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On Bordering of Regular Matrices

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ABSTRACT

Necessary and sufficient conditions are given for a commutative ring $\mathbb{R}$ to be a ring over which every regular matrix can be completed to an invertible matrix of a particular size by bordering. Such rings are precisely the projective free rings. Also, over such rings every regular matrix has a rank factorization. Using the bordering technique, we give an interesting method of computing minors of a reflexive g-inverse G of a regular matrix A when $I - AG$ and $I - GA$ have rank factorizations.

1. INTRODUCTION

Bordered matrices have been studied in literature by many workers, and these matrices also play an important role in finding generalized inverses of matrices. Generalizing the works of Goldman and Zelen [7], Blattner [5], and Ben-Israel and Greville [2], Kentaro Nomakuchi [11] presented a characterization of generalized inverses of matrices over the field of complex numbers using bordered matrices. Specifically, Nomakuchi showed that if A is an $m \times n$ matrix of rank r over the field of complex numbers, there exists an

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invertible matrix

$$T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix}$$

of size $(m + n - r) \times (m + n - r)$ where P and Q are matrices of size $m \times (m - r)$ and $(n - r) \times n$ respectively. Nomakuchi in fact showed that all g-inverses of A can be obtained by looking at the inverses of matrices T in

$$\mathscr{B}(A) = \left\{ T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} P, Q \text{ are matrices of size } m \times (m - r) \text{ and } (n - r) \times n \text{ respectively, and } T \text{ is invertible} \right\}. \quad (1.1)$$

The above results hold good even for matrices over any field. But over an arbitrary ring it may not be possible to find a bordered matrix of the above kind for every matrix, as the following example shows.

EXAMPLE. Consider the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

over the ring of integers $\mathbb{Z}$. This is a 2×2 matrix of determinantal rank 1. For this A there is no bordered matrix

$$T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix}$$

where T is an invertible 3×3 matrix over $\mathbb{Z}$, because $|T|$ is divisible by 2 whatever P, Q, and R may be. Hence $\mathscr{B}(A) = \varnothing$ over the ring of integers.

From our Lemmas 2 and 3 it will follow that if a matrix A has a bordering, then it necessarily is regular. Observe that the matrix $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ in the above example is not regular over $\mathbb{Z}$.

From our Theorem 4, it follows that over an arbitrary commutative ring every regular matrix need not admit a bordered matrix of the above type.

In Theorem 4 we shall give necessary and sufficient conditions on a commutative ring $\mathbb{R}$ with identity so that every regular matrix over $\mathbb{R}$ has a bordered matrix of the above type.

2. PRELIMINARIES

Let $\mathbb{R}$ be a commutative ring with unity. Let A be an $m \times n$ matrix, and consider the following matrix equations:

(1) $AGA = A$,
(2) $GAG = G$.

If G is an $n \times m$ matrix satisfying (1), then G is called a *generalized inverse* (*g-inverse*, 1-*inverse*) of A. A matrix A is called *regular* if it has a g-inverse. If G satisfies (1) and (2), it is called a *reflexive g-inverse* of A.

Let A be an $m \times n$ matrix, and let $\alpha = \{i_1, \dots, i_r\}$, $\beta = \{j_1, \dots, j_r\}$ be subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. We denote by A^{α}_{β} the submatrix of A determined by rows indexed by α and columns indexed by β. The determinant of a square matrix A is denoted by $|A|$, and $(\partial/\partial a_{ij})|A|$ denotes the cofactor of a_{ij} in the expansion of the determinant of A. The determinantal rank (the largest size of a nonvanishing minor) is denoted by $\rho(A)$. In this paper, we say that an $m \times n$ matrix A of rank r has a *rank factorization* if there is a left invertible matrix B of size $m \times r$ and a right invertible matrix C of size $r \times n$ such that $A = BC$. We denote by $C_r(A)$ the rth compound matrix of A with rows indexed by r-element subsets of $\{1, \dots, m\}$ and columns indexed by r-element subsets of $\{1, \dots, n\}$. At several places in this paper, α, β, are assumed to be r-element subsets of $\{1, 2, \dots, m\}$ or $\{1, 2, \dots, n\}$ without that being stated explicitly.

