Uniform Approximation by Nevai Operators

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The author establishes new direct and converse results for the weighted and unweighted uniform approximation by some rational operators of Nevai type.

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1. INTRODUCTION

Let \( \psi(x) \) be a generalized smooth Jacobi weight function (we write \( \psi \in \text{GSJ} \)) defined by

\[
\psi(x) = (1-x)^\gamma \prod_{k=1}^{q} |x-t_k|^\gamma(1+x)^\delta, \quad x \in (-1, 1),
\]

where \(-1 < t_1 < \ldots < t_q < 1, \gamma, \delta, \gamma_k > -1, k = 1, 2, \ldots, q \) and \( 0 < \psi \in \text{Lip}_M \).

Further, let \( \{p_n(\psi)\}_{n=0}^\infty \) be the corresponding system of orthonormal polynomials associated with the weight function \( \psi \) and denote by \( x_n, k = x_k \), \( k = 1, \ldots, n \), the zeros of \( p_n(\psi) \) in natural order.

Then let \( \lambda_n(x) \) be the \( n \)th Christoffel function corresponding to the weight \( \psi \) defined by

\[
\lambda_n(x) = \lambda_n(\psi; x) = \left[ \sum_{k=1}^{n} \frac{\ell_n^2(x; x)}{\lambda_n^2(x)} \right]^{-1},
\]

where \( \ell_n^2(x; x) \) is the \( n \)th Christoffel function.
where \( \ell_{n,k}(x) = p_n(x)/p'_n(x_k)(x-x_k) \), \( k = 1, \ldots, n \), are the fundamental Lagrange polynomials and \( \lambda_{n,k} = \lambda_n(u; x_k) = \lambda_k \), \( k = 1, \ldots, n \), are the corresponding Cotes numbers.

Then for every function \( f \) defined in \([-1, 1]\] consider the Nevai operator \( N_n \) given by

\[
N_n(f; x) = \frac{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_{n,k}^{s/2}} f(x_k)}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_{n,k}^{s/2}}}, \quad x \in [-1, 1], s \geq 2. \tag{3}
\]

From the definition (3), it follows that \( N_n \) is a positive operator interpolating \( f \) at the nodes \( x_k \), \( k = 1, \ldots, n \), it preserves constant functions and, if \( s \) is an even integer, \( N_n(f) \) is a rational function.

In the particular case \( s = 2 \), this operator coincides with the operator \( F_n \) introduced and studied by Nevai in [12]. When \( s = 2 \), Criscuolo et al. in [3] obtained pointwise error estimates for \( N_n \), involving the usual modulus of continuity of \( f \). Some weak asymptotic relations were also given in [3].

In [5] Della Vecchia and Mastroianni introduced a modification of \( N_n \) operator and they proved pointwise simultaneous approximation error estimates of Gopengauz-Teliakovskii type. We also remark that \( N_n \) belongs to a more general class of linear, positive, rational interpolatory operators introduced and studied by Criscuolo and Mastroianni in [2] (see also [5]). In particular in [2] a uniform convergence result of Korovkin type for \( N_n \) was established. An expression of \( N_n \) in terms of \( H_n \), with \( H_n \) the Hermite–Fejér interpolating polynomial operator, was also showed in [2]. Moreover \( N_n \) is related to Shepard operator \( S_n \) (see (34)).

Operators \( N_n \) are of interest in applications because they can be used in approximating Christoffel functions corresponding to non-classical weight functions.

In this paper we want to investigate the more general weighted approximation case, when the function \( f \) may be unbounded at \( \pm 1 \).

First we show that, similarly as for polynomials, for the operators \( N_n \) the weighted convergence with Pollaczek type weights is not guaranteed in general (Proposition 2.1). Therefore here we consider weights vanishing algebraically at \( \pm 1 \), i.e. functions having an algebraic singularity at \( \pm 1 \). For such functions we give weighted uniform approximation estimates by \( N_n \) involving a suitable modulus of smoothness. We also establish converse results (Theorem 2.1). Useful tools for our results are new weighted Markov–Bernstein inequalities for \( N_n \) (Lemmas 3.2–3.3). We also show that our results are sharp in some sense (see remarks to Theorem 2.1).

