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Groups of automorphisms of cyclic trigonal Riemann surfaces ☆

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ABSTRACT

We give the list of all groups G acting as a group of automorphisms of some cyclic trigonal compact Riemann surface X of genus $g \geqslant 5$ and containing the trigonality automorphism group. An abstract group G may act in different ways producing coverings $X \to X/G$ with different ramification type; the list of all such different ramification types is also given.

Introduction

A classical problem in the study of actions of groups on Riemann surfaces is the following: fix a family of compact Riemann surfaces and then find all groups which act as a group of automorphisms of some surface in the given family. This problem was already considered at the end of the XIX century by Riemann, Wiman and Klein, among others. After more than a hundred years, there are few families of Riemann surfaces whose groups of automorphisms have been completely described.

The first surfaces to be considered were those of low genus. It was Wiman [18,19] in the late 1800s who first obtained results on groups of automorphisms of surfaces (or, in that time language, groups of birational transformations of smooth complex algebraic curves) of genus between two and six.

As the genus increases, the situation becomes much more involved. In a series of papers, Kuribayashi et al. [11–13] deal with groups of automorphisms in genus 3, 4 and 5, although their approach is different since they classify representations of groups of automorphisms in the complex vector

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space of holomorphic abelian differentials of the surface. The aid of computer allows Breuer [4] to deal with this classification up to genus 48.

Apart from surfaces of low genus other families whose groups of automorphisms have been described are those characterized by the presence of a particular automorphism. If a compact Riemann surface X admits an involution ϕ such that the quotient $X/\langle\phi\rangle$ has genus p then X is said to be p-hyperelliptic. For $p=0$ we get the classical hyperelliptic Riemann surfaces, whilst the 1-hyperelliptics are known as elliptic–hyperelliptic surfaces. The groups of automorphisms of the hyperelliptic surfaces have been calculated by Brand and Stichtenoth [3], whilst Tyszkowska in [16] and [17] solve the same problem for the family of 1- and 2-hyperelliptic surfaces respectively.

In this paper we deal with the groups of automorphisms of a cyclic trigonal Riemann surface. A Riemann surface X is called cyclic trigonal if it admits an automorphism ϱ of order three such that the quotient surface $X/\langle\varrho\rangle$ is the Riemann sphere. The group $\langle\varrho\rangle$ is called the trigonality automorphism group. This group is unique whenever X has genus $g \geqslant 5$, as shown by Accola in [1]. Our main result is Theorem 2.1, where we give the list of groups acting as a group of automorphisms of a cyclic trigonal Riemann surface of genus $g \geqslant 5$ and containing the trigonality automorphism group. The same abstract group G may act in different ways producing coverings $X \to X/G$ with different ramification type; Theorem 2.1 also contains the list of all such different ramification types.

Trigonal Riemann surfaces (that is, three sheeted coverings of the Riemann sphere) have been studied from different points of view. It is well known that a trigonal Riemann surface admits a defining algebraic equation of the form $y^3 + A(x)y + B(x) = 0$ where A and B are polynomials in x, with $A(x)$ identically zero whenever the surface is cyclic trigonal, see [10]. On the other hand Accola [1] characterizes the cyclic ones in terms of theta-functions and the Jacobian variety of the surface. The possible gap sequences for Weierstrass points of trigonal surfaces are studied by Coppens [6]. Kato and Horiuchi [10] and Accola [2] analyze the existence of trigonal Riemann surfaces with a given type of ramification points. More recently, Costa, Izquierdo and Ying [7] have studied the Riemann surfaces with a non-unique cyclic trigonal morphism.

1. Preliminaries

We will extensively use the combinatorial theory of Fuchsian groups. For the reader's convenience we recall here the basic properties and fix the notations to be used in the paper. By a *Fuchsian group* we mean a discrete cocompact subgroup of the group of isometries of the hyperbolic plane U. The algebraic structure of a Fuchsian group Λ and the geometric structure of the quotient space U/Λ are determined by the *signature* $\sigma(\Lambda)$ of Λ:

$$\sigma(\Lambda) = (\gamma; m_1, \dots, m_r),$$

where $m_i \geqslant 2$ for all i. The integers m_i are called the *periods* of Λ and γ is its *orbit genus*. The compact quotient space U/Λ has the structure of a hyperbolic 2-orbifold of topological genus γ with r conic points of orders $m_1, \dots, m_r$. These are the branching orders of the branch points of the (ramified) covering projection $U \to U/\Lambda$.

A Fuchsian group with this signature has an abstract group presentation in terms of 2γ hyperbolic generators $a_1, b_1, \dots, a_\gamma, b_\gamma$ and r elliptic isometries $x_1, \dots, x_r$ subject to the defining relations

$$x_1^{m_1} = \dots = x_r^{m_r} = x_1 \cdots x_r [a_1, b_1] \cdots [a_\gamma, b_\gamma] = 1,$$

where $[a,b] = aba^{-1}b^{-1}$. These are called *canonical generators*. The ordering in which proper periods are written is irrelevant since any permutation of them yields an isomorphic Fuchsian group, see [14].

By the Uniformization Theorem, a compact Riemann surface X of genus $g \geqslant 2$ can be represented as the quotient U/Γ of the hyperbolic plane U under the action of a *surface Fuchsian group* Γ. This is a torsion free Fuchsian group or, equivalently, a Fuchsian group with signature $(g; -)$. An abstract group G then acts as a group of automorphisms of $X = U/\Gamma$ if and only if G is isomorphic to the quotient Λ/Γ for some Fuchsian group Λ containing Γ as a normal subgroup of index $|G|$, or equivalently, if and only if there exists an epimorphism $\theta: \Lambda \to G$ with $\ker\theta = \Gamma$. An epimorphism, as θ, whose kernel has no non-trivial elements of finite order will be called *smooth*. In this situation, if Λ has signature $(\gamma; m_1, \ldots, m_r)$ then we say that *G acts on X with ramification type* $(\gamma; m_1, \ldots, m_r)$. The relation between the signature of Λ and the branching data of the projection $U \to U/\Lambda$, which are the same as those of $X \to X/G$, justifies this terminology.

A compact Riemann surface X is said to be *trigonal* if it can be realized as a three sheeted covering of the Riemann sphere $\widehat{\mathbb{C}}$. If there exists an automorphism ϱ of X which permutes cyclically the three sheets of the covering then X is said to be *cyclic trigonal*. Using results of Accola in [1] it is not difficult to get the following characterization of cyclic trigonal Riemann surfaces in terms of Fuchsian groups. We provide a short proof for the reader's convenience.

Proposition 1.1. *Let X be a Riemann surface of genus $g \geqslant 2$ represented as U/Γ for some surface Fuchsian group Γ. Then X is cyclic trigonal if and only if there exists a Fuchsian group $\widetilde{\Gamma}$ with signature*

$$\sigma(\widetilde{\Gamma}) = (0; 3, \overset{g+2}{\ldots}, 3)$$

containing Γ as a normal subgroup of index 3.

Proof. If $X = U/\Gamma$ is cyclic trigonal and ϱ permutes the three sheets then $\langle\varrho\rangle = \widetilde{\Gamma}/\Gamma$ for some Fuchsian group $\widetilde{\Gamma}$. Since Γ is torsion-free, all the proper periods of $\widetilde{\Gamma}$ are equal to 3, so that $\widetilde{\Gamma}$ has signature $(\tilde{\gamma}; 3, \overset{r}{\ldots}, 3)$. In fact, $\tilde{\gamma} = 0$ since the orbit space $X/\langle\varrho\rangle = U/\widetilde{\Gamma}$ is the Riemann sphere. Applying now the Hurwitz–Riemann formula to Γ and $\widetilde{\Gamma}$ we obtain that the number of periods of $\widetilde{\Gamma}$ is $g+2$.

Conversely, if there exists a Fuchsian group $\widetilde{\Gamma}$ as above then the quotient group $Z_3 = \widetilde{\Gamma}/\Gamma$ acts on the Riemann surface $X = U/\Gamma$ in such a way that the orbit space has genus 0. Therefore, U/Γ is cyclic trigonal. □

By Lemma 2.1 in [1], if the cyclic trigonal surface X has genus $g \geqslant 5$ then the group $\widetilde{\Gamma}$ is unique. We call $\widetilde{\Gamma}$ *the trigonality Fuchsian group*. The quotient $\widetilde{\Gamma}/\Gamma$ is generated by the automorphism ϱ which permutes cyclically the three sheets of the covering. This group $\langle\varrho\rangle$ is called *the trigonality automorphism group*, and it is normal in the full group $\mathrm{Aut}\,X$ of conformal automorphisms of X.

We will consider surfaces of genus $g \geqslant 5$ since our approach rely upon the uniqueness of $\widetilde{\Gamma}$. However, the approach is also valid for $g = 3$ since also for this value of g the group $\widetilde{\Gamma}$ is unique [7, Theorem 7].

2. Results

All throughout the paper, G will denote a group of automorphisms of a cyclic trigonal compact Riemann surface X of genus $g \geqslant 5$ such that G contains the trigonality automorphism group $\langle\varrho\rangle$. We write $X = U/\Gamma$ for some surface Fuchsian group Γ of orbit genus g and $G = \Lambda/\Gamma$ for some Fuchsian group Λ, say with signature $(\gamma; m_1, \ldots, m_r)$. Our first goal is to determine this signature. Observe that $\gamma = 0$ since $U/\Lambda = X/G$ is the Riemann sphere because G contains ϱ.

Since the trigonality Fuchsian group $\widetilde{\Gamma}$ is unique, it is normal in Λ. Let us write $\widetilde{G} = \Lambda/\widetilde{\Gamma}$. Observe that $\widetilde{G} = G/\langle\varrho\rangle$ and so it acts on $X/\langle\varrho\rangle$; hence, $\widetilde{G}$ is a spherical group, and so it is isomorphic to Z_N, $D_{N/2}$, A_4, S_4 or A_5.

Let p_i be the order of the image of x_i in $\widetilde{G}$, where $x_1, \ldots, x_r$ are the elliptic elements in a canonical set of generators of Λ. Since $\widetilde{\Gamma}$ has signature $(0; 3, \overset{g+2}{\ldots}, 3)$ each integer m_i/p_i equals either 1 or 3. For simplicity we may assume that $m_i = 3p_i$ for $i \leqslant s$ and $m_i = p_i$ for $i > s$, for some $s \in \{1, \ldots, r\}$.