The relevant properties of $C_r(A)$ from [3] that will be used are listed below:

(i) $C_r(AB) = C_r(A)C_r(B)$.
(ii) If A is an $m \times n$ matrix with $\rho(A) = r$, then $\rho(C_r(A)) = 1$.

We follow Jacobson [8] for the notation and terminology regarding modules.

Now we shall recall a result given by Rao [4] for the construction of a g-inverse of a given regular matrix satisfying a sufficient condition.

THEOREM 1 (Rao [4, Theorem 1(ii) ⇒ (iii)]). *Let A be an $m \times n$ matrix of rank r over $\mathbb{R}$ such that for some $c^{\beta}_{\alpha} \in \mathbb{R}$*

$$\sum_{\alpha, \beta} c^{\beta}_{\alpha} |A^{\alpha}_{\beta}| = 1, \tag{2.1}$$

where the summation is over all r-element subsets α, β of $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$ respectively. Then the matrix G obtained from

$$g_{ji} = \sum_{\alpha, \beta} c_\alpha^\beta \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha| \tag{2.2}$$

is a g-inverse of A.

The proof of this theorem given in [4] is basically a finer analysis of the results of [3].

3. BORDERING AND g-INVERSES

For a real matrix A of full column rank it is always possible to find a matrix P such that $[A, P]$ is invertible. This raises the problem of finding necessary and sufficient conditions for a given $m \times n$ matrix A of rank n over a general commutative ring to admit a matrix P such that $[A, P]$ is invertible. Over a general commutative ring the result mentioned above for real matrices is no longer true. Take for example the 2×1 matrix $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ of rank 1 over $\mathbb{Z}$. This cannot be completed to a 2×2 invertible matrix over $\mathbb{Z}$. Our Lemma 2 gives some useful necessary conditions, and the second remark after Corollary 5 gives some necessary and sufficient conditions for a more general problem. For example, it follows that over a commutative ring if there is a P such that $[A, P]$ is invertible, A must be regular. This tells us that there is an inherent relation between regularity of the matrix A and the existence of a bordering of the type $[A, P]$.

We shall in fact consider for a matrix A of order $m \times n$ and of rank r the question of existence of matrices P, Q, and R such that

$$\begin{bmatrix} A & P \\ Q & R \end{bmatrix}$$

is invertible. One can easily see that P must be $m \times l$, where $l \geqslant m - r$, and Q must be $k \times n$, where $k \geqslant n - r$. So we shall consider the existence of matrices P, Q, and R such that $\begin{bmatrix} A & P \\ Q & R \end{bmatrix}$ is an invertible matrix of order $(m + n - r) \times (m + n - r)$. Again, regularity of A becomes a necessary condition, as is explained in Theorem 4 below.

First we shall start with a lemma which is crucial for our Theorem 4, which at the same time explains what we mentioned at the beginning of this section.

LEMMA 2. *Let A be an $m \times n$ matrix of rank r over $\mathbb{R}$, and suppose that P is an $m \times (m-r)$ matrix such that $T = [A, P]$ has a right inverse. Then A is regular, and P has a left inverse P_L^{-1} such that $P_L^{-1}A = 0$ and $P_L^{-1}P = I_{m-r}$. In fact, if $\begin{bmatrix} G \\ Q \end{bmatrix}$ is a right inverse of $[A, P]$, then G is a g-inverse of A and Q is P_L^{-1} satisfying above properties.*