In the particular case of the unweighted approximation, i.e., if \( f \in C([-1, 1]) \), we obtain more precise direct and converse results.
Finally the difficult problem of saturation of $N_n$ for $s \geq 2$ is investigated, and when $s > 2$, it is solved (Theorems 2.2 and 2.3).

2. MAIN RESULTS

Letting

$$w(x) = (1 - x^2)^\alpha, \quad \alpha > 0, \ x \in [-1, 1], \tag{4}$$

we consider functions $f$ locally continuous on $(-1, 1)$ ($f \in C_{loc}((-1, 1))$) such that

$$\lim_{x \to \pm 1} w(x) f(x) = 0. \tag{5}$$

Here we want to study the weighted uniform convergence of $N_n(f)$ to $f$, with $N_n$ defined by (3) and $w$ given by (4), i.e., the convergence behaviour of $w(x)f(x) - N_n(f; x)$, for $|x| \leq 1$. First we remark that we have to consider weights of type (4) for the weighted approximation by $N_n$, since for Pollaczek type weights the convergence is not guaranteed in general (cf. [6] for analogous behaviour of Shepard operator).

Indeed, putting $\|wf\|_{[a, b]} = \sup_{x \in [a, b]} w(x)|f(x)|$ and $\|wf\| = \sup_{|x| \leq 1} w(x)|f(x)|$, we have

**Proposition 2.1.** Let $u(x) = (1 - x^2)^\gamma, \ \gamma > -1$ (i.e. $u$ is given by (1) with $\gamma = \delta$ and $\gamma_1 = \gamma_2 = \cdots = \gamma_q = 0$). Moreover put $W(x) = \exp(-1/(1 - x^2))$ and $f(x) = \exp((-1 - x^2)^{-1/2})$. Then

$$\lim \sup_n \|WN_n(f)\| = +\infty. \tag{6}$$

Now let

$$\omega^\gamma(f; t)_w = \sup_{0 < h < t} \|wA_{t^2} f\|_{[-1 + t^2, 1 - 2t^2]} + \sup_{0 < h < 2t^2} \|wA_{t^2} f\|_{[-1 - t^2, 1 + 2t^2]}$$

$$+ \sup_{0 < h < 2t^2} \|wA_{h^2} f\|_{[-1 - 2t^2, 1]},$$

$$A_{t^2} f(x) = f(x + \frac{\varphi(x)}{2}) - f(x - \frac{\varphi(x)}{2}),$$

$$A_{t^2} f(x) = f(x + h) - f(x),$$

$$A_h f(x) = f(x) - f(x - h),$$

be the weighted modulus of smoothness of first order of $f$ with step function $\varphi(x) = \sqrt{1 - x^2}$ and $w$ given by (4) (cf. [8, formula (8.2.10), p. 97]).
In the following $C$ denotes a positive constant which may assume different values in different formulas. Moreover let $v \sim \mu$, for $v$ and $\mu$ two quantities depending on some parameters, if $|v/\mu|^{\pm 1} \leq C$, with $C$ independent of the parameters. Then we give the following direct and converse result.

**Theorem 2.1.** Let $s \geq 2\alpha + 2$. If $f$ satisfies condition (5), then

$$\|w[f - N_n(f)]\| \leq C \omega^s \left( f; \frac{1}{n} \right)_n$$  \hspace{1cm} (7)

and

$$\omega^s \left( f; \frac{1}{n} \right)_n \sim \|w[f - N_n(f)]\| + \frac{1}{n} \|w\varphi N_n'(f)\|.$$  \hspace{1cm} (8)

In addition

$$\|w[f - N_n(f)]\| = O(n^{-\beta}) \Leftrightarrow \omega^s(f; t)_n = O(t^\beta), \quad 0 < \beta < 1.$$  \hspace{1cm} (9)

**Remark.** From (7) we deduce the weighted uniform convergence of the operator $N_n$, if $s \geq 2\alpha + 2$. As expected, our error estimates are strongly affected by the mesh distribution (see the presence of the function $\varphi$ on the right-hand side of (7)).

We remark that such results can also be obtained by polynomial operators (cf. [8]) (which however are not positive), while classical positive operators of Bernstein-type give a poorer rate of convergence (cf. [8]) and do not interpolate.

From (8), by (7) we deduce (see formula (45))

$$\|w\varphi N_n'(f)\| \leq C n \omega^s \left( f; \frac{1}{n} \right)_n.$$  \hspace{1cm} (10)

We remark that an analogous estimate holds true for the best weighted polynomial approximation to $f$ (see [8]).