Let N be the order of $\widetilde{G}$. Then Corollary 2 in [15] gives $g+2=\sum_{i=1}^{s} N/p_i$. This equation together with the Hurwitz–Riemann formula for $(\Lambda, \widetilde{\Gamma})$ yield

$$2N-2=N\sum_{i=t+1}^{r}(1-1/p_i),$$

where t stands for the cardinality of the set of indices i for which $p_i=1$. Since $p_i \geqslant 2$ for each $i \geqslant t+1$ the sum on the right-hand side has at most three summands. Now the reader will have no trouble in verifying that the only solutions to the above equation are the following, where the first one corresponds to the trivial case $N=1$:

(A) $t=r$ and $N=1$,
(B) $t=r-2,\ p_{r-1}=p_r=N>1$,
(C) $t=r-3,\ p_{r-2}=p_{r-1}=2,\ p_r=N/2,\ N>2$ and even,
(D) $t=r-3,\ p_{r-2}=2,\ p_{r-1}=p_r=3,\ N=12$,
(E) $t=r-3,\ p_{r-2}=2,\ p_{r-1}=3,\ p_r=4,\ N=24$,
(F) $t=r-3,\ p_{r-2}=2,\ p_{r-1}=3,\ p_r=5,\ N=60$.

Since $p_i=1$ implies $m_i=3$ we get $m_i=3$ for $1 \leqslant i \leqslant t$, whilst $m_i=p_i$ or $3p_i$ for $i>t$. Consequently, the signature of Λ is one of the following:

(A) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3)$,
(B) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3,N\varepsilon_1,N\varepsilon_2)$,
(C) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3,2\varepsilon_1,2\varepsilon_2,(N/2)\varepsilon_3)$,
(D) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3,2\varepsilon_1,3\varepsilon_2,3\varepsilon_3)$, $N=12$,
(E) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3,2\varepsilon_1,3\varepsilon_2,4\varepsilon_3)$, $N=24$,
(F) $\sigma(\Lambda)=(0;3,\overset{t}{\ldots},3,2\varepsilon_1,3\varepsilon_2,5\varepsilon_3)$, $N=60$,

where $\varepsilon_i=1$ or 3 for $i=1,2,3$. To lighten notation, in the sequel we will omit the orbit genus $\gamma=0$ in the above signatures.

Observe that the spherical group $\widetilde{G}$ is trivial in case (A) and isomorphic to Z_N, $D_{N/2}$, A_4, S_4, A_5 in cases (B)–(F) respectively. As groups with signature, these are (N,N), $(2,2,N/2)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ respectively. In the language of maps, these correspond to the regular maps on the sphere, and the cyclic trigonal surfaces correspond to 3-sheeted covers of these regular maps. The parameter t in the above signatures is the number of branch points lying outside the fixed points of the automorphism groups of the regular maps, while ε_1, ε_2 and ε_3 are the number of branch points lying above the vertices, edge-centers or face-centers. We will not follow this approach here; instead, we will use the combinatorial theory of Fuchsian groups.

The rest of the paper is devoted to analyze separately each of the above signatures. We proceed as follows. We first fix a signature σ and assume the existence of a smooth epimorphism $\theta:\Lambda\to G$ from a Fuchsian group Λ with signature σ onto a finite group G such that its kernel uniformizes a cyclic trigonal Riemann surface and G contains the trigonality automorphism. This will yield a list of candidates to be a group of automorphisms acting with fixed ramification type σ on a cyclic trigonal Riemann surface. For each such group we obtain a presentation and we then have to find under which conditions such a presentation provides a group of the appropriate order, namely, order $3N$ where N is the order of $\widetilde{G}$. Moreover, we will also have to check whether such conditions yield a group action with ramification type σ on a cyclic trigonal Riemann surface since it may happen that such conditions do not allow to define properly the corresponding smooth epimorphism.

We will write $N\equiv k\ (m)$ to denote that m divides $N-k$.

(A) In this case G is simply Z_3, the group generated by the trigonality automorphism. The genus of the cyclic trigonal Riemann surfaces on which it acts with signature (A) equals $g=t-2$ by Proposition 1.1.

(B) Here Λ has one of the following signatures:

$$(\mathrm{B1}) = (3, \overset{t}{\ldots}, 3, N, N), \qquad (\mathrm{B2}) = (3, \overset{t}{\ldots}, 3, N, 3N), \qquad (\mathrm{B3}) = (3, \overset{t}{\ldots}, 3, 3N, 3N),$$

where $t > 0$ and $N > 1$.

Let us consider first the case (B1). Let $\theta : \Lambda \to G$ be a smooth epimorphism from a Fuchsian group Λ with signature (B1) onto a finite group G such that its kernel uniformizes a cyclic trigonal Riemann surface and G contains the trigonality automorphism. Let m_i be a proper period in $\sigma(\Lambda)$ and let x_i be the corresponding canonical elliptic generator of Λ. With the notations at the beginning of this section we know that $\theta(x_i)^{p_i}$ belongs to the trigonality automorphism group $\langle \varrho \rangle$. Moreover, if $m_i = 3p_i$ then $\theta(x_i)^{p_i}$ equals ϱ or ϱ^2. In particular, for $i = 1, \ldots, t$ we get $\theta(x_i) = \varrho$ or ϱ^2. On the other hand, $\theta(x_{t+1}) = a$ and $\theta(x_{t+2}) = b$ are two elements of order N in G. Clearly a, b, ϱ generate G and the relation $x_1 \cdots x_{t+2} = 1$ gives $ab = \varrho^\alpha$ for $\alpha = 0, 1$ or 2. So actually G is generated by a and ϱ. Since $\langle \varrho \rangle$ is normal in G, we get that $a\varrho a^{-1}$ equals either ϱ or ϱ^{-1} and therefore $G = \mathrm{Z}_N \oplus \mathrm{Z}_3$ or G is the semidirect product

$$\langle a, \varrho \mid a^N, \varrho^3, a\varrho a^{-1}\varrho \rangle,$$

where in the last group N is even for it to have order $3N$.

Now we shall study whether the above groups actually act with ramification type $(3, \overset{t}{\ldots}, 3, N, N)$ on a cyclic trigonal Riemann surface. For that we have to define the appropriate smooth epimorphism $\theta : \Lambda \to G$ from a Fuchsian group Λ with this signature.

Let us write $G = \mathrm{Z}_N \oplus \mathrm{Z}_3 = \langle a \rangle \oplus \langle \varrho \rangle$. If $t > 1$ then we can choose $\theta(x_1), \ldots, \theta(x_t) \in \{\varrho, \varrho^2\}$ in such a way that the product $\theta(x_1) \cdots \theta(x_t)$ equals 1 and so choosing $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = a^{-1}$ we obtain a smooth epimorphism $\theta : \Lambda \to G$ defining the action of G on the Riemann surface $U/\ker\theta$. This surface is cyclic trigonal. Indeed, the Fuchsian group $\theta^{-1}(\langle \varrho \rangle)$, which contains $\ker\theta$ as a normal subgroup of index 3, has signature $(0; 3, \overset{g+2}{\ldots}, 3)$; this is easy to see using [15, Corollary 2] to compute the number of proper periods of $\theta^{-1}(\langle \varrho \rangle)$ and the Hurwitz–Riemann formula to compute its orbit genus. Therefore, $U/\ker\theta$ is a cyclic trigonal surface by Proposition 1.1. In the sequel, whenever a smooth epimorphism θ is defined, the proof that the surface $U/\ker\theta$ is cyclic trigonal is the same as the above, and will be omitted.

If $t = 1$ then N has to be divisible by 3. Otherwise $G = \mathrm{Z}_N \oplus \mathrm{Z}_3$ would be cyclic and, for such values of N, there exists no smooth epimorphism from a Fuchsian with signature $(3, N, N)$ onto Z_{3N}; this also follows from Harvey's conditions in [9, Theorem 4]. So $N \equiv 0\ (3)$ and hence the element $a\varrho$ has order N. Choosing $\theta(x_1) = \varrho$, $\theta(x_2) = (a\varrho)^{-1}$ and $\theta(x_3) = a$ yields a smooth epimorphism defining the action of G on a cyclic trigonal Riemann surface.

Assume now that G has presentation $\langle a, \varrho \mid a^N, \varrho^3, a\varrho a^{-1}\varrho \rangle$. The elements $a\varrho^r$ have order N for all r. So the assignment $\theta(x_1) = \cdots = \theta(x_t) = \varrho$, $\theta(x_{t+1}) = a^{-1}$ and $\theta(x_{t+2}) = a\varrho^{-t}$ induces an epimorphism $\theta : \Lambda \to G$ defining an action of G on a cyclic trigonal Riemann surface.

So in case $(\mathrm{B1}) = (3, \overset{t}{\ldots}, 3, N, N)$ we have obtained the following groups of automorphisms of cyclic trigonal Riemann surfaces of genus g, where the value of g can be found using the Hurwitz–Riemann formula:

Case	Presentation	Conditions	g
B1.1	$\mathrm{Z}_N \oplus \mathrm{Z}_3$	$t > 1$ or $t = 1$, $N \equiv 0\ (3)$	$Nt - 2$
B1.2	$\langle a, \varrho \mid a^N, \varrho^3, a\varrho a^{-1}\varrho \rangle$	$N \equiv 0\ (2)$	$Nt - 2$

In cases $(\mathrm{B2}) = (3, \overset{t}{\ldots}, 3, N, 3N)$ and $(\mathrm{B3}) = (3, \overset{t}{\ldots}, 3, 3N, 3N)$ the element $\theta(x_{t+2})$ has order $3N$. So G is a cyclic group. The relation $x_1 \cdots x_{t+2} = 1$ yields that G is generated by $\theta(x_1), \ldots, \theta(x_{t+1})$. In case of signature (B2) this forces $N \not\equiv 0\ (3)$, whilst no extra condition is required in case (B3). This also follows from [9, Theorem 4].

Let us show that Z_{3N} does act on cyclic trigonal Riemann surfaces with these ramification types. We write $Z_{3N} = \langle a\rangle$ and denote $\varrho = a^N$. In case of signature (B2), we define $\theta(x_{t+1}) = a^{3n}$, $\theta(x_{t+2}) = a$ if $N = 3n + 1$, and $\theta(x_{t+1}) = a^{3n+3}$, $\theta(x_{t+2}) = a^{-1}$ if $N = 3n + 2$; in both cases we choose $\theta(x_1), \dots, \theta(x_t) \in \{\varrho, \varrho^2\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = \varrho^2 = a^{2N}$. It is then easy to check that in both cases $\theta : \Lambda \to Z_{3N}$ is a well-defined smooth epimorphism and that it yields an action of Z_{3N} on a cyclic trigonal Riemann surface.

