On the cardinality of certain Hausdorff spaces*

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1. Proof of Theorem 1

Let $P_\kappa$ denote the set $\omega\kappa$ endowed with the product topology of $D(\kappa)^\omega$.

**Lemma 1.** Assume $X$ is weakly $\kappa$-good, $|X| > \kappa$ and $\lambda \leq \kappa^+$. Then there are, a $Y \subset X$ and a bijection $\Phi$ of $Y$ onto $P_\lambda$ such that $\Phi$ sends each compact subset of $Y$ onto a closed subset of $P_\lambda$.

Note that if $\lambda^\omega > \kappa$, $Y$ is a weakly $\kappa$-good space of cardinality $>\kappa$ as well. In the proof of Theorem 1 we will need the case $\lambda = \kappa^+$.

**Proof.** For each $n < \omega$ and $\varphi \in {}^\omega\lambda$ we define a compact subset $C_\varphi$ of $X$, $|C_\varphi| > \kappa$, by induction on $n$ as follows. $C_0$ is an arbitrary compact subset of $X$ with $|C_0| > \kappa$. Assume $C_\varphi$ is defined for all $\varphi \in {}^\omega\lambda$ in such a way that $|C_\varphi| > \kappa$. Choose a partition $C_\varphi = \bigcup_{\alpha < \lambda} C_{\varphi, \alpha}$ of $C_\varphi$ with $|C_{\varphi, \alpha}| > \kappa$,

and for each $\alpha < \lambda$ let $C_{\varphi, \alpha}$ be a compact subset of cardinality greater than $\kappa$ of $C_{\varphi, \alpha}$. This defines the subsets $C_\varphi$ for $\varphi \in {}^{n+1}\lambda$. For $f \in P_\lambda$ let $C_f = \bigcap_{n \in \omega} C_{f|n}$. The sets $C_f$ are non-empty and pairwise disjoint. Let $y_f \in C_f$, $Y = \{y_f : f \in P_\lambda\}$ and $\Phi(y_f) = f$ for $f \in P_\lambda$.

Let now $Z \subset Y$ be a compact set, $A = \Phi(Z)$. We prove that $A = \Phi(Z)$ is closed in $P_\lambda$. Indeed, assume $f \in P_\lambda$ is in the closure of $A$. For $n \in \omega$ choose an $f_n \in A$ with $f_n|n = f|n$. Then $\Phi^{-1}(f_n) = y_{f_n} \in Z$ for $n \in \omega$. We may assume that $\{y_{f_n} : n \in \omega\}$ is infinite, hence it has an accumulation point $z$ in $Z \subset Y$. Then $z = y_g$ for some $g \in P_\lambda$. For every fixed $m \in \omega$, $\{y_{f_n} : m \leq n < \omega\} \subset C_{f|m}$, hence $z \in C_{f|m}$, and consequently $z = y_g \in C_f$. It follows that $f = g$ and $f \in \Phi(Z) = A$. □

**Definition.** For $f, g \in {}^{\omega}\lambda$ write $f < g$ iff $\{n \in \omega : f(n) \geq g(n)\}$ is finite.

**Lemma 2.** Assume $\omega \leq \tau$. Let $A \subset P_\lambda$ be a subset such that $A$ is well-ordered by $<$ defined above, and $\text{typ } A(<) = \tau$. Then $A$ is either not closed or not $\tau$-dense in itself (in the topology of $P_\lambda$).

**Proof.** Assume indirectly that $A$ is both closed and $\tau$-dense in itself. We define for $n \in \omega$ an integer $k_n$ and functions $g^n_i \in {}^{k_i+1}\lambda$ for $i \leq n$ by induction on $n$ as follows.