Proof. Suppose T has a right inverse. Then there exists a linear combination $\sum_\alpha |T_\alpha^m| c^\alpha$ of $m \times m$ minors of T which equals one, i.e.,

$$\sum_\alpha |T_\alpha^m| c^\alpha = 1. \tag{3.1}$$

Since $\rho(A) = r$, $\rho([A, P]) = m$, and A, P are of size $m \times n$, $m \times (m-r)$ respectively, we get that $\rho(P)$ is $m-r$; also, $|T_\alpha^m|$ can be nonzero only if α contains the indices $n+1, n+2, \dots, m+n-r$. Let

$$\alpha' = \alpha \setminus \{n+1, n+2, \dots, m+n-r\}$$

whenever $|T_\alpha^m|$ is nonzero. Then

$$|T_\alpha^m| = \sum_\gamma \operatorname{sgn}(\gamma)\, |P_{m-r}^\gamma|\, |A_{\alpha'}^{\gamma^c}|$$

where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{m-r})$ is an $(m-r)$-element subset of $\{1, 2, \dots, m\}$, $\operatorname{sgn}(\gamma) = (-1)^{\sum_{i=1}^{m-r} \gamma_i + (n+i)}$, and $\gamma^c = \{1, 2, \dots, m\} \setminus \gamma$ (by Laplace expansion). Hence by considering only the nonzero $|T_\alpha^m|$, (3.1) can be rewritten as

$$\sum_\alpha \left(\sum_\gamma \operatorname{sgn}(\gamma)\, |P_{m-r}^\gamma|\, |A_{\alpha'}^{\gamma^c}| \right) c^\alpha = 1. \tag{3.2}$$

So

$$\sum_\gamma \left(\sum_\alpha \operatorname{sgn}(\gamma)\, |A_{\alpha'}^{\gamma^c}|\, c^\alpha \right) |P_{m-r}^\gamma| = 1, \tag{3.3}$$

and the matrix P_L^{-1} is obtained as

$$(P_L^{-1})_{ij} = \sum_\gamma \left(\sum_\alpha \operatorname{sgn}(\gamma)\, |A_{\alpha'}^{\gamma^c}|\, c^\alpha \right) \frac{\partial}{\partial p_{ji}} |P_{m-r}^\gamma| \tag{3.4}$$

$$= \sum_\alpha c^\alpha \frac{\partial}{\partial t_{j,n+i}} |T_\alpha^m| \tag{3.5}$$

Clearly, by Theorem 1, P_L^{-1} obtained by (3.4) is a left inverse of P. Since the matrix T^* obtained by replacing the $(n+i)$th column with the kth column of A is of rank strictly less than m, we get

$$\left(P_L^{-1}A\right)_{ik} = \sum_j \left(\sum_\alpha c^\alpha \frac{\partial}{\partial t_{j,n+i}} |T_\alpha^m| \right) a_{jk} = \sum_\alpha |T_\alpha^{*m}| c^\alpha = 0,$$

i.e., $P_L^{-1}A = 0$. Since the left hand side in (3.2) is a linear combination of $r \times r$ minors of A, we get that A is regular.

If $\begin{bmatrix} G \\ Q \end{bmatrix}$ is a right inverse of $[A, P]$, we get that $AG + PQ = I_m$. By multiplying the above equation by P_L^{-1} we get that $Q = P_L^{-1}$. So we get $QA = 0$, which gives $AGA = A$. Hence the proof. ■

LEMMA 3. *Let A be an $m \times n$ matrix of rank r over $\mathbb{R}$, and let Q be an $(n-r) \times n$ matrix such that*

$$T = \begin{bmatrix} A \\ Q \end{bmatrix}$$

has a left inverse. Then A is regular, and Q has a right inverse Q_R^{-1} such that $AQ_R^{-1} = 0$ and $QQ_R^{-1} = I_{n-r}$. In fact, if $[G, P]$ is a left inverse of $\begin{bmatrix} G \\ Q \end{bmatrix}$, then G is a g-inverse of A and P is Q_R^{-1} satisfying above properties.

The proof is similar to that of Lemma 2.

In the following theorem we shall characterize commutative rings with identity over which every regular matrix A has nonempty $\mathscr{B}(A)$. Here we obtain that such rings are projective free (i.e., every finitely generated projective module is free).