In case of signature (B3), if $t > 1$ then we choose $\theta(x_1), \dots, \theta(x_t) \in \{\varrho, \varrho^2\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = 1$ and define $\theta(x_{t+1}) = a^{-1}$, $\theta(x_{t+2}) = a$. If $t = 1$ then we define $\theta(x_1) = \varrho = a^N$, $\theta(x_2) = a^{2N-1}$, $\theta(x_3) = a$ if $N \equiv 1$ (3), and $\theta(x_1) = \varrho^2 = a^{2N}$, $\theta(x_2) = a^{N-1}$, $\theta(x_3) = a$ if $N \not\equiv 1$ (3). In both cases these definitions yield an action of Z_{3N} on a cyclic trigonal Riemann surface.

Summarizing, the unique group acting in cases (B2) and (B3) is Z_{3N}, and it acts under the following conditions:

Case	Ramification type	Presentation	Conditions	g
B2	$(3, \overset{t}{\dots}, 3, N, 3N)$	Z_{3N}	$N \not\equiv 0$ (3)	$Nt - 1$
B3	$(3, \overset{t}{\dots}, 3, 3N, 3N)$	Z_{3N}	None	Nt

(C) Here Λ has one of the following signatures:

$$(C1) = (3, \overset{t}{\dots}, 3, 2, 2, N/2), \qquad (C2) = (3, \overset{t}{\dots}, 3, 2, 2, 3N/2),$$
$$(C3) = (3, \overset{t}{\dots}, 3, 2, 6, N/2), \qquad (C4) = (3, \overset{t}{\dots}, 3, 2, 6, 3N/2),$$
$$(C5) = (3, \overset{t}{\dots}, 3, 6, 6, N/2), \qquad (C6) = (3, \overset{t}{\dots}, 3, 6, 6, 3N/2),$$

with $t > 0$ in cases (C1) and (C2), and $N > 2$ and even in all of them. We consider first the case in which Λ has signature (C1). Let $\theta : \Lambda \to G$ be a smooth epimorphism describing the action of a finite group G on a cyclic trigonal Riemann surface. Then $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $i = 1, \dots, t$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = c$ are three elements of orders 2, 2 and $N/2$ respectively. The relation $x_1 \cdots x_{t+3} = 1$ gives $abc = \varrho^\alpha$ for some α and so in particular G is generated by a, b and ϱ. If $\varrho \in \langle a, b\rangle$, then $G = D_{3N/2}$. If $\varrho \notin \langle a, b\rangle$ then $\langle a, b\rangle = D_{N/2}$ and G is a semidirect product $Z_3 \rtimes D_{N/2}$. So $G = D_{N/2} \oplus Z_3$ or G is one of the groups

$$\text{(a)} \quad \langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho\rangle$$

or

$$\text{(b)} \quad \langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho^{-1}\rangle,$$

where in the last group $N \equiv 0$ (4) for it to have order $3N$.

Let us study now whether the above groups act or not with ramification type (C1) on a cyclic trigonal Riemann surface. Let Λ be a Fuchsian group with signature (C1).

First let $G = D_{3N/2}$. The conditions given in [5, Theorem 2.3] for the existence of a smooth epimorphism from a Fuchsian group onto a dihedral group yield $N \not\equiv 0$ (6). We show that for such values of N the dihedral group $D_{3N/2}$ does act on a cyclic trigonal Riemann surface. Let us write $D_{3N/2} = \langle a, b \mid a^2, b^2, (ab)^{3N/2}\rangle$ and denote $\varrho = (ab)^{N/2}$, which generates a normal subgroup of order 3. Let us define $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and let us choose $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $1 \leqslant i \leqslant t$ in such a way that $\theta(x_1)\cdots\theta(x_t)$ equals either $(ab)^{N/2}$ if $N \equiv 4$ (6), or $(ab)^N$ if $N \equiv 2$ (6). It is straightforward to check that in both cases $\theta(x_{t+3})$ is an element of order $N/2$ and that this definition of θ yields an action of $D_{3N/2}$ on a cyclic trigonal Riemann surface.

Now for $G = D_{N/2} \oplus Z_3 = \langle a, b \mid a^2, b^2, (ab)^{N/2}\rangle \oplus \langle\varrho\rangle$ an epimorphism $\theta : \Lambda \to G$ defining a suitable action exists if $t > 1$. Indeed, $\theta(x_1), \dots, \theta(x_t)$ can be chosen in $\{\varrho, \varrho^2\}$ such that $\theta(x_1)\cdots\theta(x_t) = 1$ and then we can define $\theta(x_{t+1}) = a$ and $\theta(x_{t+2}) = b$. The same epimorphism works for the groups with presentation (a) or (b).

So let $t = 1$. If $G = \mathrm{D}_{N/2} \oplus \mathrm{Z}_3$ then the involutions $\theta(x_2)$ and $\theta(x_3)$ generate the factor $\mathrm{D}_{N/2}$ whilst $\theta(x_1)$ generates the cyclic one. So, for the product $\theta(x_1)\theta(x_2)\theta(x_3)$ to have order $N/2$ it has to be $\theta(x_3)^{N/2} = 1$, that is, $N \equiv 0$ (6). In this case, the assignment $\theta(x_1) = \varrho$, $\theta(x_2) = a$ and $\theta(x_3) = b$ is a well-defined smooth epimorphism. Now for a group with presentation (a) or (b) an appropriate smooth epimorphism is given by $\theta(x_1) = \varrho$, $\theta(x_2) = \varrho^{-1} b \varrho$ and $\theta(x_3) = \varrho^{-1} a$.

Summing up, in case $(\mathrm{C1}) = (3, \overset{t}{\ldots}, 3, 2, 2, N/2)$ with $t > 0$, we have obtained the following groups of automorphisms:

Case	Presentation	Conditions	g
C1.1	$\mathrm{D}_{3N/2}$	$t > 0,\ N \not\equiv 0$ (6)	$Nt - 2$
C1.2	$\mathrm{D}_{N/2} \oplus \mathrm{Z}_3$	$t > 1$ or $N \equiv 0$ (6)	$Nt - 2$
C1.3	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho \rangle$	$t > 0$	$Nt - 2$
C1.4	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho^{-1} \rangle$	$t > 0,\ N \equiv 0$ (4)	$Nt - 2$

The case $(\mathrm{C2}) = (3, \overset{t}{\ldots}, 3, 2, 2, 3N/2)$ where $t > 0$, gives rise to groups with the same presentation as those in (C1) since the same reasonings work here. So let us study whether these groups act with this signature.

First let $G = \mathrm{D}_{3N/2} = \langle a, b \mid a^2, b^2, (ab)^{3N/2} \rangle$ and observe that either $(ab)^{N/2+1}$ or $(ab)^{N+1}$ has order $3N/2$. So we define $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and choose $\theta(x_1), \ldots, \theta(x_t) \in \{(ab)^{N/2}, (ab)^N\}$ in such a way that $\theta(x_1) \cdots \theta(x_{t+2})$ equals either $(ab)^{N/2+1}$ if this element has order $3N/2$ or $(ab)^{N+1}$ otherwise.

Now, for an epimorphism $\theta : \Lambda \to G$ defining a suitable action for all remaining groups to exist, the group G must have an element of order $3N/2$. It is easy to check that in cases $G = \mathrm{D}_{N/2} \oplus \mathrm{Z}_3$ or G is the group with presentation (a) this forces $N \not\equiv 0$ (6), whilst in case G is the group with presentation (b) it has to be $N = 4$. In this last case G is the dihedral group D_6, a group which has already been considered. For the other two cases with $N \not\equiv 0$ (6), we define $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and choose $\theta(x_1), \ldots, \theta(x_t) \in \{\varrho, \varrho^2\}$ such that $\theta(x_1) \cdots \theta(x_t) = \varrho$.

Summing up, in case (C2) we have obtained the following groups:

Case	Presentation	Conditions	g
C2.1	$\mathrm{D}_{3N/2}$	$t > 0$	Nt
C2.2	$\mathrm{D}_{N/2} \oplus \mathrm{Z}_3$	$t > 0, N \not\equiv 0$ (6)	Nt
C2.3	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho \rangle$	$t > 0, N \not\equiv 0$ (6)	Nt

Let now Λ have signature $(\mathrm{C3}) = (3, \overset{t}{\ldots}, 3, 2, 6, N/2)$. Then $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $i = 1, \ldots, t$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = c$ are three elements of orders 2, 6 and $N/2$ respectively. The difference with cases (C1) and (C2) is that now $\theta(x_{t+2})^2$ equals ϱ or ϱ^2 (since, with the notations at the beginning of the section, $p_{t+2} = 2$). The relation $x_1 \cdots x_{t+3} = 1$ gives $abc = \varrho^\alpha$ for some α and so in particular

$$1 = c^{-N/2} = \left(\varrho^{-\alpha} ab\right)^{N/2} = (ab)^{N/2} \varrho^\beta$$

for some β. Now $\langle b^2 \rangle = \langle \varrho \rangle$ which is normal and so in particular $ab^2a = b^2$ or $ab^2a = b^4$. Therefore G is a factor group of one of the following groups G':

(a) $\langle a, b \mid a^2, b^6, ab^2ab^4, (ab)^{N/2}b^4 \rangle$, (b) $\langle a, b \mid a^2, b^6, ab^2ab^4, (ab)^{N/2}b^2 \rangle$,

(c) $\langle a, b \mid a^2, b^6, ab^2ab^4, (ab)^{N/2} \rangle$, (d) $\langle a, b \mid a^2, b^6, ab^2ab^2, (ab)^{N/2}b^4 \rangle$,

(e) $\langle a, b \mid a^2, b^6, ab^2ab^2, (ab)^{N/2}b^2 \rangle$, (f) $\langle a, b \mid a^2, b^6, ab^2ab^2, (ab)^{N/2} \rangle$.

Observe that the order of such an abstract group G' is at most $3N$ since the element b^2 generates a normal subgroup such that $G'/\langle b^2\rangle$ is a dihedral group of order at most N. Now we shall find which of these groups have actually order $3N$, in which case G' will be the group G we are looking for.

Observe that in cases (a), (b), (d) or (e), the relation $(ab)^{N/2} = b^2$ or b^4 implies that $\langle b^2\rangle$ is central in G. This, which makes the relation $ab^2a = b^2$ redundant in (a) and (b), allows us to discard (d) and (e). Indeed, in these groups it is $ab^2a = b^4$ and so $b^4 = b^2$, which would yield a group of order N.