From (8), since [8]

$$\omega^s \left( f; \frac{1}{n} \right)_n \sim K^s \left( f; \frac{1}{n} \right)_n$$  \hspace{1cm} (11)

with $K^s(f)_n$ the weighted $K$-functional, it follows that

$$\inf_{h \in C^0_{c,n}((-1,1)) \atop \|w[h]\| < +\infty} \{ \|w[f - h]\| + \frac{1}{n} \|w\varphi h'\| \} \sim \|w[f - N_n(f)]\| + \frac{1}{n} \|w\varphi N_n'(f)\|.$$  \hspace{1cm} (12)
in other words the infimum at the left-hand side in (12) is essentially realized by \( N_n(f) \).

Moreover, (7) cannot be improved because of (9). In a sense, equivalence relation (9) characterizes the class of functions satisfying (5) and having a given behaviour near \( \pm 1 \) by the order of approximation by \( N_n \) operator.

When the function \( f \) is continuous on the whole interval \([-1, 1]\), we can give more precise direct and converse results, solving the saturation problem of \( N_n \), for \( s > 2 \). Indeed

**Theorem 2.2.** Let \( s > 2 \) and \( f \in C([-1, 1]) \). Then

\[
\|f - N_n(f)\| \leq C \omega^\phi \left( f; \frac{1}{n} \right) \tag{13}
\]

and

\[
\omega^\phi \left( f; \frac{1}{n} \right) \sim \|f - N_n(f)\| + \frac{1}{n} \|\phi N_n(f)\|. \tag{14}
\]

In addition if \( f \neq \text{constant} \)

\[
\limsup_{n \to +\infty} \frac{\|N_n(f) - f\|}{\omega^\phi \left( f; \frac{1}{n} \right)} \sim 1, \tag{15}
\]

where the sign \( \sim \) does not depend on \( f \).

Moreover

\[
\|N_n(f) - f\| = o \left( \frac{1}{n} \right) \iff f \text{ is a constant}, \tag{16}
\]

\[
\|N_n(f) - f\| = O \left( \frac{1}{n} \right) \iff \omega^\phi(f; t) \leq Ct. \tag{17}
\]

**Remarks.** First note that direct estimate (13) cannot be improved because of (15).

Estimation (15) is a counterpart of (13) and has a character similar to the result of Totik [16, (1.2)]

\[
\|B_n(f) - f\|_{[0, 1]} \sim \omega^\phi_2 \left( f; \frac{1}{\sqrt{n}} \right), \tag{18}
\]

with \( B_n(f) \) the \( n \)th Bernstein polynomial, \( f \in C([0, 1]) \), \( \| \cdot \|_{[0, 1]} \) the usual supremum norm on \([0, 1]\) and \( \omega^\phi_2 \) the second modulus of smoothness of
Ditzian and Totik with \( \psi(x) = \sqrt{x(1-x)} \). However, due to the interpolatory character of \( N_n \), we cannot get the estimation (15) with “lim” (instead of “lim sup”) as a consequence of a result stated by Della Vecchia et al. in [7, p. 77] (cf. also [17, Theorem 2.1, p. 310]).

Estimation (15) combined with the equivalence relation (see, e.g., [8]) \( \omega^*(f; t) \sim K^*(f; t) \), with \( K^*(f) \) the \( K \)-functional with step-function \( \varphi \), can serve as a characterization of such \( K \)-functionals.

Finally we remark that (16)–(17) handle the saturation problem for \( N_n \) with \( s > 2 \).

We remark that the assumption \( s > 2 \) in Theorem 2.2 is essential: indeed the case \( s = 2 \) presents additional difficulties because we do not have strong localization theorems like for the case \( s > 2 \). Some contributions to the saturation problem of \( N_n(f) \) if \( s = 2 \) were given by Criscuolo et al. in [3]. In particular (cf. [3])

\[
\begin{align*}
\omega(f; t) &\leq Ct^a, \ 0 < a < 1 \Rightarrow \|f - N_n(f)\| = O(n^{-b}), \\
\omega(f; t) &\leq Ct \Rightarrow \|f - N_n(f)\| = O\left(\frac{\log n}{n}\right),
\end{align*}
\]  

with \( \omega(f) \) the usual modulus of continuity of \( f \).

