrt{t}}, \quad \frac{1+\sqrt{\alpha}}{2} \leqslant \theta < 1. \end{aligned}$$

Since $0 \leqslant s \leqslant (1-\alpha)t$, we have:

$$K_1 = \int\limits_{\alpha t}^{t} |\Delta G|_{\xi = \tilde{a}_1} \mathrm{d}\tau = -\int\limits_{0}^{(1-\alpha)t} \Delta G(p+s, s)\,\mathrm{d}s = K_{10} + K_{11},$$

where

$$K_{10} = -\int\limits_{s<p} \Delta G(p+s, s)\,\mathrm{d}s, \qquad K_{11} = -\int\limits_{s>p} \Delta G(p+s, s)\,\mathrm{d}s.$$

Note that $s<p$ iff $s<\varepsilon\sqrt{t}(1-\frac{\tilde a_1}{2\theta\sqrt{t}})^{-1}$ and $\frac{\tilde a_1}{2\theta\sqrt{t}}<1$. Hence, inequality $s<p$ implies $s<2\varepsilon\sqrt{t}$, if $t>t_0$ and inequality $s>p$ implies $s>\varepsilon\sqrt{t}$, if $t>t_0$.

Using Lemma 6(ii), we obtain:

$$K_{10}\leqslant\int_0^{2\varepsilon\sqrt{t}}\frac{1}{\sqrt{s}}\mathrm{e}^{-s/4}\,\mathrm{d}s\leqslant\int_0^{\infty}\frac{1}{\sqrt{s}}\mathrm{e}^{-s/4}\,\mathrm{d}s\leqslant A_2,$$

$$K_{11}\leqslant A\int_{\varepsilon\sqrt{t}}^{(1-\alpha)t}s^{-3/2}p\mathrm{e}^{-p^2/(4s)}\,\mathrm{d}s$$

$$\left(\text{putting } s=\eta\cdot t \text{ and } p=\sqrt{t}\left(\varepsilon+\frac{\tilde a_1\eta}{2\theta}\right)\right)$$

$$\leqslant A\int_{\varepsilon/\sqrt{t}}^{1-\alpha}\eta^{-3/2}\left(\varepsilon+\frac{\tilde a_1\eta}{2\theta}\right)\exp\left(-\left(\varepsilon+\frac{\tilde a_1\eta}{2\theta}\right)^2/(4\eta)\right)\mathrm{d}\eta$$

$$\leqslant A\left(\int_0^1\frac{\tilde a_1}{2\theta\sqrt{\eta}}\exp\left(-\frac{\tilde a_1^2\eta}{16}\right)\mathrm{d}\eta+\int_0^1\varepsilon\eta^{-3/2}\mathrm{e}^{-\varepsilon^2/(4\eta)}\,\mathrm{d}\eta\right)$$

$$\left(\text{putting } \eta=r\tilde a_1^{-2} \text{ or } \eta=\rho\varepsilon^2 \text{ respectively}\right)$$

$$\leqslant A\left(\frac{1}{2\theta}\int_0^{\infty}r^{-1/2}\mathrm{e}^{-r/16}\,\mathrm{d}r+\int_0^{\infty}\rho^{-3/2}\mathrm{e}^{-1/(4\rho)}\,\mathrm{d}\rho\right)\leqslant A_2.$$

Inequality (3.21) is proved.