THEOREM 4. *The following are equivalent over any commutative ring $\mathbb{R}$:*

(i) *Every finitely generated projective module over $\mathbb{R}$ is free.*
(ii) *Every regular matrix has a rank factorization.*
(iii) *For every regular matrix A, $\mathscr{B}(A) \neq \varnothing$.*

Proof. (i) ⇒ (ii): Let every finitely generated projective module over $\mathbb{R}$ be free. Let A be an $m \times n$ regular matrix of rank k. Consider A as a module homomorphism from $\mathbb{R}^n$ into $\mathbb{R}^m$. Since A is regular, there exists a matrix $G: \mathbb{R}^m \to \mathbb{R}^n$ such that $AGA = A$. We observe that AG is an idempotent linear map on $\mathbb{R}^m$ into $\mathbb{R}^m$ and Range(A) = Range(AG) (= S, say). Of course, AG is identity on S. Now observe that for any idempotent linear map $T:\mathbb{R}^m \to \mathbb{R}^m$, Range$(T)$ is projective. So we get that S is projective, and by the hypothesis it is free. Suppose that S is isomorphic to $\mathbb{R}^p$ for some integer p through an isomorphism $\phi: S \to \mathbb{R}^p$. Let $C = \phi A$ and $B = i\phi^{-1}$, where $i: S \to \mathbb{R}^m$ is the inclusion map. Observe that $A = BC$, where B is an $m \times p$ matrix and C is a $p \times n$ matrix. Now we shall see that B has a left inverse, C has a right inverse, and $k = p$.

Observe that $\phi^{-1}: \mathbb{R}^p \to S$ is a linear mapping onto S and $AG:\mathbb{R}^m \to \mathbb{R}^m$ is a projection onto S. From this we obtain that $AGi\phi^{-1} = \phi^{-1}$. In other words, $\phi AGi\phi^{-1}$ is an identity on $\mathbb{R}^p$. Now clearly the matrix C' obtained from $Gi\phi^{-1}$ is a right inverse of C, and the matrix B' obtained from ϕAG is a left inverse of B. Now $A = BC$ and $B'AC' = B'BCC' = I_p$ give us $\rho(A) = k = p$. Hence $A = BC$ is a rank factorization.

(ii) $\Rightarrow$ (iii): Suppose every regular matrix has a rank factorization. We shall prove that for every regular matrix A, $\mathscr{B}(A)$ is nonempty. Let A be an $m \times n$ matrix of rank r, and G be a reflexive g-inverse of A.

We first consider the case $r < \min\{m, n\}$. Then $I_m - AG$ and $I_n - GA$ are idempotent matrices, and so they are regular of rank $m - r$ and $n - r$ respectively. [In fact, if an idempotent matrix E has a rank factorization $E = PQ$, then $\rho(E)$ = trace E. Since P has a left inverse and Q has a right inverse, we get that $PQPQ = PQ \Rightarrow QP = I_r$ and

$$r = \text{trace } I_r = \text{trace } QP = \text{trace } PQ = \text{trace } E.]$$

Since every regular matrix over $\mathbb{R}$ has a rank factorization and AG and GA are idempotent matrices of rank r, and since $r < \min(m, n)$, we get that $I_m - AG$ and $I_n - GA$ are nonzero idempotent matrices of rank $m - r$ and $n - r$ respectively. Let

$$I_m - AG = B_{m\times(m-r)}C_{(m-r)\times m} \tag{3.6}$$

and

$$I_n - GA = P_{n\times(n-r)}Q_{(n-r)\times n} \tag{3.7}$$

be rank factorizations. Since $I_m - AG$ and $I_n - GA$ are idempotent matrices, we get that $CB = I_{m-r}$ and $QP = I_{n-r}$. Using (3.6) and (3.7), we also get that $CA = 0$, $GB = 0$, $AP = 0$, and $QG = 0$. Hence we get that

$$\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}$$

is an $(m + n - r) \times (m + n - r)$ matrix with inverse

$$\begin{bmatrix} G & P \\ C & 0 \end{bmatrix}.$$

Hence $\mathscr{B}(A)$ is nonempty. For the case $r = \min(m, n)$, a slight adjustment in the above proof will give the result.