Let $h_0^n$ be an arbitrary element of $A$, $k_0 = 0$ and $g_0^n = h_0^1|1$. Assume $g^n_i : i \leq n$ has already been defined in such a way that for $U^n_i = \{f \in A : f|k_n + 1 = g^n_i\}$
we have

\[ |I_i^n| = \tau \quad \text{for } i \leq n. \]

Let \( h^n_{n+1} \) be an arbitrary element of \( A \). By the assumption, we can choose

\( h^n_{n+1} \in U^n_i \) for \( i \leq n \) in such a way that \( h^n_{n+1} < \cdots < h^n_0 \) holds. Then there is an integer \( k_{n+1} > k_n \) such that

\[ h^n_{n+1}(k_{n+1}) < \cdots < h^n_0(k_{n+1}) \]

holds. Let \( g^{n+1}_i = h^{n+1}_i \mid (k_{n+1} + 1) \) for \( i \leq n + 1 \).

As \( A \) is \( \tau \)-dense in itself, we have \( |U^{n+1}_i| = \tau \) for \( i \leq n + 1 \) and the definition is complete. It follows now by induction on \( n \) that:

(i) \( g^n_i \subset g^{n+1}_i \) for \( i \leq n \)

(ii) \( g^n_i(k_i) > g^n_j(k_i) \) for \( i < j \leq n \).

By (i), we can define \( f_i = \bigcup \{ g^n_i : i \leq n < \omega \} \) for \( i < \omega \). \( A \) being closed, \( f_i \in A \) for \( i < \omega \). On the other hand, for \( i < j < \omega \) we have

\[ f_i(k_i) > f_j(k_i) \]

hence for infinitely many \( k_i \). Then \( f_j < f_i \) for \( i < j < \omega \), contradicting the fact that \( A \) is well-ordered by <. 0

**Lemma 3.** There is a subset \( B \subset P_{\kappa^+}, |B| = \kappa^+ \), such that \( B \) is not the union of \( \kappa \) closed subsets of \( P_{\kappa^+} \).

**Proof.** Let \( T = \{ \alpha < \kappa^+: \text{cf}(\alpha) = \omega \} \). \( T \) is a stationary subset of \( \kappa^+ \). For each \( \alpha \in T \) choose an \( f_\alpha \in P_{\kappa^+} \) such that \( f_\alpha(0) < \cdots < f_\alpha(n) < \cdots \)

and

\[ \sup\{ f_\alpha(n) : n \in \omega \} = \alpha. \]

Let \( B = \{ f_\alpha : \alpha \in T \} \). Note that the set \( B = B_{\kappa^+} \) was first used for an interesting combinatorial argument in Baumgartner [2].

For a subset \( S \subset T \), let \( \Phi(S) = \{ f_\alpha : \alpha \in S \} \subset B \). As \( \Phi \) is a bijection of \( \mathcal{P}(T) \) onto \( \mathcal{P}(B) \), it is clearly sufficient to prove that for a stationary set \( S \subset T \), \( \Phi(S) \) can not be closed. Note first that \( B \) is well-ordered by < in the order type \( \kappa^+ \), hence the same holds for \( \Phi(S) \) provided \( |S| = \kappa^+ \). Assume now indirectly that for some stationary \( S \subset T \), \( C = \Phi(S) \) is closed. For a \( \varphi \in \kappa^+ \), \( n < \omega \), let \( U_{\varphi} = \{ f \in P_{\kappa^+} : f \upharpoonright n = \varphi \} \) be the open set induced by \( \varphi \).

Let \( D = C \cap \bigcup \{ U_{\varphi} \cap C : \Phi^{-1}(U_{\varphi} \cap C) \) is nonstationary in \( \kappa^+ \} \). \( D \) is clearly closed in \( P_{\kappa^+} \). We claim that \( |D| = \kappa^+ \) and \( D \) is \( \kappa^+ \)-dense in itself. As \( D = \Phi(S') \) for some \( S' \subset S \), it is sufficient to see that \( S \setminus S' \) is nonstationary. For each \( \alpha \in S \setminus S' \), choose a \( \varphi_{\alpha} \subset f_{\alpha} \) such that \( \Phi^{-1}(U_{\varphi_{\alpha}} \cap C) \) is nonstationary. Then \( \varphi_{\alpha} \subset f_{\alpha} \), and range \( (\varphi_{\alpha}) \subset \alpha \) is finite. If we assume indirectly that \( S \setminus S' \) is stationary, then by Fodor's theorem there is a \( \varphi \) such \( \varphi_{\alpha} = \varphi \) for stationary many \( \alpha \), a contradiction to the definition of \( \varphi_{\alpha} \).