Let us prove now (3.20). Let us find interval of variable s in which $\bar\xi_-=a_-<\tilde a_1$, i.e.,

$$a_-=\bar x\sqrt{t/\tau}-\frac{1}{2\sqrt{\tau}}-\sqrt{(t-\tau)/\tau+1/(4\tau)}<\tilde a_1,\quad \text{i.e.,}$$

$$\bar x\sqrt{t/\tau}-\sqrt{(t-\tau)/\tau}<\bar a_1,\quad \text{where } \bar a_1=\tilde a_1\left(1+\mathrm{O}\left(\frac{1}{\sqrt{\tau}}\right)\right).$$

Put $\eta=\frac{t-\tau}{t}=\frac{s}{t}$. We obtain:

$$\bar x-\sqrt{\eta}<\bar a_1\sqrt{1-\eta},\quad \text{i.e.,}$$

$$\bar x^2-2\bar x\sqrt{\eta}+\eta<\bar a_1^2(1-\eta),\quad \text{i.e.,}$$

$$\left(\sqrt{\eta}-\frac{\bar x}{1+\bar a^2}\right)^2<\frac{\bar a_1^2(1+\bar a_1^2-\bar x^2)}{(1+\bar a_1^2)^2},\quad \text{i.e.,}$$

$$\sqrt{\eta_1} < \sqrt{\eta} < \sqrt{\eta_2}, \quad \text{where}$$
$$\sqrt{\eta_1} = \frac{\bar{x}}{1+\bar{a}_1^2} - \frac{\bar{a}_1\sqrt{1+\bar{a}_1^2-\bar{x}^2}}{1+\bar{a}_1^2}; \qquad \sqrt{\eta_2} = \frac{\bar{x}}{1+\bar{a}_1^2} + \frac{\bar{a}_1\sqrt{1+\bar{a}_1^2-\bar{x}^2}}{1+\bar{a}_1^2}.$$

The interval is not empty if $\bar{x} \leqslant \sqrt{1+\bar{a}_1^2}$. In addition we have:

$$\bar{x} - \bar{a}_1\sqrt{1+\bar{a}_1^2-\bar{x}^2} \geqslant \bar{x} - \bar{a}_1\left(1 + \frac{\bar{a}_1^2-\bar{x}^2}{2}\right)$$
$$= (\bar{x}-\bar{a}_1)\left(1+\frac{\bar{a}_1(\bar{x}+\bar{a}_1)}{2}\right) \geqslant (\bar{x}-\bar{a}_1)\big(1+\bar{a}_1^2\big).$$

Hence $\sqrt{\eta_1} > \bar{x} - \bar{a}_1$. The condition $\bar{\xi}_- = a_-$ implies that

$$y = (x-\xi) = (t-\tau) + \sqrt{t-\tau} + \mathrm{O}(1) = s + \sqrt{s} + \mathrm{O}(1).$$

From Lemmas 5, 6, we deduce:

$$-\Delta G|_{\bar{\xi}=a_-} \leqslant \frac{\sqrt{\mathrm{e}}}{s\sqrt{2\pi}}\left(1+\mathrm{O}\left(\frac{1}{\sqrt{s}}\right)\right) \leqslant \frac{A}{s}.$$

Hence,

$$K^- \leqslant \int_{\alpha t}^{t} |\Delta_x G|_{\bar{\xi}=a_-}\,\mathrm{d}\tau \leqslant \int_{\eta_1 t}^{(1-\alpha)t} \frac{A}{s}\,\mathrm{d}s \leqslant A_2 \ln_+ \frac{\sqrt{1-\alpha}}{(\bar{x}-\tilde{a}_1)}, \quad t \geqslant t_0.$$

Let us prove (3.22). Using definition of K^+ and (3.14), we obtain:

$$K^+ = -\int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi}\in[a_1,\tilde{a}_1]} \Delta_x G(x-\xi, t-\tau)\,\mathrm{d}\xi \leqslant \int_{\alpha t}^{t} G(x-\xi, t-\tau)|_{\bar{\xi}=\tilde{a}_1}\,\mathrm{d}\tau.$$

Put (as in the proof of (3.21)) $\varepsilon = \bar{x} - \tilde{a}_1$, $y = x - \xi$, $s = t - \tau$, $p = y - s$.