Consider first the group (a). Here

$$(ba)^{N/2} = a(ab)^{N/2}a = ab^2a = b^2 = (ab)^{N/2}$$

and so in particular $(ab)^{N/2+1} = (ba)^{N/2-1}b^2 = b^2(ba)^{N/2-1}$. Thus inductively $(ab)^N = b^N$. But since $(ab)^N = b^4$ we obtain $N \equiv 4$ (6). Similarly one can show that $N \equiv 2$ (6) in case (b). In case (c) we have $(ba)^{N/2} = 1$. So in particular $(ab)^{N/2} = (ba)^{N/2}$ which as before gives $1 = (ab)^N = b^N$ and so $N \equiv 0$ (6).

Finally consider the group (f). Here we shall show that $N/2$ must be even. Indeed assume to get a contradiction that $N/2$ is odd. Here $(ab)^{N/2} = 1 = (ba)^{N/2}$ and so $(ab)^{N/2+1} = (ba)^{N/2-1}b^2 = b^2(ba)^{N/2-1}$. Now $(ab)^{N/2+2} = b^2(ba)^{N/2-2}b^2 = b^2b^4(ba)^{N/2-2} = (ba)^{N/2-2}$. So inductively $(ab)^{N-1} = ba$ which gives $1 = (ab)^N = b^2$ and so the group (f) has order N for $N/2$ odd.

It remains to show that the groups (a), (b), (c) and (f) for $N \equiv 4$ (6), $N \equiv 2$ (6), $N \equiv 0$ (6) and $N \equiv 0$ (4) respectively do have order $3N$.

Let Δ be a Fuchsian group with signature $(0; 2, 6, N/2)$ and let x and y be canonical generators of orders 2 and 6 respectively. Then the normal closure of y^2 in Δ is a subgroup Δ' such that Δ/Δ' is a dihedral group of order N. Moreover, Δ' is a Fuchsian group with signature $(0; 3, \overset{N/2}{\ldots}, 3)$. Now let

$$A_i = (xy)^i y^2 (xy)^{-i}, \quad \text{for } i = 0, \ldots, N/2 - 1.$$

We shall see how Δ acts on these elements. The reader can check easily that

$$xA_0x = A_1, \qquad xA_1x = A_0 \quad \text{and} \quad xA_ix = v_iA_{N/2+1-i}v_i^{-1},$$

where $v_i = A_0A_{N/2-1}\cdots A_{N/2+2-i}$ for $i = 2, \ldots, N/2 - 1$ and

$$yA_0y^{-1} = A_0, \qquad yA_1y^{-1} = A_0A_{N/2-1}A_0^{-1} \quad \text{and} \quad yA_iy^{-1} = v_{i+1}A_{N/2-i}v_{i+1}^{-1}$$

for $i = 1, \ldots, N/2 - 1$.

So $\langle A_0, \ldots, A_{N/2-1}\rangle$ is a normal subgroup of Δ with factor group $D_{N/2}$ and therefore, since each A_i has order 3 and $A_0A_{N/2-1}A_{N/2-2}\cdots A_1 = 1$, we have that $A_0, \ldots, A_{N/2-1}$ form a system of canonical generators for Δ'.

Now it is clear that for $N \equiv 0$ (6) the normal closure Δ'' of

$$\{A_i^{-1}A_{i+1} \mid i = 0, \ldots, N/2 - 2\}$$

in Δ' is a subgroup such that $\Delta'/\Delta'' \cong Z_3$ and is invariant with respect to the action of x and y. So Δ'' is a normal subgroup of Δ and Δ/Δ'' has order $3N$. Finally observe that $A_0^{-1}A_1 = y^4xy^2x$ and so the group Δ/Δ'' is a factor group of (c). Thus the last has order $3N$.

For $N \equiv 0$ (4) the normal closure Δ'' of

$$\{A_iA_{i+1} \mid i = 0, \ldots, N/2 - 2\}$$

in Δ' is a subgroup such that $\Delta'/\Delta'' \cong Z_3$ and is invariant with respect to the action of x and y. So Δ'' is a normal subgroup of Δ and Δ/Δ'' has order $3N$. Finally observe that $A_0A_1 = y^2xy^2x$ and so the group Δ/Δ'' is a factor group of (f). Thus the last has order $3N$.

Now let Δ be a Fuchsian group with signature $(0; 2, 6, 3N/2)$ and let x and y be canonical generators of orders 2 and 6. Then the normal closure of $\{y^2, (xy)^{N/2}\}$ in Δ is a subgroup Δ' such that Δ/Δ' is a dihedral group of order N. Moreover, Δ' is a Fuchsian group with signature $(0; 3, \overset{2+N/2}{\ldots}, 3)$. Now let

$$B_1 = (xy)^{-N/2}, \qquad B_2 = (yx)^{-N/2} \quad \text{and} \quad A_i = (xy)^i y^2 (xy)^{-i} \quad \text{for } i = 0, \ldots, N/2 - 1.$$

We shall see how Δ acts on these elements. The reader can check that

$$xB_1x = B_2, \qquad xB_2x = B_1, \qquad xA_0x = A_1, \qquad xA_1x = A_0, \qquad xA_ix = w_iA_{N/2+1-i}w_i^{-1}$$

for $i = 2, \ldots, N/2 - 1$, where $w_i = (A_0B_1A_{N/2-1} \cdots A_1)^{-2}(A_{N/2-i} \cdots A_1)^{-1}$. Similarly

$$yB_1y^{-1} = B_2, \qquad yB_2y^{-1} = A_0B_1A_0^{-1}$$

and

$$yA_0y^{-1} = A_0, \qquad yA_{N/2-1}y^{-1} = B_2^{-1}A_1B_2, \qquad yA_iy^{-1} = w_{i+1}A_{N/2-i}w_{i+1}^{-1}$$

for $i = 1, \ldots, N/2 - 2$. So $\langle B_1, B_2, A_0, \ldots, A_{N/2-1}\rangle$ is a normal subgroup of Δ with factor group $\mathrm{D}_{N/2}$. Now B_1, B_2 and each A_i have order 3. Furthermore, $B_2A_0B_1A_{N/2-1} \cdots A_1 = 1$. Thus $B_1, B_2, A_0, \ldots, A_{N/2-1}$ form a system of canonical generators for Δ'.

Now it is clear that for $N \equiv 4$ (6) the normal closure Δ'' of

$$\left\{B_1B_2^{-1}, A_iB_1 \;\middle|\; i = 0, \ldots, N/2 - 1\right\}$$

in Δ' is a subgroup such that $\Delta'/\Delta'' \cong \mathrm{Z}_3$ and is invariant with respect to the action of x and y. So Δ'' is a normal subgroup of Δ and Δ/Δ'' has order $3N$. Observe also that $A_0B_1 = y^2(xy)^{-N/2} = ((xy)^{N/2}y^4)^{-1}$ and so the group Δ/Δ'' is a factor group of (a). Thus the last has order $3N$.

For $N \equiv 2$ (6) the normal closure Δ'' of

$$\left\{B_1B_2^{-1}, A_iB_1^{-1} \;\middle|\; i = 0, \ldots, N/2 - 1\right\}$$

in Δ' is a subgroup such that $\Delta'/\Delta'' \cong \mathrm{Z}_3$ and is invariant with respect to the action of x and y. So Δ'' is a normal subgroup of Δ and Δ/Δ'' has order $3N$. Finally observe that $A_0B_1^{-1} = y^2(xy)^{N/2}$ and so the group Δ/Δ'' is a factor group of (b). Thus the last has order $3N$.

Now we shall study whether each of the above groups acts or not with ramification type $(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$ on a cyclic trigonal Riemann surface. Let Λ be a Fuchsian group with this signature.

Let us consider first a group G with presentation (a). For $t > 0$ we can choose $\theta(x_1), \ldots, \theta(x_t) \in \{b^2, b^4\}$ in such a way that $\theta(x_1) \cdots \theta(x_t) = b^2$ and so choosing $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^3a$ we obtain a suitable epimorphism $\theta : \Lambda \to G$ since b^3a is an element of order $N/2$ in G. However, we claim that for $t = 0$ such an epimorphism does not exist. Indeed, it is not hard to see that the elements of order $N/2$ in G are those of the form $(ab)^{3m}$ with $\gcd(N/2, m) = 1$. So $\theta(x_3)$ has this form. Now $\theta(x_1)$ is an element of order two which together with $(ab)^{3m}$ has to generate the whole G. It is easy to see that then $\theta(x_1) = a(ab)^{3r}$ for some r. But in such a case, $\theta(x_2) = \theta(x_1)\theta(x_3)^{-1} = a(ab)^{3(r-m)}$ which does not have the required order 6. Therefore, G does not act with signature $(2, 6, N/2)$, as claimed.

For a group G with presentation (b), it also happens that G does not contain two generators of orders 2 and $N/2$ such that their product has order 6. So again an epimorphism $\theta : \Lambda \to G$ defining the suitable action does not exist if $t = 0$. However, for $t > 0$ we can choose the same epimorphism as above.

Finally, a group G with presentation either (c) or (f) does act with signature $(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$ for any value of t. Indeed, if $t = 0$ then $\theta(x_1) = a$, $\theta(x_2) = b$, $\theta(x_3) = (ab)^{-1}$ yields a suitable group action since ab has order $N/2$ in both cases. If $t > 0$ then we can choose $\theta(x_1), \ldots, \theta(x_t) \in \{b^2, b^4\}$ in such a way that $\theta(x_1) \cdots \theta(x_t) = b^2$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^3 a$ in case (c) or $\theta(x_{t+3}) = ba$ in case (f).

Summing up, in case $(C3) = (3, \overset{t}{\ldots}, 3, 2, 6, N/2)$ we have obtained the following groups:

Case	Presentation	Conditions	g
C3.1	$\langle a, b \mid a^2, b^6, (ab)^{N/2} b^4 \rangle$	$N \equiv 4\ (6)$, $t > 0$	$N(2t+1)/2 - 2$
C3.2	$\langle a, b \mid a^2, b^6, (ab)^{N/2} b^2 \rangle$	$N \equiv 2\ (6)$, $t > 0$	$N(2t+1)/2 - 2$
C3.3	$\langle a, b \mid a^2, b^6, ab^2ab^4, (ab)^{N/2} \rangle$	$N \equiv 0\ (6)$	$N(2t+1)/2 - 2$
C3.4	$\langle a, b \mid a^2, b^6, ab^2ab^2, (ab)^{N/2} \rangle$	$N \equiv 0\ (4)$	$N(2t+1)/2 - 2$

The case $(C4) = (3, \overset{t}{\ldots}, 3, 2, 6, 3N/2)$ gives rise to groups with the same presentation as those in (C3) since the same reasonings work here. So let us study whether these groups act with this signature.