(iii) $\Rightarrow$ (ii): Let X be a finitely generated projective module with $X \oplus Y \cong \mathbb{R}^n$ for some module Y and some integer n. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the natural projection onto X, and $\rho(A) = r$. Then A and $B = I - A$ are idempotent matrices, and so B is regular. From (iii) we get that $\mathscr{B}(B)$ is nonempty. Let

$$T = \begin{bmatrix} B & P \\ Q & S \end{bmatrix} \in \mathscr{B}(B)$$

with inverse

$$T^{-1} = \begin{bmatrix} G & E \\ F & H \end{bmatrix}.$$

Then we get that $\begin{bmatrix} B \\ Q \end{bmatrix}$ has a left inverse. By Lemma 3 we can obtain a right inverse Q_R^{-1} of Q such that $BQ_R^{-1} = 0$ and $QQ_R^{-1} = I_{n-r}$. Since $[G \;\; E]$ is a left inverse of $\begin{bmatrix} B \\ Q \end{bmatrix}$, we get that

$$GB + EQ = I. \tag{3.8}$$

By multiplying both sides of (3.8) on the right by Q_R^{-1} we get that

$$E = Q_R^{-1}. \tag{3.9}$$

Since $(I - GB)(I - B) = I - B$ and $(I - B)(I - GB) = I - GB$, we get that $\text{Range}(I - GB) = \text{Range}(I - B)$, and this in turn gives us that

$\text{Range}(I - GB) = \text{Range}(A) = X$. From (3.8) we get that

$$\text{Range}(I - GB) = \text{Range}(EQ) = \text{Range}(E),$$

the last equality because Q has a right inverse. But $\text{Range}(E)$ is free because E has a left inverse. Thus, X is free. ■

COROLLARY 5. *Over a commutative ring $\mathbb{R}$ with identity, if every finitely generated projective module is free, then every $m \times k$ regular matrix of rank k can be completed to an $m \times m$ invertibe matrix.*

The proof follows from (i) ⇒ (iii) of the above theroem.

REMARK. Over a commutative ring, if for a matrix A there is a bordering

$$T = \begin{bmatrix} A & P \\ Q & S \end{bmatrix}$$

that is invertible, then T^{-1} is of the form

$$\begin{bmatrix} G & E \\ F & 0 \end{bmatrix}.$$

This follows from Lemmas 2 and 3 and their use in the proof of (iii) ⇒ (i) of Theorem 4.

REMARK. Clearly, for a regular matrix A, if G is a g-inverse of A, then $\text{Ker } A \cong \text{Range}(I - GA)$ and $\text{Coker}(A) \cong \text{Range } I - AG$. From the proof of the above theorem, it is clear that $\mathscr{B}(A)$ is nonempty if and only if $I_n - GA$ and $I_m - AG$ have rank factorizations for any g-inverse G of A. If $I_m - AG = B_{m \times (m-r)} C_{(m-r) \times m}$ and $I_n - GA = P_{n \times (n-r)} Q_{(n-r) \times n}$,

$$\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}$$

gives an invertible bordering of A. In other words, a regular matrix A over $\mathbb{R}$ has nonempty $\mathscr{B}(A)$ if and only if its kernel and cokernel are free.