We have:

$$\int_{\alpha t}^{t} G|_{\bar{\xi}=\tilde{a}_1}\,\mathrm{d}\tau = \int_{s<p} G(p+s, s)\,\mathrm{d}s + \int_{s>p} G(p+s, s)\,\mathrm{d}s.$$

Because $s < p$ implies $s < 2\varepsilon\sqrt{t}$, $t \geqslant t_0$, and using Lemma 5(i), we obtain:

$$\int_{s<p} G(p+s,s)\,\mathrm{d}s \leqslant \int_0^{2\varepsilon\sqrt{t}} \frac{1}{\sqrt{2\pi(p+s)}} \exp\left(-\frac{p^2}{2(p+s)}\right)\mathrm{d}s$$
$$\leqslant \int_0^{2\varepsilon\sqrt{t}} \frac{1}{\sqrt{2\pi p}}\mathrm{e}^{-p/4}\,\mathrm{d}s \leqslant \int_0^{2\varepsilon\sqrt{t}} \frac{1}{\sqrt{2\pi s}}\mathrm{e}^{-s/4}\,\mathrm{d}s \leqslant A_2.$$

Because $s>p$ implies $s\in(\varepsilon\sqrt{t},(1-\alpha)t)$ and using Lemma 5(i), we obtain:

$$\int_{s>p} G(p+s,s)\,\mathrm{d}s \leqslant \int_{\varepsilon\sqrt{t}}^{(1-\alpha)t} \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{p^2}{4s}\right)\mathrm{d}s$$
$$\leqslant \frac{1}{\sqrt{2\pi}} \int_0^{(1-\alpha)t} s^{-1/2} \exp\left(-\frac{p^2}{4s}\right)\mathrm{d}s.$$

Using $p=\sqrt{t}(\varepsilon+\frac{\tilde{a}_1 s}{2\theta t})$ and putting $\rho=\tilde{a}_1^2\frac{s}{t}$, we obtain further,

$$\int_{s>p} G(p+s,s)\,\mathrm{d}s \leqslant \frac{\sqrt{t}}{\sqrt{2\pi}\tilde{a}_1} \int_0^{\tilde{a}_1^2(1-\alpha)} \frac{1}{\sqrt{\rho}}\mathrm{e}^{-\rho/16}\,\mathrm{d}\rho$$
$$\leqslant \sqrt{\frac{t}{2\pi}}\inf\left\{2\sqrt{1-\alpha}, \frac{1}{\tilde{a}_1}\int_0^{\infty} \frac{1}{\sqrt{\rho}}\mathrm{e}^{-\rho/16}\,\mathrm{d}\rho\right\}.$$

Hence, $K^+\leqslant A\sqrt{t}\inf\{\sqrt{1-\alpha}, \frac{1}{\sqrt{t}}+\frac{1}{\tilde{a}_1}\}$.

Lemma 9 is proved. □

Proof of Theorem 2(ii). From formula (3.4) and estimates (3.5), (3.17)–(3.19) we deduce the following inequality under condition that $\bar{x}\in(\tilde{a}_1, a_2+\sigma_0\sqrt{t})$, $\sigma_0<\sigma$, $t\geqslant\tilde{a}_1^2$ and $\alpha>\frac{1+\sigma_0}{1+\sigma}$:

$$\Delta u(x,t)\leqslant \frac{A_3\Gamma\cdot\bar{x}}{t}\left[\frac{1}{\sqrt{1-\alpha}}+\frac{\sqrt{1-\alpha}}{\delta^2}+\frac{1}{\delta}\right.$$
$$\left.+\left(\gamma_0\Gamma\cdot\tilde{a}_1+\frac{1}{\delta}\right)\left(1+\ln_+\frac{\sqrt{1-\alpha}}{\bar{x}-\tilde{a}_1}\right)\right]$$
$$+\gamma_0\Gamma\cdot\bar{x}\int_{\alpha t}^{t}\mathrm{d}\tau\int_{\bar{\xi}\geqslant\tilde{a}_1}\left|\Delta_x G(x-\xi,t-\tau)\right|\frac{|\Delta u(\xi,\tau)|}{\sqrt{\tau}}\,\mathrm{d}\xi. \qquad (3.23)$$