Let us consider first the case (a). Here ab is an element of order $3N/2$ and so an epimorphism defining the suitable action exists if $t \neq 1$. For $t = 1$ we can choose $\theta(x_1) = b^4$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(x_4) = ba$. In a similar way we can show that a suitable epimorphism exists in case (b) for an arbitrary value of t.

However, there exists no element of order $3N/2$ in case (c) and case (f) with $N \neq 4$, as is easy to prove. So such an epimorphism never exists for these values of N. In case (f) with $N = 4$ the group is dihedral and there is no smooth epimorphism from a Fuchsian group with signature $(3, \overset{t}{\ldots}, 3, 2, 6, 6)$ onto a dihedral group [5, Theorem 2.3].

Summing up, in case (C4) we have obtained the following groups:

Case	Presentation	Conditions	g
C4.1	$\langle a, b \mid a^2, b^6, (ab)^{N/2} b^4 \rangle$	$N \equiv 4\ (6)$	$N(2t+1)/2$
C4.2	$\langle a, b \mid a^2, b^6, (ab)^{N/2} b^2 \rangle$	$N \equiv 2\ (6)$	$N(2t+1)/2$

Let now Λ have signature $(C5) = (3, \overset{t}{\ldots}, 3, 6, 6, N/2)$. Then $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $i = 1, \ldots, t$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = c$ are three elements of orders 6, 6 and $N/2$ respectively such that $a^2, b^2 \in \{\varrho, \varrho^2\}$. In particular, either $a^2 = b^2$ or $a^2 = b^4$. The relation $x_1 \cdots x_{t+3} = 1$ gives $abc = \varrho^\alpha$ for some α and so

$$1 = c^{-N/2} = \left(\varrho^{-\alpha} ab\right)^{N/2} = (ab)^{N/2} \varrho^\beta$$

for some β. Hence $(ab)^{N/2} \in \{1, a^2, a^4\}$. Therefore G is a factor group of one of the following groups:

(a) $\langle a, b \mid a^6, b^6, a^2b^4, (ab)^{N/2} \rangle$, (b) $\langle a, b \mid a^6, b^6, a^2b^2, (ab)^{N/2} \rangle$,
(c) $\langle a, b \mid a^6, b^6, a^2b^4, (ab)^{N/2} a^2 \rangle$, (d) $\langle a, b \mid a^6, b^6, a^2b^2, (ab)^{N/2} a^2 \rangle$,
(e) $\langle a, b \mid a^6, b^6, a^2b^4, (ab)^{N/2} a^4 \rangle$, (f) $\langle a, b \mid a^6, b^6, a^2b^2, (ab)^{N/2} a^4 \rangle$.

Observe that the order of any of these abstract groups is at most $3N$ since the element a^2 generates a normal subgroup such that the corresponding factor group is dihedral of order at most N. Now we shall find which of these groups have actually order $3N$, in which case it will be the group G we are looking for.

Consider first the group (a). Here $1 = (ab)^{N/2} = (a^2 a^{-1} b^{-1} b^2)^{N/2} = a^N b^N (a^{-1} b^{-1})^{N/2} = a^N b^N = a^{2N}$ and thus 3 divides N. So $N \equiv 0\ (6)$ since N is even.

In case (c) we have $a^4 = (a^2 a^{-1} b^{-1} b^2)^{N/2} = a^N b^N (a^{-1} b^{-1})^{N/2} = a^{2N} a^2$ and so $a^{2N-2} = 1$ which gives that 3 divides $N - 1$. So $N \equiv 4\ (6)$ since N is even.

In case (d) we have $a^4 = (a^2a^{-1}b^{-1}b^2)^{N/2} = a^Nb^N(a^{-1}b^{-1})^{N/2} = a^{3N}a^2$. Thus 6 divides $3N-2$ which is impossible. So (d) cannot stand as a group of automorphisms of a cyclic trigonal Riemann surface. Similarly we can rule out the group (f).

In case (e) we have $a^2 = (a^2a^{-1}b^{-1}b^2)^{N/2} = a^Nb^N(a^{-1}b^{-1})^{N/2} = a^{2N}a^4$ and so $a^{2N+2}=1$ which gives that 3 divides $N+1$. So $N \equiv 2$ (6) since N is even.

Finally, let $x=a$, $y=b^{-1}$ in case (b). Then $1=a^2b^2=x^2y^4$ and

$$1=(ab)^{N/2}=\left(xy^5\right)^{N/2}=(xy)^{N/2}y^{2N}=(xy)^{N/2}x^{2N}=\begin{cases}(xy)^{N/2} & \text{if } N\equiv 0\ (6),\\ (xy)^{N/2}x^4 & \text{if } N\equiv 2\ (6),\\ (xy)^{N/2}x^2 & \text{if } N\equiv 4\ (6).\end{cases}$$

So for $N\equiv 0$, 2 or 4 (6), the group (b) is isomorphic to the group (a), (e) or (c) respectively. Therefore, we just have to deal with the group (b) and show that it has the appropriate order $3N$ for any even value of N. This can be done in a similar way as in case (C3) and we omit its proof.

We now show that the group (b) acts with ramification type $(3,\overset{t}{\ldots},3,6,6,N/2)$ on a cyclic trigonal Riemann surface. Indeed for $t\neq 1$ we choose $\theta(x_1),\ldots,\theta(x_t)\in\{b^2,b^4\}$ in such a way that $\theta(x_1)\cdots\theta(x_t)=1$ and define $\theta(x_{t+1})=a$, $\theta(x_{t+2})=b$, $\theta(x_{t+3})=(ab)^{-1}$. For $t=1$ we can choose $\theta(x_1)=a^2$, $\theta(x_2)=a^{-1}$, $\theta(x_3)=b$ and $\theta(x_4)=(ab)^{-1}$.

Summing up, in case $(C5)=(3,\overset{t}{\ldots},3,6,6,N/2)$ we have obtained the following group:

Case	Presentation	Conditions	g
C5.1	$\langle a,b \mid a^6,b^6,a^2b^2,(ab)^{N/2}\rangle$	$N\equiv 0$ (2)	$N(t+1)-2$

Finally, the case $(C6)=(3,\overset{t}{\ldots},3,6,6,3N/2)$ gives rise to the same group as (C5). If $N\equiv 0$ (6) then this group has no element of order $3N/2$ and so it does not act with the above signature. If $N\not\equiv 0$ (6) then ab^3 has order $3N/2$ and a suitable epimorphism can be defined. Indeed, if $t>0$ then we choose $\theta(x_1),\ldots,\theta(x_t)\in\{b^2,b^4\}$ such that $\theta(x_1)\cdots\theta(x_t)=b^2$, and we define $\theta(x_{t+1})=a$, $\theta(x_{t+2})=b$ and $\theta(x_{t+3})=(ab^3)^{-1}$. If $t=0$ then we define $\theta(x_1)=b$, $\theta(x_2)=bab$ and $\theta(x_3)=(ab^3)^{-1}$.

Summing up, in case (C6) we have obtained the following group:

Case	Presentation	Conditions	g
C6.1	$\langle a,b \mid a^6,b^6,a^2b^2,(ab)^{N/2}\rangle$	$N\equiv 2$ or 4 (6)	$N(t+1)$

(D) Here Λ has one of the following signatures:

(D1) $(3,\overset{t}{\ldots},3,2,3,3)$, (D2) $(3,\overset{t}{\ldots},3,2,3,9)$, (D3) $(3,\overset{t}{\ldots},3,2,9,9)$,
(D4) $(3,\overset{t}{\ldots},3,6,3,3)$, (D5) $(3,\overset{t}{\ldots},3,6,3,9)$, (D6) $(3,\overset{t}{\ldots},3,6,9,9)$,

with $t>0$ in case (D1), and G must have order 36. In all these cases we have $\theta(x_i)\in\{\varrho,\varrho^2\}$ for $i=1,\ldots,t$, where $\theta:\Lambda\to G$ is an epimorphism defining an action of G on a cyclic trigonal Riemann surface. Let us define $a=\theta(x_{t+1})$, $b=\theta(x_{t+2})$ and $c=\theta(x_{t+3})$. The different orders of these elements yield the above different cases. Observe that the relation $x_1\cdots x_{t+3}=1$ gives $abc=\varrho^\alpha$ for some α and so in particular G is generated by a,b and ϱ. Moreover, the element $(ab)^3=(\varrho^\alpha c^{-1})^3$ belongs to the normal subgroup $\langle\varrho\rangle$ since in all cases we have $c^3\in\langle\varrho\rangle$.

Let us consider first the case (D1). Here a, b and c have orders 2, 3 and 3 respectively. If $\varrho\in\langle a,b\rangle$, then $G=\langle a,b\rangle$ and ab has order 9 since otherwise G would be the alternating group A_4, which does not have the required order 36. As $(ab)^3$ generates a normal subgroup of G we have that $(ba)^3=a^{-1}(ab)^3a$ equals either $(ab)^3$ or $(ab)^6$ and therefore G is a factor group of $\langle a,b \mid a^2,b^3,(ab)^9,(ba)^3(ab)^m\rangle$ for some $m\in\{3,6\}$. This abstract presentation yields a group of order 12 or 36 since $(ab)^3$ represents an element generating a normal subgroup which is either trivial or has order 3 and the corresponding factor group is the alternating group A_4 of order 12. Using then the system for computational discrete algebra GAP [8] we see that both groups have order 12.

So $\varrho \notin \langle a, b\rangle$ and then G is a semidirect product $Z_3 \rtimes A_4$. So G is a group with presentation $\langle a, b, \varrho \mid a^2, b^3, (ab)^3, \varrho^3, a\varrho a\varrho^{\varepsilon}, b\varrho b^{-1}\varrho^{\delta}\rangle$ for some $\varepsilon, \delta \in \{1, 2\}$. It is easy to see that just for the values $\varepsilon = \delta = 2$ the above is a semidirect product of the appropriate order 36. So G is the group

$$\langle a, b, \varrho \mid a^2, b^3, (ab)^3, \varrho^3, a\varrho a\varrho^{-1}, b\varrho b^{-1}\varrho^{-1}\rangle = Z_3 \oplus A_4.$$

Now the epimorphism $\theta : \Lambda \to Z_3 \oplus A_4 = \langle\varrho\rangle \oplus \langle a, b\rangle$ given by

$$\theta(x_1) = \cdots = \theta(x_t) = \varrho, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b, \qquad \theta(x_{t+3}) = (ab)^2\varrho^{-t}$$

defines an action of $Z_3 \oplus A_4$ on a cyclic trigonal Riemann surface whose genus can be calculated from the Hurwitz–Riemann formula and equals $g = 12t - 2$.