EXAMPLE. Let $\mathbb{R}$ be an integral domain on which not every stably free module is free. Let S be such a stably free but not free module, i.e., there is a free module $\mathbb{R}^k$ such that $S \oplus \mathbb{R}^k = \mathbb{R}^n$. Let A be an $n \times n$ matrix obtained by projection from $\mathbb{R}^n$ onto S. Clearly, by considering A to be a matrix over the quotient field of $\mathbb{R}$ we get that $\rho(A) = n - k$. Since $I - A$ is projection onto $\mathbb{R}^k$, we get a rank factorization $I - A = PQ$, where P is $n \times k$ and Q is $k \times n$. Here A is an example of a matrix which does not have rank factorization [because Range$(A) = S$ is not free] but has nonempty $\mathscr{B}(A)$ which contains

$$\begin{bmatrix} A & P \\ Q & 0 \end{bmatrix}.$$

The matrix $I - A$ is an example of a matrix which has rank factorization but no bordering of the required type. In fact, if

$$\begin{bmatrix} I - A & R \\ S & T \end{bmatrix}$$

is an invertible matrix of size $2n - k \times 2n - k$ with inverse

$$\begin{bmatrix} G & X \\ Y & Z \end{bmatrix},$$

by looking into the proof of (iii) $\Rightarrow$ (i) we get that the range of A is free, a contradiction. This example shows that for a given matrix C, the existence of a rank factorization of C neither implies nor is implied by $\mathscr{B}(C) \neq \emptyset$.

COROLLARY 6. *Over any commutative ring $\mathbb{R}$ with identity, the statement that every regular matrix is of the form*

$$M\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}N$$

where M and N are invertible matrices over $\mathbb{R}$ is equivalent to any of (i), (ii), *and* (iii) *of Theorem* 4.

Proof. Any matrix of the form

$$M\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}N$$

where M and N are invertible matrices is easily vertified to have a rank factorization. Thus, the statement given in the corollary implies that every regular matrix over $\mathbb{R}$ has a rank factorization. Conversely, if A is a regular matrix, then from condition (ii) of Theorem 4, A has a rank factorization, say $A = BC$. Then from Corollary 5, the matrix B can be completed to an invertible matrix P of size $m \times m$, the matrix C can be completed to an invertible matrix Q of size $n \times n$, and we get that

$$A = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q.$$

Hence the corollary. ∎

REMARK. Part (iii) of Theorem 4, namely, "for every regular matrix A, $\mathscr{B}(A)$ is nonempty," is equivalent to the statement that for every $m \times n$ regular matrix A of rank r, there is an $m \times (m - r)$ matrix P such that $[A, P]$ is right invertible.

REMARK. Corollary 6 generalizes a result (Theorem 1 of [13]) of Sontag. In [13] Sontag showed that over the ring of polynomials in several variables with complex coefficients, being regular is the same as having Smith normal form with $S = I_r$.

As an application of our result we shall prove the following result.

THEOREM 7. *If $\mathbb{R}$ is a commutative ring with the property that for every finitely generated subring $\mathbb{A}$ of $\mathbb{R}$ with identity there is a projective free subring $\mathbb{R}'$ such that $\mathbb{A} \subset \mathbb{R}' \subset \mathbb{R}$, then $\mathbb{R}$ is projective free.*

Proof. Consider an $\mathbb{R}$-projective module P, and let Q be an $\mathbb{R}$-module such that $P \oplus Q \simeq \mathbb{R}^n$. Let A be an $n \times n$ matrix such that $A : \mathbb{R}^n \to P$ is the natural projection map. Consider the subring $\mathbb{A}$ generated by 1 and a_{ij} $(1 \leqslant i, j \leqslant n)$. By hypothesis, there is a subring $\mathbb{R}'$ such that $\mathbb{A} \subset \mathbb{R}' \subset \mathbb{R}$, and $\mathbb{R}'$ is projective free. Since $\mathbb{R}'$ is projective free and A is regular over $\mathbb{R}$ and $\mathbb{R}'$, from (i) $\Leftrightarrow$ (ii) of Theorem 4 we get that A has a rank factorization over $\mathbb{R}'$. Since $\mathbb{R}'$ is a subring of $\mathbb{R}$, A has rank factorization over $\mathbb{R}$, which in turn implies that P is free. Hence we get that $\mathbb{R}$ is projective free. ∎

Now, using our results and Quillen and Suslin's theorem, we shall show that $\mathbb{P}[X_1, X_2, \dots]$ is projective free for any principal ideal domain $\mathbb{P}$, thus extending Quillen and Suslin's result to infinitely many variables.