Put:

$$v(t) = t \cdot \max_{\bar{x} \in (\tilde{a}_1, a_2 + \sigma_0 \sqrt{t})} \frac{\Delta u(x,t)}{g(\bar{x})},$$

where

$$g(\bar{x}) = B_1 + B_2\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right),$$

$$B_1 = \bar{x}\left(\frac{1}{\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\delta^2} + \frac{1}{\delta}\right); \qquad B_2 = \bar{x}\left(\gamma_0 \Gamma \cdot \tilde{a}_1 + \frac{1}{\delta}\right).$$

Then we have $\Delta u(x,t) \leqslant \frac{v(t) \cdot g(\bar{x})}{t}$. From this relation and from (3.23), we obtain:

$$v(t) \leqslant A_3 \Gamma + \frac{\gamma_0 \Gamma \cdot t}{g(\bar{x})} \int_{\alpha t}^{t} \frac{v(\tau)}{\tau^{3/2}} \int_{\bar{\xi} > \tilde{a}_1} |\Delta_x G| \cdot g(\bar{\xi})\, \mathrm{d}\xi.$$

By Lemma 6(v), we have:

$$\int_{\bar{\xi} > \tilde{a}_1} |\Delta_x G| \cdot g(\bar{\xi})\, \mathrm{d}\xi \leqslant \frac{A_4 g(\bar{x})}{\sqrt{t-\tau}} \left(1 + \sqrt{(1-\alpha)/\alpha}\right)\left(1/\sqrt{\alpha}\right).$$

From the last two inequalities, putting $\tau = \rho t$, we get:

$$v(t) \leqslant A_3 \Gamma + A_4 \gamma_0 \Gamma \int_{\alpha}^{1} \frac{v(\rho t)\, \mathrm{d}\rho}{\rho^{3/2}\sqrt{1-\rho}} \left(1 + \sqrt{(1-\alpha)/\alpha}\right)\left(1/\sqrt{\alpha}\right).$$

Choose α_1 so close to 1 that $\alpha_1 > \frac{1+\sigma_0}{1+\sigma}$ and

$$\left(1 + \sqrt{(1-\alpha_1)/\alpha_1}\right)\left(1/\sqrt{\alpha_1}\right) A_4 \gamma_0 \Gamma \int_{\alpha_1}^{1} \frac{\mathrm{d}\rho}{\rho^{3/2}\sqrt{1-\rho}} < 1.$$

It means that $\frac{1}{\sqrt{1-\alpha_1}}$ must be of order $\mathrm{O}(\frac{\sqrt{1+\sigma}}{\sqrt{\sigma-\sigma_0}} + \gamma_0 \Gamma)$.

Using Lemma A.2 of Appendix A we obtain:

$$\Delta u \leqslant \frac{v(t) \cdot g(\bar{x})}{t} \leqslant \frac{A_5 \Gamma}{t}\left(B_1 + B_2\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right)\right),$$

where $\bar{x} \in (\tilde{a}_1, a_2 + \sigma_0\sqrt{t})$, $t \geqslant t_0 \geqslant \tilde{a}_1^2$. Put now $\sqrt{1-\alpha} = \min\{\delta, \sqrt{1-\alpha_1}\}$.