Here we have

**Theorem 2.3.** Let \( s = 2 \). Then the following implications hold

\[
\begin{align*}
\omega^*(f; t) &\leq Ct^b, \ 0 < b < 1 \Rightarrow \|f - N_n(f)\| \leq C n^{-b}, \\
\omega^*(f; t) &\leq Ct \Rightarrow \|f - N_n(f)\| \leq C \frac{\log n}{n},
\end{align*}
\]

\[
\|f - N_n(f)\| = O(n^{-1}) \Rightarrow \omega^*(f; t) \leq Ct^b, \quad \forall \beta \in (0, 1).
\]

**Remark.** The estimate (22) seems exact, in the sense that, following [11, pp. 11–14] we can find a function \( f \) such that \( \omega^*(f; t) \leq Ct \) implies that

\[
\|f - N_n(f)\| \geq C \frac{\log n}{n}.
\]

3. PROOFS OF THE MAIN RESULTS

First we give some preliminary results which will be useful in the sequel.
We recall that if $x_0 = -1$, $x_{n+1} = 1$ and $x_k$, $k = 1, \ldots, n$, denote the zeros of $p_n(u) = p_n$ with $u$ given by (1), then $x_k = \cos \theta_k$, $\theta_k \in [0, \pi]$, $k = 0, 1, \ldots, n+1$, and

$$\theta_k - \theta_{k+1} \sim \frac{1}{n}, \quad k = 0, \ldots, n$$  \hspace{1cm} (24)

(see, e.g., [12, 13]).

Moreover [13, p. 673]

$$\frac{1}{|p_n'(x_k)|} \sim \lambda_k |p_{n-1}(x_k)|.$$  \hspace{1cm} (25)

In addition [13, p. 673] for $k = 1, \ldots, n$

$$|p_{n-1}(x_k)| \sim (1-x_k)^{-\gamma/2+1/4} (1+x_k)^{-\delta/2+1/4} \prod_{j=1}^g (|t_j - x_k| + n^{-1})^{-\gamma/2}$$  \hspace{1cm} (26)

and [13]

$$\lambda_k \sim \frac{1}{n} (1-x_k)^{\gamma+1/2} (1+x_k)^{\delta+1/2} \prod_{j=1}^g (|t_j - x_k| + n^{-1})^{\gamma}, \quad k = 1, \ldots, n.$$  \hspace{1cm} (27)

Consequently

$$\frac{|\ell_{n,k}(x)|}{\lambda_k^{1/2}} \sim \frac{|p_n(x)| (1-x_k^{2})^{1/2}}{\sqrt{n |x-x_k|}}, \quad k = 1, \ldots, n.$$  \hspace{1cm} (28)

We also recall that if $x_j$ denotes the closest zero to $x$, then [13, p. 673]

$$\frac{1-x_k^{2}}{n^2 (x-x_k)^2} \leq \frac{1}{(k-j)^2}, \quad k \neq j,$$  \hspace{1cm} (29)

$$|x-x_k| \sim \frac{|j-k|}{n} \sqrt{|1-x^2| + \frac{(j-k)^2}{n^2}}, \quad k \neq j,$$  \hspace{1cm} (30)

and

$$|x-x_j| \leq C \left( \frac{\sqrt{|1-x^2|} + \frac{1}{n^2}}{n} \right).$$  \hspace{1cm} (31)

Moreover [13, p. 673]

$$1-x_k \sim 1+x_1 \sim \frac{1}{n^2}.$$  \hspace{1cm} (32)
\[
\max_{|x| \leq 1} n \lambda_n(x) p^2_n(x) \leq C. \tag{33}
\]

Consequently by (28)–(32)
\[
N_n(f; x) \sim \frac{\sum_{k=1}^{n} \frac{(1-x^2)^{s/2}_k}{|x-x_k|^s} |f(x_k)|}{\sum_{k=1}^{n} \frac{(1-x^2)^{s/2}_k}{|x-x_k|^s}} \\
\leq C \frac{|x-x_j|^s}{(1-x^2)^{s/2}_j} \left[ \sum_{k \neq j} \frac{|f(x_k)|}{|j-k|^s} n^s \right] + |f(x_j)| \\
\leq C \left[ \sum_{k \neq j} \frac{|f(x_k)|}{|j-k|^s} + |f(x_j)| \right]. \tag{34}
\]

This shows that \( N_n \) is related to Shepard operator \( S_n \) in some sense (see [2]). Since the weighted behaviour of Shepard-type operators is unknown in general (it is the subject of a future paper) and our demonstration techniques are based on direct estimates for \( N_n \) (see, e.g., the proofs of Lemmas 3.2 and 3.3 and Theorem 2.2), here we had to work directly on \( N_n \) operators.