In case Λ has signature (D2) we get again the direct product $Z_3 \oplus A_4$. Now however, a smooth epimorphism $\theta : \Lambda \to G$ does not exist since there is no element of order 9 in $Z_3 \oplus A_4$.

In case Λ has signature (D3) the elements a, b and c have orders 2, 9 and 9 respectively. Now $b^3 \in \{\varrho, \varrho^2\}$ and so in particular $ab^3a = b^{-m}$ for some $m \in \{3, 6\}$ and $(ab)^3 = b^{-n}$ for some $n \in \{0, 3, 6\}$. Therefore G is a factor group of $\langle a, b \mid a^2, b^9, (ab)^9, ab^3ab^m, (ab)^3b^n\rangle$ for some $m \in \{3, 6\}$, $n \in \{0, 3, 6\}$. It is obvious that for all such values of m and n the group has order 12 or 36 while GAP asserts that G has order 36 just for $m = n = 6$. This yields the group with presentation

$$\langle a, b \mid a^2, b^9, (ab)^9, (ab)^3b^6\rangle,$$

where we have omitted the relation ab^3ab^6 since it is a consequence of the others. The epimorphism $\theta : \Lambda \to G$ given by

$$\theta(x_1) = \cdots = \theta(x_t) = b^3, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b, \qquad \theta(x_{t+3}) = b^{-(3t+1)}a$$

defines an action of this group G on a cyclic trigonal Riemann surface whose genus, by the Hurwitz–Riemann formula, equals $g = 12t + 6$.

Let now Λ have signature (D4). Here a, b and c have orders 6, 3 and 3 respectively, with $a^2 \in \{\varrho, \varrho^2\}$. So $ba^2b^{-1} = a^{-m}$ for some $m \in \{2, 4\}$ and $(ab)^3 = a^{-n}$ for some $n \in \{0, 2, 4\}$. Therefore G is a factor group of $\langle a, b \mid a^6, b^3, (ab)^9, ba^2b^{-1}a^m, (ab)^3a^n\rangle$ for some $m \in \{2, 4\}$ and $n \in \{0, 2, 4\}$. It is obvious that for all such values of m and n the group has order 12 or 36 while GAP asserts that G has order 36 just for $m = 4$ and $n = 0$, i.e., for

$$\langle a, b \mid a^6, b^3, ba^2b^{-1}a^4, (ab)^3\rangle. \tag{1}$$

The epimorphism $\theta : \Lambda \to G$ defined by

$$\theta(x_1) = \cdots = \theta(x_t) = a^2, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b, \qquad \theta(x_{t+3}) = b^{-1}a^{-(2t+1)}$$

yields an action of this group G on a cyclic trigonal Riemann surface of genus $g = 12t + 4$.

In case Λ has signature (D5) we get again the group with presentation (1). Now however, a smooth epimorphism $\theta : \Lambda \to G$ does not exist since there is no element of order 9 in this group.

Finally, let Λ have signature (D6). Then a, b and c have orders 6, 9 and 9 respectively and $a^2, b^3 \in \{\varrho, \varrho^2\}$. In particular, $a^2 = b^{-m}$ and $(ab)^3 = a^{-n}$ for some $m \in \{3, 6\}$ and $n \in \{0, 2, 4\}$. Therefore G is a factor group of $\langle a, b \mid a^6, b^9, (ab)^9, a^2b^m, (ab)^3a^n\rangle$, which has order 12 or 36 for all such values of m and n. GAP asserts that G has order 36 just for $m = 3$, $n = 2$ or $m = 6, n = 4$ i.e., for $\langle a, b \mid$

$a^6, b^9, (ab)^9, a^2b^3, (ab)^3a^2\rangle$ or $\langle a, b \mid a^6, b^9, (ab)^9, a^2b^6, (ab)^3a^4\rangle$. However, the mapping $a \mapsto a$, $b \mapsto b^2$ induces an isomorphism between these groups. So in case (D6) we get the group

$$\langle a, b \mid a^6, b^9, (ab)^9, a^2b^3, (ab)^3a^2\rangle. \tag{2}$$

The epimorphism $\theta : \Lambda \to G$ defined by

$$\theta(x_1) = \cdots = \theta(x_t) = a^2, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b, \qquad \theta(x_{t+3}) = b^{-1}a^{-(2t+1)}$$

yields an action of this group G on a cyclic trigonal Riemann surface of genus $g = 12t + 12$.

Summarizing, case (D) gives the following groups of automorphisms:

Case	Ramification type	Presentation	g
D1.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 3)$, $t > 0$	$\langle a, b \mid a^2, b^3, (ab)^3\rangle \oplus \langle \varrho \mid \varrho^3\rangle$	$12t - 2$
D3.1	$(3, \overset{t}{\ldots}, 3, 2, 9, 9)$	$\langle a, b \mid a^2, b^9, (ab)^9, (ab)^3b^6\rangle$	$12t + 6$
D4.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 3)$	$\langle a, b \mid a^6, b^3, (ab)^3, ba^2b^{-1}a^4\rangle$	$12t + 4$
D6.1	$(3, \overset{t}{\ldots}, 3, 6, 9, 9)$	$\langle a, b \mid a^6, b^9, (ab)^9, a^2b^3, (ab)^3a^2\rangle$	$12t + 12$

(E) Here Λ has one of the following signatures:

(E1) $(3, \overset{t}{\ldots}, 3, 2, 3, 4)$, (E2) $(3, \overset{t}{\ldots}, 3, 2, 3, 12)$, (E3) $(3, \overset{t}{\ldots}, 3, 6, 3, 12)$,
(E4) $(3, \overset{t}{\ldots}, 3, 6, 3, 4)$, (E5) $(3, \overset{t}{\ldots}, 3, 6, 9, 4)$, (E6) $(3, \overset{t}{\ldots}, 3, 6, 9, 12)$,
(E7) $(3, \overset{t}{\ldots}, 3, 2, 9, 4)$, (E8) $(3, \overset{t}{\ldots}, 3, 2, 9, 12)$,

with $t > 0$ in case (E1), and G must have order 72. The considerations here are similar as in case (D). In all these cases we have $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $i = 1, \ldots, t$, where $\theta : \Lambda \to G$ is an epimorphism defining an action of G on a cyclic trigonal Riemann surface. Throughout this case we define $a = \theta(x_{t+1})$, $b = \theta(x_{t+2})$ and $c = \theta(x_{t+3})$. The different orders of these elements yield the above eight different cases. The relation $x_1 \cdots x_{t+3} = 1$ yields $abc = \varrho^\alpha$ for some α and so in particular G is generated by a, b and ϱ. Moreover, the element $(ab)^4 = (\varrho^\alpha c^{-1})^4$ belongs to the normal subgroup $\langle \varrho \rangle$ since in all cases we have $c^4 \in \langle \varrho \rangle$.

Consider first the case (E1). Here a, b and c have orders 2, 3 and 4 respectively. If $\varrho \in \langle a, b\rangle$ then $G = \langle a, b\rangle$. So ab has order 12 since otherwise G would be the symmetric group S_4, which does not have the required order 72. Therefore $(ab)^4 \in \{\varrho, \varrho^2\}$ and since it generates a normal subgroup we have that the element $(ba)^4 = a^{-1}(ab)^4a$ equals either $(ab)^4$ or $(ab)^8$. Therefore in this case G is a factor group of $\langle a, b \mid a^2, b^3, (ab)^{12}, (ba)^4(ab)^m\rangle$ for some $m \in \{4, 8\}$. This presentation yields a finite group for both values of m since $(ab)^4$ represents an element generating a normal subgroup which is either trivial or has order 3 and the corresponding factor group is the symmetric group S_4 of order 24. Using GAP we see that it has order 72 just for $m = 8$ i.e., for G being

$$\langle a, b \mid a^2, b^3, (ab)^{12}, (ba)^4(ab)^8\rangle. \tag{3}$$

This group is isomorphic to $Z_3 \oplus S_4$ and does act on cyclic trigonal Riemann surfaces as we will see.

Assume now that $\varrho \notin \langle a, b\rangle$. Then G is a semidirect product $Z_3 \rtimes S_4$ and so G is a group with presentation $\langle a, b, \varrho \mid a^2, b^3, (ab)^4, \varrho^3, a\varrho a\varrho^\varepsilon, b\varrho b^{-1}\varrho^\delta\rangle$ for some $\varepsilon, \delta \in \{1, 2\}$. It is easy to see that for this group to be a semidirect product of the appropriate order 72 it has to be $\delta = 2$ whilst ε can be 1 or 2. So G is either

$$\langle a, b, \varrho \mid a^2, b^3, (ab)^4, \varrho^3, a\varrho a\varrho^{-1}, b\varrho b^{-1}\varrho^{-1}\rangle = Z_3 \oplus S_4 \tag{4}$$

or

$$\langle a, b, \varrho \mid a^2, b^3, (ab)^4, \varrho^3, a\varrho a\varrho, b\varrho b^{-1}\varrho^{-1}\rangle. \tag{5}$$

As said above, the group with presentation (3) is isomorphic to $Z_3 \oplus S_4$. Indeed, an isomorphism is given by the application $a \mapsto a,\ b \mapsto b\varrho$, where for $Z_3 \oplus S_4$ we have chosen presentation (4). Now the epimorphism $\theta : \Lambda \to Z_3 \oplus S_4$ given by

$$\theta(x_1) = \cdots = \theta(x_t) = \varrho, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b\varrho^{-t}, \qquad \theta(x_{t+3}) = (ab)^{-1}$$

defines an action of $Z_3 \oplus S_4$ on a cyclic trigonal Riemann surface while an epimorphism induced by

$$\theta(x_1) = \cdots = \theta(x_t) = \varrho, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b\varrho^{t}, \qquad \theta(x_{t+3}) = (ab)^{-1}$$

defines an action of the group with presentation (5). The genus of the corresponding surface equals $24t - 2$.

The case (E2) leads to the same groups as (E1). Now for $t = 0$ the assignment $\theta(x_1) = a$, $\theta(x_2) = b\varrho$, $\theta(x_3) = (ab)^{-1}\varrho^2$ yields an action of $Z_3 \oplus S_4$ with this ramification type. For $t > 0$ we obtain an action choosing $\theta(x_1), \ldots, \theta(x_t) \in \{\varrho, \varrho^2\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = \varrho$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$, $\theta(x_{t+3}) = (ab)^{-1}\varrho^2$. The group with presentation (5) does not act with ramification type (E2) since this group has no element of order 12.