Then we obtain:

$$\Delta u \leqslant \frac{A_5 \Gamma \cdot \bar{x}}{t}\left[\frac{1}{\sqrt{1-\alpha}} + \left(\gamma_0 \Gamma \cdot \tilde{a}_1 + \frac{1}{\delta}\right)\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1}\right)\right].$$

Now let $\bar{x} > a_1$ be fixed and take $\tilde{a}_1 = \frac{a_1 + \bar{x}}{2}$, $d = \frac{\delta}{2}$. We obtain:

$$\Delta u \leqslant \frac{A_6 \Gamma \cdot \bar{x}}{t}\left[\frac{\sqrt{1+\sigma}}{\sqrt{\sigma - \sigma_0}} + \gamma_0 \Gamma + \left(\gamma_0 \Gamma \cdot a_1 + \frac{1}{d}\right)\right].$$

Theorem 2(ii) is proved. □

Sketch of the proof of Theorem 2(ii)′. *Step* 1. Let function u satisfy Eq. (1.2) in Ω_0 with $\varphi(0) = C = 1$ and $\varepsilon = 1$. Put $\varphi_0 = \varphi - C$. We use again the Green–Poisson type representation formulas for u of type (3.2), (3.4), where $\chi = \chi_0(\bar{x})$, $\bar{x} = \frac{x-t}{\sqrt{t}}$, $\chi_0 : \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function such that $0 \leqslant \chi_0 \leqslant 1$, $\chi_0|_{[\tilde{a}_1, \tilde{a}_2]} \equiv 1$, $\chi|_{(-\infty, a_1)} \equiv 0$, $\chi|_{(a_2, \infty)} \equiv 0$, $0 < a_1 < \tilde{a}_1 < \tilde{a}_2 < a_2$, inequalities (3.1) are valid with $\delta = \min\{\tilde{a}_1 - a_1, a_2 - \tilde{a}_2\}$. We obtain representation ($\bar{x} \in [\tilde{a}_1, \tilde{a}_2]$, $t > \alpha t$):

$$\Delta u = I_0 u + I_1 u + I_2 u + I_3 u + I_4 u, \tag{3.4$'$}$$

where

$$I_0 u = -\int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi} \in (\tilde{a}_1, \tilde{a}_2)} \Delta G \cdot \varphi_0(u) \cdot \Delta u \, \mathrm{d}\xi,$$

$$I_1 u = -\int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi} \in [a_1, a_2] \setminus [\tilde{a}_1, \tilde{a}_2]} \Delta G \cdot \varphi_0(u) \cdot \Delta u \chi \, \mathrm{d}\xi,$$

$$I_2 u = \int_{\bar{\xi} \in [a_1, a_2]} \Delta G(x - \xi, t - \alpha t) u(\xi, \alpha t) \chi(\xi, \alpha t) \, \mathrm{d}\xi,$$

$$I_3 u = \int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi} \in [a_1, a_2] \setminus [\tilde{a}_1, \tilde{a}_2]} \Delta G (u \chi' + u \Delta \chi) \, \mathrm{d}\xi,$$

$$I_4 u = -\int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi} \in [a_1, a_2] \setminus [\tilde{a}_1, \tilde{a}_2]} \Delta G \cdot \Delta u \cdot \Delta \chi \, \mathrm{d}\xi.$$

Step 2. Let u satisfy conditions of Theorem 2(ii)′ and Δu be represented in Ω_0 by formula (3.4)′, $\alpha > 1/2$. Using Lemmas 4–6, we obtain Lemmas 7′ and 9′:

Lemma 7′. *For $\bar{x} \in [a_1, a_2]$ and $t \geqslant t_0$ the following estimates are valid*:

$$\left|I_2 u(x,t)\right| \leqslant \frac{A_2 \Gamma}{\sqrt{(1-\alpha)t}}, \tag{3.5$'$}$$

$$\left|I_3 u(x,t)\right| \leqslant \frac{A_2 \Gamma}{t}\left(\frac{1}{\delta^2} + \frac{\tilde{a}_2}{2\delta}\right)\sqrt{1-\alpha}. \tag{3.6$'$}$$