COROLLARY 8. *If $\mathbb{P}$ is a principal ideal domain, then $\mathbb{P}[X_1, X_2, \dots]$ is projective free.*

Proof. Quillen in [12] proved that $\mathbb{P}[X_1, X_2, \dots, X_n]$ is projective free (Serre's conjecture) for all n. For any finitely generated subring $\mathbb{A}$ of $\mathbb{P}[X_1, X_2, \dots]$ we can find indices $i_1, i_2, \dots, i_k$ such that $\mathbb{A}$ is in $\mathbb{P}[X_{i_1}, X_{i_2}, \dots, X_{i_k}]$. So, from the above Theorem 7, we get that $\mathbb{P}[X_1, X_2, \dots]$ is projective free. ∎

4. MINORS OF REFLEXIVE GENERALIZED INVERSES

Now we shall give a method of computing the minors of a reflexive g-inverse G of an $m \times n$ matrix A of rank r. In the process, surprisingly, we obtain a formula that is similar to (2.2) for computing any $k \times k$ minor $(k \leqslant r)$ of G. In particular, in the case of a 1×1 minor of G we obtain Theorem 3 of [1]. In fact, if

$$T = \begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}$$

is an invertible matrix of size $(m + n - r) \times (m + n - r)$ with inverse

$$\begin{bmatrix} G & P \\ C & 0 \end{bmatrix}$$

such that $I - AG = BC$ and $I - GA = PQ$ as in (ii) ⇒ (iii) of Theorem 4, then we obtain that the $r \times r$ minor $|G_\alpha^\beta|$ of G is

$$(-1)^{s(\alpha)+s(\beta)+(m-r)(n-r)} |B^{\alpha^c}| |Q_{\beta^c}|,$$

where $s(\alpha) = \sum_{i=1}^{r} \alpha_i$, $s(\beta) = \sum_{i=1}^{r} \beta_i$, $\alpha^c = \{1, 2, \dots, m\} \setminus \alpha$, and $\beta^c = \{1, 2, \dots, n\} \setminus \beta$.

In the following theorem, for convenience, we shall consider the case $\rho(A) < \min\{m, n\}$, in which case $I - AG$ and $I - GA$ are nonzero.

THEOREM 9. *Let A be an $m \times n$ matrix of rank $r < \min\{m, n\}$. Let G be a reflective g-inverse of A. If BC, PQ are rank factorizations of $I - AG$ and $I - GA$ respectively, then:*

(i) *The determinant*

$$\det\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix} = \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)+(m-r)(n-r)} |B^{\alpha'}||Q_{\beta'}||A_\beta^\alpha|$$

is a linear combination of $r \times r$ minors of A.

(ii) *for any k-element subsets γ of $(1, 2, \ldots, m)$ and δ of $(1, 2, \ldots, n)$ $(k < r)$,*

$$|G_\gamma^\delta| = \left(\det\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}\right)^{-1}$$

$$\times \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)+(m-r)(n-r)} |B^{\alpha'}||Q_{\beta'}| \frac{\partial}{\partial |A_\delta^\gamma|} |A_\beta^\alpha|,$$

where $(\partial/\partial|A_\delta^\gamma|)|A_\beta^\alpha|$ is the cofactor of $|A_\delta^\gamma|$ in the determinantal expansion of $|A_\beta^\alpha|$. In particular,

$$|G_\alpha^\beta| = \left(\det\begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}\right)^{-1} (-1)^{s(\alpha)+s(\beta)+(m-r)(n-r)} |B^{\alpha'}||Q_{\beta'}|,$$

the (j, i)th element of G is

$$g_{ji} = \sum_{\alpha, \beta} |G_\alpha^\beta| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|,$$

and

$$|G_\gamma^\delta| = \sum |G_\alpha^\beta| \frac{\partial}{\partial |A_\delta^\gamma|} |A_\beta^\alpha|.$$