First we prove Proposition 2.1.

**Proof.** Let \( n \) be even. From (28) we obtain
\[
w(0) N_n(f; 0) = \exp(-1) \sum_{k=1}^{n} \frac{|f_{n,k}(0)|^{s/2} \exp \left( \frac{1}{\sqrt{1-x^2_k}} \right)}{\lambda_{n,k}} \\
\sim \sum_{k=1}^{n} \frac{(1-x^2_k)^{s/2} \exp \left( \frac{1}{\sqrt{1-x^2_k}} \right)}{|x_k|^s}. \\
\geq \frac{C(1-x^2)^{s/2} \exp \left( \frac{1}{\sqrt{1-x^2}} \right)}{\sum_{k=1}^{n} (1-x^2_k)^{s/2} |x_k|^s}. \tag{35}
\]
Since \( |x_k| \geq \frac{C}{x}, k = \frac{n}{2}; \frac{n}{2} + 1 \), by (29) we deduce
\[
\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x_k|^{r'}} \leq C \left\{ n^s \sum_{k \neq \frac{n}{2}; \frac{n}{2} + 1} \frac{1}{|k-n/2|^r + n^r} \right\} \\
\leq Cn^r.
\]

Therefore from (32), (35), and (36),
\[
w(0)|N_n(f; 0)| \geq C \exp(Cn) \left( 1-x_n^2 \right)^{s/2} \left( \frac{1}{n^s} \right)
\geq C \exp(Cn) \frac{n^{2s}}{n^{2s}}
\]
which is unbounded when \( n \to + \infty \). The assertion follows.  

The following lemma will be useful in the sequel. It establishes the boundedness of the operator \( N_n \) in the weighted norm.

**Lemma 3.1.** Let \( s \geq 2a+1 \). Then for every function \( f \) defined on \([-1, 1]\) we have
\[
\|wN_n(f)\| \leq C \|w\| \|wN_n\left( \frac{1}{w} \right)\| \leq C \|w\| \tag{37}
\]
with \( C \) a positive constant independent of \( f \) and \( n \).

**Proof.** Because of the interpolatory property of \( N_n \) at \( x_k, k = 1, \ldots, n \), we may assume \( x \neq x_k, k = 1, \ldots, n \). Assume \( x \geq 0 \). Similarly we work if \( x < 0 \).

We distinguish three cases.

**Case 1.** \( x > x_k > 0 \).

Then \( w(x) \sim (1-x)^a < (1-x_k)^a \sim w(x_k) \), therefore
\[
\sum_{\theta < x < \alpha} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{r/2}} \frac{|f(x_k)|}{w(x_k)} \leq \|w\| \|w\| \sum_{\theta < x < \alpha} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{r/2}} \frac{1}{w(x_k)}
\leq C \|w\|.
\]
Case 2. \( 0 < x < x_k \).

Here by (34)

\[
T := w(x) \sum_{x_k > x} \frac{|f_{n,k}(x)|^s}{\lambda_k^{s/2}} |f(x_k)| \sum_{k=1}^n \frac{|f_{n,k}(x)|^s}{\lambda_k^{s/2}} w(x_k)
\leq \|wf\|w(x) \sum_{k=1}^n \frac{|f_{n,k}(x)|^s}{\lambda_k^{s/2}} \frac{1}{w(x_k)}
\leq C \|wf\|(1-x)^s \left\{ \sum_{k=j+1}^{(n+j)/2} + \sum_{k=(n+j)/2+1}^n \right\} \frac{1}{(k-j)^s} \frac{1}{(1-x_k)^s}
= C \|wf\|(1-x)^s \left( \sum_{k=j+1}^{(n+j)/2} + \sum_{k=(n+j)/2+1}^n \right) \frac{1}{(k-j)^s} \frac{1}{(1-x_k)^s}
= C \|wf\|(1-x)^s \left( \sum_{k=j+1}^{(n+j)/2} + \sum_{k=(n+j)/2+1}^n \right) \frac{1}{(k-j)^s} \frac{1}{(1-x_k)^s}
:= \|wf\| (S_1 + S_2),
\]

where again \( x_j \) denotes the closest knot to \( x \).