In case Λ has signature (E3) the elements a, b and c have orders 6, 3 and 12 respectively. Now $a^2 \in \{\varrho, \varrho^2\}$ and so $(ab)^4 = a^{-n}$ for some $n \in \{0, 2, 4\}$ and $ba^2b^{-1} = a^m$ for some $m \in \{2, 4\}$. Therefore G is a factor group of $\langle a, b \mid a^6, b^3, (ab)^{12}, (ab)^4a^n, ba^2b^{-1}a^m\rangle$ for some $n \in \{0, 2, 4\}$ and $m \in \{2, 4\}$. This presentation yields a finite group since its quotient under the normal subgroup $\langle a^2\rangle$ is isomorphic to S_4. GAP asserts that G has the appropriate order 72 just for $m = 4$ and any of the above values of n. Therefore, G is any of the following groups G_n:

$$G_n = \langle a, b \mid a^6, b^3, (ab)^{12}, (ab)^4a^n, ba^2b^{-1}a^4\rangle$$

for $n = 0$, 2 or 4. However, it is easy to see that the application $a \mapsto a^{-1},\ b \mapsto b$ induces an isomorphism between the groups G_4 and G_0, while $a \mapsto a,\ b \mapsto a^2b$ induces an isomorphism between G_2 and G_4. So we just have to consider, for instance, the group $G = G_2$ with presentation

$$\langle a, b \mid a^6, b^3, a^2(ab)^4\rangle \tag{6}$$

where we have omitted the relations $(ab)^{12} = 1$ and $ba^2b^{-1}a^4 = 1$ since they are consequence of the others.

Now, if $t = 0$ then an epimorphism $\theta : \Lambda \to G$ defining the action of this group can be defined by $\theta(x_1) = a$, $\theta(x_2) = b$ and $\theta(x_3) = (ab)^{-1}$. For $t > 0$ we can choose $\theta(x_1), \ldots, \theta(x_t) \in \{a^2, a^4\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = a^4$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^{-1}a$.

Signature (E4) leads to the same group as (E3). For $t = 0$ an appropriate epimorphism can be defined by $\theta(x_1) = a$, $\theta(x_2) = a^2b$ and $\theta(x_3) = b^{-1}a^3$. For $t > 0$ we can choose $\theta(x_1), \ldots, \theta(x_t) \in \{a^2, a^4\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = a^2$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^{-1}a^3$.

In cases (E5) and (E6) an analogous reasoning as above shows that G is a factor group of $\langle a, b \mid a^6, b^9, (ab)^{12}, a^2b^m, (ab)^4a^n\rangle$ for some $m \in \{3, 6\}$ and $n \in \{0, 2, 4\}$. Now it is obvious that for any of these values of m and n such group has order 24 or 72 while GAP asserts that G has order 24 for any such m and n. So we discard them.

Finally, cases (E7) and (E8) leads to a factor group of $\langle a, b \mid a^2, b^9, (ab)^{12},\ ab^3ab^m,\ (ab)^4b^n\rangle$ for some $m \in \{3, 6\}$ and $n \in \{0, 3, 6\}$ which by GAP has order 72 just for $m = 3$, $n = 0$ i.e.,

$$\langle a, b \mid a^2, b^9, (ab)^4, ab^3ab^3\rangle. \tag{7}$$

An epimorphism $\theta : \Lambda \to G$ defining the action of this group exists for arbitrary t in the case (E7): if $t = 0$ we can define $\theta(x_1) = a$, $\theta(x_2) = b$ and $\theta(x_3) = (ab)^{-1}$; if $t > 0$ we can choose $\theta(x_1), \dots, \theta(x_t) \in \{b^3, b^6\}$ in such a way that $\theta(x_1)\cdots\theta(x_t) = b^3$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b^{-1}$ and $\theta(x_{t+3}) = (ab^5)^{-1}$. However, this group cannot act with ramification type (E8) since it contains no element of order 12.

Summing up, case (E) gives the following groups of automorphisms:

Case	Ramification type	Presentation	g
E1.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 4)$, $t > 0$	$\langle a, b \mid a^2, b^3, (ab)^4\rangle \oplus \langle \varrho \mid \varrho^3\rangle$	$24t - 2$
E1.2	$(3, \overset{t}{\ldots}, 3, 2, 3, 4)$, $t > 0$	$\langle a, b \mid a^2, b^3, (ab)^4, \varrho^3, a\varrho a\varrho, b\varrho b^{-1}\varrho^2\rangle$	$24t - 2$
E2.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 12)$	$\langle a, b \mid a^2, b^3, (ab)^4\rangle \oplus \langle \varrho \mid \varrho^3\rangle$	$24t + 4$
E3.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 12)$	$\langle a, b \mid a^6, b^3, a^2(ab)^4\rangle$	$24t + 16$
E4.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 4)$	$\langle a, b \mid a^6, b^3, a^2(ab)^4\rangle$	$24t + 10$
E7.1	$(3, \overset{t}{\ldots}, 3, 2, 9, 4)$	$\langle a, b \mid a^2, b^9, (ab)^4, ab^3ab^3\rangle$	$24t + 6$

(F) Here Λ has one of the following signatures:

(F1) $(3, \overset{t}{\ldots}, 3, 2, 3, 5)$, (F2) $(3, \overset{t}{\ldots}, 3, 2, 3, 15)$, (F3) $(3, \overset{t}{\ldots}, 3, 6, 3, 15)$,
(F4) $(3, \overset{t}{\ldots}, 3, 6, 3, 5)$, (F5) $(3, \overset{t}{\ldots}, 3, 6, 9, 5)$, (F6) $(3, \overset{t}{\ldots}, 3, 6, 9, 15)$,
(F7) $(3, \overset{t}{\ldots}, 3, 2, 9, 5)$, (F8) $(3, \overset{t}{\ldots}, 3, 2, 9, 15)$,

with $t > 0$ in case (F1), and G must have order 180. The considerations here are similar as in case (E). In all these cases we have $\theta(x_i) \in \{\varrho, \varrho^2\}$ for $i = 1, \dots, t$, and we again define $a = \theta(x_{t+1})$, $b = \theta(x_{t+2})$ and $c = \theta(x_{t+3})$. The relation $x_1 \cdots x_{t+3} = 1$ yields $abc = \varrho^\alpha$ for some α and so in particular G is generated by a, b and ϱ. Moreover, the element $(ab)^5 = (\varrho^\alpha c^{-1})^5$ belongs to the normal subgroup $\langle \varrho\rangle$ since in all cases we have $c^5 \in \langle \varrho\rangle$.

Consider the case (F1). Here a, b and c have orders 2, 3 and 5 respectively. If $\varrho \in \langle a, b\rangle$, then $G = \langle a, b\rangle$ and so ab has order 15 since otherwise G would be the alternating group A_5. Therefore $(ab)^5 \in \{\varrho, \varrho^2\}$ and so the element $(ba)^5 = a^{-1}(ab)^5a$ equals either $(ab)^5$ or $(ab)^{10}$. Hence in this case G is a factor group of the finite group $\langle a, b \mid a^2, b^3, (ab)^{15}, (ba)^5(ab)^m\rangle$ for some $m \in \{5, 10\}$. Using GAP we see that the last has order 180 just for $m = 10$, i.e., for G being

$$\bigl\langle a, b \bigm| a^2, b^3, (ab)^{15}, (ba)^5(ab)^{10}\bigr\rangle. \tag{8}$$

This group is isomorphic to $Z_3 \oplus A_5$ and does act on cyclic trigonal Riemann surfaces as we will see.

Assume now that $\varrho \notin \langle a, b\rangle$. Then G is a semidirect product $Z_3 \rtimes A_5$ and so G is a group with presentation $\langle a, b, \varrho \mid a^2, b^3, (ab)^5, \varrho^3, a\varrho a\varrho^\varepsilon, b\varrho b^{-1}\varrho^\delta\rangle$ for some $\varepsilon, \delta \in \{1, 2\}$. In fact, just for $\varepsilon = \delta = 2$ this group has order 180, i.e., for G being the group

$$\bigl\langle a, b, \varrho \bigm| a^2, b^3, (ab)^5, \varrho^3, a\varrho a\varrho^{-1}, b\varrho b^{-1}\varrho^{-1}\bigr\rangle = Z_3 \oplus A_5. \tag{9}$$

Observe that the assignment given by $a \mapsto a$, $b \mapsto b\varrho$ induces an isomorphism between the groups with presentations (8) and (9). Now the epimorphism $\theta : \Lambda \to Z_3 \oplus A_5$ given by

$$\theta(x_1) = \cdots = \theta(x_t) = \varrho, \qquad \theta(x_{t+1}) = a, \qquad \theta(x_{t+2}) = b\varrho^{-t}, \qquad \theta(x_{t+3}) = (ab)^{-1}$$

defines an action of $Z_3 \oplus A_5$ on a cyclic trigonal Riemann surface whose genus, by the Hurwitz–Riemann formula, equals $60t - 2$.

The case (F2) leads also to the group $Z_3 \oplus A_5$. Now for $t = 0$ the assignment $\theta(x_1) = a$, $\theta(x_2) = b\varrho$, $\theta(x_3) = (ab)^{-1}\varrho^2$ gives an action of $Z_3 \oplus A_5$. For $t > 0$ we obtain an action choosing $\theta(x_1), \dots, \theta(x_t) \in \{\varrho, \varrho^2\}$ in such a way that $\theta(x_1) \cdots \theta(x_t) = \varrho$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$, $\theta(x_{t+3}) = (ab)^{-1}\varrho^2$.

We now consider the case (F3). Here a, b and c have orders 6, 3 and 15 respectively. Now $a^2 \in \{\varrho, \varrho^2\}$ and so $(ab)^5 = a^{-n}$ for some $n \in \{0, 2, 4\}$ and $ba^2b^{-1} = a^m$ for some $m \in \{2, 4\}$. Therefore G is a factor group of $\langle a, b \mid a^6, b^3, (ab)^{15}, (ab)^5 a^n, ba^2b^{-1}a^m \rangle$ for some $n \in \{0, 2, 4\}$ and $m \in \{2, 4\}$. This presentation yields a finite group and in fact GAP asserts that G has order 180 just for $m = 4$ and any of the above values of n. Therefore, G is any of the following groups G_n:

$$G_n = \left\langle a, b \;\middle|\; a^6, b^3, (ab)^{15}, (ab)^5 a^n, ba^2b^{-1}a^4 \right\rangle$$

for $n = 0$, 2 or 4. However, it is easy to see that the application $a \mapsto a^{-1}$, $b \mapsto b$ induces an isomorphism between the groups G_2 and G_0, while $a \mapsto a$, $b \mapsto a^4 b$ induces an isomorphism between G_2 and G_4. So we just have to consider, for instance, the group $G = G_2$ with presentation

$$\left\langle a, b \;\middle|\; a^6, b^3, a^2(ab)^5 \right\rangle \tag{10}$$

where we have omitted the relations $(ab)^{15} = 1$ and $ba^2b^{-1}a^4 = 1$ since they are consequence of the others.