Proof. Since $I - AG$ $(= BC)$ and $I - GA$ $(= PQ)$ have rank factorizations, from (ii) $\Rightarrow$ (iii) of Theorem 4 we get that

$$T = \begin{bmatrix} A & B \\ Q & 0 \end{bmatrix}$$

is invertible with

$$T^{-1} = \begin{bmatrix} G & P \\ C & 0 \end{bmatrix}.$$

Since B is a full column rank matrix and Q is a full row rank matrix, using Laplace expansion (refer to Grantmacher [6]), we get that

$$|T| = \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)+(m-r)(n-r)} |B^{\alpha^c}| \, |Q_{\beta^c}| \, |A^{\alpha}_{\beta}|.$$

Let γ be any k-element subset of $\{1, 2, \dots, m\}$, and δ be any k-element subset of $\{1, 2, \dots, n\}$; then

$$|(T^{-1})^{\delta}_{\gamma}| = |T^{-1}| \left(\frac{\partial}{\partial \, |T^{\gamma}_{\delta}|} |T| \right).$$

Again from the structure of T and Laplace expansion, we can prove part (ii) of the theorem. ∎

REMARK. If $I - AG = 0$, it can be seen easily as in the previous theorem that

$$T = \begin{bmatrix} A \\ Q \end{bmatrix}$$

is invertible,

$$\det \begin{bmatrix} A \\ Q \end{bmatrix} = \sum_{\alpha, \beta} (-1)^{s(\beta)} |Q_{\beta^c} \mid A_{\beta}|,$$

and

$$|G^{\delta}| = \left(\det \begin{bmatrix} A \\ Q \end{bmatrix} \right)^{-1} \sum_{\beta} (-1)^{s(\beta)} |Q_{\beta^c}| \frac{\partial}{\partial \, |A_{\delta}|} |A_{\beta}|.$$

The case $I - GA = 0$ is also similar.

REMARK. Theorem 9 generalizes Theorem 3 of [9]. This is because [9] deals with only the matrices over fields or integral domains.

REFERENCES

1 R. B. Bapat, K. P. S. Bhaskara Rao, and K. Manjunatha Prasad, Generalized inverses over integral domains, *Linear Algebra Appl.* 140:181–196 (1990).

2 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses and Applications*, Wiley, 1974.

3 K. P. S. Bhaskara Rao, On generalized inverses of matrices over integral domains, *Linear Algebra Appl.* 49:179–189 (1983).

4 K. P. S. Bhaskara Rao, Generalized inverses of matrices over rings, unpublished manuscript.

5 J. W. Blattner, Bordered matrices, *J. Soc. Indust. Appl. Math.* 19:528–536 (1962).

6 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1977.

7 A. J. Goldman and M. Zelen, Weak generalized inverse and minimum variance linear unbiased estimation, *J. Res. Nat. Bur. Standards Sect. B* 68:151–172 (1964).

8 N. Jacobson, *Basic Algebra*, Vols. I, II, Freeman, 1974.

9 Jianming Miao, Reflexive generalized inverses and their minors, *Linear and Multilinear Algebra*, to appear.

10 T. Muir, *A Treatise on the Theory of Determinants*, Dover, 1960.

11 K. Nomakuchi, On the characterization of generalized inverses by bordered matrices, *Linear Algebra Appl.* 33:1–8 (1980).

12 Daniel Quillen, Projective modules over polynomial rings, *Invent. Math.* 36:167–171 (1976).

13 E. D. Sontag, On generalized inverse of polynomial and other matrices, *IEEE Trans. Automat. Control* AC-25(3):514–517 (1980).

14 C. R. Rao and S. K. Mitra, *Generalized Inverses of Matrices and Applications*, Wiley, 1974.

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