Now \( D \) is closed, \( \kappa^+ \)-dense in itself (since it is even stationary-dense in itself) and well-ordered by < in ordertype \( \kappa^+ \). By Lemma 2, this is not possible. 0
Proof of Theorem 1. Assume indirectly that $X$ is $\kappa$-good and $|X| > \kappa$. By Lemma 1, applied with $\lambda = \kappa^+$, every subset of $P_\kappa$ is the union of $\kappa^+$ closed subsets of $P_\kappa$. However, by Lemma 3, this is false for $B$. \quad \square

2. Some results on weakly $\kappa$-good spaces

Let us first mention that the following is the strongest possible generalization of Theorem 1.

$\ast\ast(\kappa)$ Assume $X$ is an infinite Hausdorff space, $|X| = \kappa \geq \omega$. Then there is a $Y \subset X$, $|Y| = \kappa$ such that all compact subspaces of $Y$ have cardinality smaller than $\kappa$.

We formulate a set of theoretical principle.

$\ast\ast(\alpha)$ $P_\alpha$ contains a subset $A = \{f_\alpha: \alpha < \lambda^\omega\}$ such that for $\alpha < \beta < \lambda^\omega$ there is an $n \in \omega$ with $f_\alpha(n) < f_\beta(n)$.

It follows from results of Shelah [6, Chapter XIII § 5] that $\ast\ast(\lambda)$ holds provided $\text{cf}(\lambda) = \omega$ and $\lambda$ is smaller than the first fixed point of the $\kappa$ function, i.e., the smallest $\alpha$ with $\kappa_\alpha = \alpha$. To be a little more explicit, let $D$ be an ultrafilter on $\omega$ and write

$$f \leq_D g \text{ for } f, g \in \omega^\omega \iff \{n \in \omega: f(n) < g(n)\} \in D.$$ 

It is proved in [6] that under the above conditions on $\lambda$, there are an $A = \{f_\alpha: \alpha < \lambda^\omega\} \subset P_\lambda$ and an ultrafilter $D$ such that $f_\alpha <_D f_\beta$ for $\alpha < \beta < \lambda^\omega$ and this is a much stronger statement than $\ast\ast(\lambda)$.

**Theorem 2.** Assume $\ast\ast(\lambda)$ holds for all $\omega \leq \lambda < \kappa$ with $\text{cf}(\lambda) = \omega$. Then $\ast\ast(\kappa)$ holds.

**Corollary 1.** If $\kappa$ is smaller than the first fixed point of the $\kappa$ function then every weakly $\kappa$-good Hausdorff space $X$ has cardinality at most $\kappa$.

To prove Theorem 2, we need the following.

**Lemma 4.** Assume $|X| = \kappa = \kappa^\omega$ and $\lambda^\omega < \kappa$ for $\lambda < \kappa$. Then there is a $Y \subset X$, $|Y| = \kappa$ such that $Y$ has no compact subspace of size $\kappa$.

**Proof.** This is essentially Theorem 3 of [3]. For the convenience of the reader we outline a proof.

For a countable $D \subset X$ choose a $p(D) \in \bar{D} - D$ if such a point exists. By the assumptions $\lambda^\omega < \kappa = \kappa^\omega$ for $\lambda < \kappa$, we can choose a sequence $\{p_\alpha: \alpha < \kappa\}$ in such a way that for

$$P_\alpha = \{p_\beta: \beta < \alpha\} \quad \text{and} \quad Q_\alpha = \{p(D): D \in [P_\alpha]^\omega\}, \quad p_\alpha \notin P_\alpha \cup Q_\alpha.$$ 

Let $Y = \{p_\alpha: \alpha < \kappa\}$. We may assume that $Y$ has no subset of size $\kappa$ which is right
separated or else we are home. Let now $Z$ be a compact subset of $Y$ of size $\kappa$. Note that each initial segment of $Z$ is countably compact. Then, by Theorem 1 of [4], (see also the remark on p. 61 in [4]), $Z$ is right separated, a contradiction. 