Lemma 9′. *For $\bar{x} \in [\tilde{a}_1, \tilde{a}_2]$ and $t \geqslant t_0$ the following estimates are valid*:

$$\left|I_1 u(x,t)\right| \leqslant A_2 \frac{\gamma_0 \Gamma^2}{t}\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1-\alpha}}{\tilde{a}_2 - \bar{x}}\right), \tag{3.18$'$}$$

$$\left|I_4 u(x,t)\right| \leqslant A_2 \frac{A_0 \Gamma}{\delta t}\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1-\alpha}}{\tilde{a}_2 - \bar{x}}\right). \tag{3.19$'$}$$

Step 3. From formula (3.4)′ and estimates (3.5)′, (3.6)′, (3.18)′, (3.19)′ we deduce the following inequality ($\bar{x} \in [\tilde{a}_1, \tilde{a}_2]$):

$$\begin{aligned}\Delta u \leqslant \frac{A_3 \Gamma}{t}\Bigg[&\frac{1}{\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\delta^2} + \frac{\tilde{a}_2\sqrt{1-\alpha}}{2\delta} \\ &+ \left(\gamma_0 \Gamma + \frac{1}{\delta}\right)\left(1 + \ln_+ \frac{\sqrt{1-\alpha}}{\bar{x} - \tilde{a}_1} + \ln_+ \frac{\sqrt{1-\alpha}}{\tilde{a}_2 - \bar{x}}\right)\Bigg] \\ &- \int_{\alpha t}^{t} \mathrm{d}\tau \int_{\bar{\xi} \in (\tilde{a}_1, \tilde{a}_2)} \Delta G \varphi_0(u) \Delta u \,\mathrm{d}\xi.\end{aligned} \tag{3.23$'$}$$

By assumption of Theorem 2(ii)′ we have $\Delta_\xi u(\xi, \tau) \geqslant 0$. If in assumptions of Theorem 2(ii)′ we have additional positivity conditions $\varphi'(0) \geqslant 0$ and $u \geqslant 0$ then we can replace the integral term in (3.23)′ by the following bigger one:

$$-\gamma_0 \Gamma \int_{\alpha t}^{t} \frac{\mathrm{d}\tau}{\sqrt{\tau}} \int_{\bar{\xi}:\ \Delta G < 0} \Delta_x G \cdot \Delta u \,\mathrm{d}\xi.$$

Following further the proof of Theorem 2(ii) and applying again Lemma 6(v), we obtain the statement of Theorem 2(ii)′ with constant $B = B_0(a_2 + \frac{1}{d} + \frac{\gamma_0 \Gamma}{C})$. □

Without additional positivity conditions the statement of Theorem 2(ii)′ is also valid but for the proof of it more hard version of Lemma 6(v) is needed where the weight $(1 + \ln_+ \frac{1}{\bar{\xi} - \tilde{a}_1})$ is replaced by $(1 + \ln_+ \frac{1}{\tilde{a}_2 - \bar{\xi}})$.

Lemma 6(v)′. *Let* $0 < \bar{x} < \tilde{a}_2$. *Then*

$$\int\limits_{\bar{\xi}<\tilde{a}_2} \left|\Delta G(x-\xi, t-\tau)\right|\left(1+\ln_+ \frac{1}{\tilde{a}_2-\bar{\xi}}\right) d\xi \leqslant \frac{A_1'}{\sqrt{t-\tau}}\left(1+\ln_+ \tilde{a}_2 + \ln_+ \frac{1}{\tilde{a}_2-\bar{x}}\right).$$

Appendix A. Integral inequalities

Lemma A.1. *Let* $0 \leqslant \psi(x) = O(\frac{1}{x})$, $x \geqslant 0$, *and* $\int_0^\infty \psi(x)\, dx < \infty$. *Then*

$$\int\limits_0^a \psi(x) \ln_+ \frac{b}{a-x}\, dx \leqslant A_\psi \left(1+\ln_+ \frac{b}{a}\right).$$