Now if $t = 0$ then an epimorphism $\theta : \Lambda \to G$ defining the action of this group can be defined by $\theta(x_1) = a$, $\theta(x_2) = b$ and $\theta(x_3) = (ab)^{-1}$. For $t > 0$ we can choose $\theta(x_1), \dots, \theta(x_t) \in \{a^2, a^4\}$ in such a way that $\theta(x_1) \cdots \theta(x_t) = a^2$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^{-1}a^3$.

The signature (F4) leads to the same group as (F3). For $t = 0$ we can choose $\theta(x_1) = a^{-1}$, $\theta(x_2) = b$ and $\theta(x_3) = b^{-1}a$ whilst for $t > 0$ we can choose $\theta(x_1), \dots, \theta(x_t) \in \{a^2, a^4\}$ in such a way that $\theta(x_1) \cdots \theta(x_t) = a^4$ and $\theta(x_{t+1}) = a$, $\theta(x_{t+2}) = b$ and $\theta(x_{t+3}) = b^{-1}a$.

Cases (F5) and (F6) give rise to the group with presentation $\langle a, b \mid a^6, b^9, (ab)^{15}, a^2b^m, (ab)^5 a^n \rangle$ for some $m \in \{3, 6\}$ and $n \in \{0, 2, 4\}$. However, GAP asserts that it has order 60 for such values of m and n.

Finally, cases (F7) and (F8) lead to the group with presentation $\langle a, b \mid a^2, b^9, (ab)^{15}, ab^3ab^m, (ab)^5 b^n \rangle$ for some $m \in \{3, 6\}$ and $n \in \{0, 3, 6\}$. However, it has order 60 for all such values of m and n.

Summing up, case (F) gives the following groups of automorphisms:

Case	Ramification type	Presentation	g
F1.1	$(3, \overset{t}{\dots}, 3, 2, 3, 5)$, $t > 0$	$\langle a, b \mid a^2, b^3, (ab)^5 \rangle \oplus \langle \varrho \mid \varrho^3 \rangle$	$60t - 2$
F2.1	$(3, \overset{t}{\dots}, 3, 2, 3, 15)$	$\langle a, b \mid a^2, b^3, (ab)^5 \rangle \oplus \langle \varrho \mid \varrho^3 \rangle$	$60t + 10$
F3.1	$(3, \overset{t}{\dots}, 3, 6, 3, 15)$	$\langle a, b \mid a^6, b^3, a^2(ab)^5 \rangle$	$60t + 40$
F4.1	$(3, \overset{t}{\dots}, 3, 6, 3, 5)$	$\langle a, b \mid a^6, b^3, a^2(ab)^5 \rangle$	$60t + 28$

We have proved the following theorem.

Theorem 2.1. *Let $g \geqslant 5$ be an integer. An abstract group G acts as a group of automorphisms of a cyclic trigonal compact Riemann surface X of genus g and containing the trigonality automorphism group $\langle \varrho \rangle$ if and only if G is one of the groups occurring in Table* 1. *The ramification type of the normal covering $X \to X/G$ is shown in the second column, whilst the conditions for this action to happen there appear in the fifth column. Moreover, $N > 1$ in cases* (B), *$N > 2$ and even in cases* (C) *and the values of N and t in the last column make the values of g to be an integer $\geqslant 5$.*

Table 1
Groups of automorphisms of cyclic trigonal compact Riemann surfaces of genus $g \geqslant 5$.

Case	Ramification type	Presentation	$\lvert G\rvert$	Conditions	g
A	$(3, \overset{t}{\ldots}, 3)$	Z_3	3	None	$t-2$
B1.1	$(3, \overset{t}{\ldots}, 3, N, N)$	$Z_N \oplus Z_3$	$3N$	$t > 1$ or $t = 1, N \equiv 0\ (3)$	$Nt-2$
B1.2	$(3, \overset{t}{\ldots}, 3, N, N)$	$\langle a, \varrho \mid a^N, \varrho^3, a\varrho a^{-1}\varrho\rangle$	$3N$	$N \equiv 0\ (2)$	$Nt-2$
B2	$(3, \overset{t}{\ldots}, 3, N, 3N)$	Z_{3N}	$3N$	$N \not\equiv 0\ (3)$	$Nt-1$
B3	$(3, \overset{t}{\ldots}, 3, 3N, 3N)$	Z_{3N}	$3N$	None	Nt
C1.1	$(3, \overset{t}{\ldots}, 3, 2, 2, N/2)$	$D_{3N/2}$	$3N$	$t > 0,\ N \not\equiv 0\ (6)$	$Nt-2$
C1.2	$(3, \overset{t}{\ldots}, 3, 2, 2, N/2)$	$D_{N/2} \oplus Z_3$	$3N$	$t > 1$ or $N \equiv 0\ (6)$	$Nt-2$
C1.3	$(3, \overset{t}{\ldots}, 3, 2, 2, N/2)$	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho\rangle$	$3N$	$t > 0$	$Nt-2$
C1.4	$(3, \overset{t}{\ldots}, 3, 2, 2, N/2)$	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho^{-1}\rangle$	$3N$	$t > 0,\ N \equiv 0\ (4)$	$Nt-2$
C2.1	$(3, \overset{t}{\ldots}, 3, 2, 2, 3N/2)$	$D_{3N/2}$	$3N$	$t > 0$	Nt
C2.2	$(3, \overset{t}{\ldots}, 3, 2, 2, 3N/2)$	$D_{N/2} \oplus Z_3$	$3N$	$t > 0, N \not\equiv 0\ (6)$	Nt
C2.3	$(3, \overset{t}{\ldots}, 3, 2, 2, 3N/2)$	$\langle a, b, \varrho \mid a^2, b^2, \varrho^3, (ab)^{N/2}, a\varrho a\varrho, b\varrho b\varrho\rangle$	$3N$	$t > 0, N \not\equiv 0\ (6)$	Nt
C3.1	$(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$	$\langle a, b \mid a^2, b^6, (ab)^{N/2}b^4\rangle$	$3N$	$N \equiv 4\ (6),\ t > 0$	$N(2t+1)/2-2$
C3.2	$(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$	$\langle a, b \mid a^2, b^6, (ab)^{N/2}b^2\rangle$	$3N$	$N \equiv 2\ (6),\ t > 0$	$N(2t+1)/2-2$
C3.3	$(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$	$\langle a, b \mid a^2, b^6, ab^2ab^4, (ab)^{N/2}\rangle$	$3N$	$N \equiv 0\ (6)$	$N(2t+1)/2-2$
C3.4	$(3, \overset{t}{\ldots}, 3, 2, 6, N/2)$	$\langle a, b \mid a^2, b^6, ab^2ab^2, (ab)^{N/2}\rangle$	$3N$	$N \equiv 0\ (4)$	$N(2t+1)/2-2$
C4.1	$(3, \overset{t}{\ldots}, 3, 2, 6, 3N/2)$	$\langle a, b \mid a^2, b^6, (ab)^{N/2}b^4\rangle$	$3N$	$N \equiv 4\ (6)$	$N(2t+1)/2$
C4.2	$(3, \overset{t}{\ldots}, 3, 2, 6, 3N/2)$	$\langle a, b \mid a^2, b^6, (ab)^{N/2}b^2\rangle$	$3N$	$N \equiv 2\ (6)$	$N(2t+1)/2$
C5.1	$(3, \overset{t}{\ldots}, 3, 6, 6, N/2)$	$\langle a, b \mid a^6, b^6, a^2b^2, (ab)^{N/2}\rangle$	$3N$	$N \equiv 0\ (2)$	$N(t+1)-2$
C6.1	$(3, \overset{t}{\ldots}, 3, 6, 6, 3N/2)$	$\langle a, b \mid a^6, b^6, a^2b^2, (ab)^{N/2}\rangle$	$3N$	$N \equiv 2$ or $4\ \ (6)$	$N(t+1)$
D1.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 3)$	$A_4 \oplus Z_3$	36	$t > 0$	$12t-2$
D3.1	$(3, \overset{t}{\ldots}, 3, 2, 9, 9)$	$\langle a, b \mid a^2, b^9, (ab)^9, (ab)^3b^6\rangle$	36	None	$12t+6$
D4.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 3)$	$\langle a, b \mid a^6, b^3, (ab)^3, ba^2b^{-1}a^4\rangle$	36	None	$12t+4$
D6.1	$(3, \overset{t}{\ldots}, 3, 6, 9, 9)$	$\langle a, b \mid a^6, b^9, (ab)^9, a^2b^3, (ab)^3a^2\rangle$	36	None	$12t+12$
E1.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 4)$	$S_4 \oplus Z_3$	72	$t > 0$	$24t-2$
E1.2	$(3, \overset{t}{\ldots}, 3, 2, 3, 4)$	$\langle a, b \mid a^2, b^3, (ab)^4, \varrho^3, a\varrho a\varrho, b\varrho b^{-1}\varrho^2\rangle$	72	$t > 0$	$24t-2$
E2.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 12)$	$S_4 \oplus Z_3$	72	None	$24t+4$
E3.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 12)$	$\langle a, b \mid a^6, b^3, a^2(ab)^4\rangle$	72	None	$24t+16$
E4.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 4)$	$\langle a, b \mid a^6, b^3, a^2(ab)^4\rangle$	72	None	$24t+10$
E7.1	$(3, \overset{t}{\ldots}, 3, 2, 9, 4)$	$\langle a, b \mid a^2, b^9, (ab)^4, ab^3ab^3\rangle$	72	None	$24t+6$
F1.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 5)$	$A_5 \oplus Z_3$	180	$t > 0$	$60t-2$
F2.1	$(3, \overset{t}{\ldots}, 3, 2, 3, 15)$	$A_5 \oplus Z_3$	180	None	$60t+10$
F3.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 15)$	$\langle a, b \mid a^6, b^3, a^2(ab)^5\rangle$	180	None	$60t+40$
F4.1	$(3, \overset{t}{\ldots}, 3, 6, 3, 5)$	$\langle a, b \mid a^6, b^3, a^2(ab)^5\rangle$	180	None	$60t+28$

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