**Proof of Theorem 2.** Assume $|A| = \kappa$, and assume indirectly that all $Y \subset \kappa$ with $|Y| = \kappa$ contains a compact subspace of size $\kappa$. It follows just like in the proof of Lemma 1 that $\kappa^\omega \leq |X| = \kappa$, hence $\kappa^\omega = \kappa$. If $\lambda^\omega < \kappa$ for $\lambda < \kappa$ then we are home by Lemma 4. Hence $\lambda^\omega = \kappa$ for some $\lambda < \kappa$. The minimal such $\lambda$ has cofinality $\omega$, hence we may assume $\text{cf}(\lambda) = \omega$. By Lemma 1, then each subspace of size $\kappa$ of $P_\lambda$ contains a closed subset of size $\kappa$. We will show that this can not be the case. We distinguish two cases.

**Case (i):** $\lambda^+ = \kappa$. It is well known and easy that by $\text{cf}(\lambda) = \omega$, there is an $A = \{ f_\alpha : \alpha < \kappa \} \subset P_\lambda$ such that $f_\alpha < f_\beta$ for $\alpha < \beta < \kappa$. $A$ contains no closed subset of size $\kappa$, because the weight of $P_\lambda$ is $\lambda < \kappa$ and each closed subset $B$ of $A$ of size $\kappa$ contains a closed $C \subset B$ $\kappa$-dense in itself. The existence of such a $C$ contradicts Lemma 2.

**Case (ii):** $\lambda^+ < \kappa$. Let $A = \{ f_\alpha : \alpha < \kappa \}$ satisfy the requirements of (3)(\lambda). Assume indirectly that $B = \{ f_\alpha : \alpha < K \}$ is closed for some $K | = \kappa$. Let $F(\alpha) = \{ \beta \in K : \forall n \in \text{w}_B(n) \leq f_\alpha(n) \}$.

By the choice of $A$, $F(\alpha) \in \alpha + 1$, hence $|F(\alpha)| < \kappa$. On the other hand, $\{ f_\beta : \beta \in F(\alpha) \}$ is a closed subset of $P_\lambda$. As all closed subsets of $P_\lambda$ of size $\lambda^\omega < \kappa$ have cardinality $\kappa^\omega = \kappa$, we get that $|F(\alpha)| \leq \lambda$ for $\alpha < \kappa$.

By Hajnal’s set mapping theorem [5] and by $\lambda^+ < \kappa$ there is a subset $K_0 \subset K$ such that $\beta \notin F(\alpha)$ for $\alpha, \beta \in K_0$ and $\beta < \alpha$. We may assume that $2^\omega < \kappa$ otherwise $2^\omega = 2^\omega$ and the theorem becomes trivial. Let $K_1 \subset K_0$, $|K_1| = (2^\omega)^+$. We define a partition $\Phi$ of the pairs $\{ \alpha, \beta \} \in K$, $\beta < \alpha$ with countably many colours:

$$\Phi(\{ \alpha, \beta \}) = n \leftrightarrow n = \min(m \in \omega : f_\beta(m) > f_\alpha(m)).$$

By the Erdős–Rado theorem we have $(2^\omega)^+ \rightarrow (\omega)^2$ and we get a sequence $\{ f_\alpha : k < \omega \}$, $\alpha_0 < \cdots < \alpha_k < \cdots$ and an $n < \omega$ such that $\{ \alpha_k : k < \omega \}$ is homogeneous for the partition $\Phi$ in the colour $n$, i.e., $f_\alpha(n) > f_{\alpha_k}(n)$ for $k < \omega$.

This is a contradiction to the indirect assumption. 

**References**