Proof. Let $a < b$. Then

$$\begin{aligned}\int\limits_0^a \psi(x)\ln_+ \frac{b}{a-x}\, dx &= \int\limits_0^{a/2} \psi(x)\ln_+ \frac{b}{a-x}\, dx + \int\limits_{a/2}^a \psi(x)\ln_+ \frac{b}{a-x}\, dx \\ &\leqslant \ln_+ \frac{2b}{a}\int\limits_0^\infty \psi(x)\, dx + \max_{x>a/2} \psi(x)\int\limits_0^a \ln_+ \frac{b}{a-x}\, dx \\ &= A_\psi\left(\frac{1}{2}\ln_+ \frac{2b}{a} + \ln_+ \frac{b}{a} + 1\right) \leqslant A_\psi\left(\ln_+ \frac{b}{a} + 1\right).\end{aligned}$$

Let $a > b$. Then

$$\begin{aligned}\int\limits_0^a \psi(x)\ln_+ \frac{b}{a-x}\, dx &= \int\limits_{a-b}^a \psi(x)\ln_+ \frac{b}{a-x}\, dx \\ &= \int\limits_{a-b}^{a-b/2} \psi(x)\ln_+ \frac{b}{a-x}\, dx + \int\limits_{a-b/2}^a \psi(x)\ln_+ \frac{b}{a-x}\, dx \\ &\leqslant \ln_+ 2\int\limits_0^\infty \psi(x)\, dx + \max_{x>a/2} \psi(x)\int\limits_0^{b/2} \ln_+ \frac{b}{x}\, dx \leqslant A_\psi. \qquad \square\end{aligned}$$

Lemma A.2. *Let* $v(t)$ *be a continuous function satisfying the inequality*:

$$v(t) \leqslant A + \int\limits_\alpha^1 h(\rho) v(\rho t)\, d\rho, \quad t \geqslant t_0,$$

where

$$0 < \int_\alpha^1 h(\rho)\,\mathrm{d}\rho < 1, \quad h \geqslant 0,\ \alpha \in (0,1).$$

Then $\exists m > 0$, $M > 0$ *such that* $v(t) \leqslant A_1 + Mt^{-m}$, $t \geqslant t_0$, *where*

$$A_1 = A\left(1 - \int_\alpha^1 h(\rho)\,\mathrm{d}\rho\right)^{-1}.$$

Proof. Find $A_1 \in \mathbb{R}$ such that $v_1(t) = A_1$ satisfies the equation:

$$v_1(t) = A + \int_\alpha^1 h(\rho) v_1(\rho t)\,\mathrm{d}\rho.$$

We get:

$$A_1 = A\left(1 - \int_\alpha^1 h(\rho)\,\mathrm{d}\rho\right)^{-1}.$$

Let us find $m > 0$ such that $v_0(t) = 1/t^m$ satisfies the equation:

$$v_0(t) = \int_\alpha^1 h(\rho) v_0(\rho t)\,\mathrm{d}\rho.$$

This holds iff $\int_\alpha^1 \frac{h(\rho)}{\rho^m}\,\mathrm{d}\rho = 1$. Since $I(m) = \int_\alpha^1 \frac{h(\rho)}{\rho^m}\,\mathrm{d}\rho$ is a continuous function of m, $I(m) \to +\infty$ as $m \to +\infty$, $I(0) < 1$, then there exists m such that $I(m) = 1$.

Choose M large enough such that

$$V(t) = v(t) - v_1(t) - Mv_0(t) < 0,$$

for $t_0 < t \leqslant t_0/\alpha = t_1$. We claim that $V(t) < 0$ $\forall t \geqslant t_0$.

Indeed, let $t^* = \sup\{t \geqslant t_0\colon\ V(t) < 0\}$. By the choice of M and continuity of V we have $t^* > t_1$.

If t^* is finite, then

$$V(t^*) \leqslant \int_\alpha^1 h(\rho) V(\rho t^*)\,\mathrm{d}\rho < 0.$$

Since V is continuous, $V < 0$ holds in a neighborhood of t^*, but this contradicts the definition of t^*. □

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