Now by (30)

\[
S_1 \leq C (1-x)^s \sum_{k=j+1}^{(n+j)/2} \frac{1}{(k-j)^s (n-k)^{2s}} = C (1-x)^s \sum_{k=j+1}^{(n+j)/2} \frac{1}{(n-(n+j)/2)^{2s}} \frac{1}{(k-j)^s} \leq C (1-x)^s (1-x_j)^s \leq C.
\]

(39)

On the other hand by (32) and (30)

\[
S_2 \leq C (1-x)^s \sum_{k=(n+j)/2+1}^n \frac{1}{(k-j)^s (1-x_k)^s} = C (1-x)^s \sum_{k=(n+j)/2+1}^n \frac{1}{(n-j+1/2)^{s-1}} \leq C
\]

(40)

if \( s \geq 2x + 1 \).

Hence from (38), by (39) and (40)

\[
T \leq C \|wf\|
\]

if \( s \geq 2x + 1 \).
Case 3. \( x_k < 0 \).

Here by (30) and (34)

\[
W := w(x) \sum_{k=1}^{n} \frac{|f(x_k)|}{\lambda_k^{s/2}} \frac{|f(x_k)|}{w(x_k)} \leq \|wf\|w(x) \sum_{k=1}^{n} \frac{|f(x_k)|}{\lambda_k^{s/2}} \frac{1}{w(x_k)} \leq C\|wf\||(1-x)^{s/2} \left( \sum_{k=1}^{n/2} \frac{1}{k^{(j-k)^2}} + \sum_{k=n/2+1}^{n} \frac{1}{(j/2+1)^{s-1}(j-n/2)^{s-1}} \right) \leq C\|wf\|
\]

if \( s-1 \geq 2\alpha \).

Hence the assertion is proved.

Note that Lemma 3.1 does not need the assumption (5).

The following lemmas are useful to prove Theorems 2.1 and 2.3. In particular they are interesting in themselves because they establish some weighted Markov–Bernstein type inequalities for the operator \( N_n \).

**Lemma 3.2.** If \( s \geq 2\alpha + 1 \), then

\[
\|w\varphi N_n(f)\| \leq Cn\|wf\|,
\]

with \( \varphi = \sqrt{1-x^2} \) and \( C \) independent of \( f \) and \( n \).

**Proof.** Since \( N_n(f; x_k) = 0 \), \( k = 1, \ldots, n \), we assume \( x \neq x_k \), \( k = 1, \ldots, n \). From (3)

\[
N_n(f; x) = \sum_{k=1}^{n} \frac{f(x_k)}{|x-x_k|^{s/2}} \left| \frac{p_s(x_k)}{|x-x_k|^{s/2}} \right| \leq \sum_{k=1}^{n} \frac{1}{|x-x_k|^{s/2}} \left| \frac{p_s(x_k)}{|x-x_k|^{s/2}} \right|.
\]

(41)
Hence
\[
N'_n(f; x) = -\frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right]
\]

and
\[
|N'_n(f; x)| \leq \frac{C}{\left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right]^2} \\
\left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right] \\
+ \frac{1}{C} \sum_{k=1}^{n} \frac{f(x_k)}{[x-x_k]^{\lambda/2}} \sum_{l=1}^{n} \frac{1}{[x-x_l]^{\lambda/2}} \left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right]
\]

\[:= S_1 + S_2 + S_3 + S_4 + S_5. \tag{42}\]

First we note that by (25)–(27)
\[
\frac{1}{\left[ \sum_{k=1}^{n} \frac{1}{[x-x_k]^{\lambda/2}} \right]^2} \leq \frac{C|x-x_1|^2}{\lambda |p_{n-1}(x)|^2}. \tag{43}
\]

First we estimate \|w\varphi S_n\|. 

\[\]
From (25)–(27), (29)–(31), and (43)

\[ w(x) \varphi(x)|S_i| \leq C \frac{w(x) \varphi(x)|x-x_j|^{2s}}{\lambda_j^s |p_{n-1}(x_j)|^{2s}} \frac{|p_{n-1}(x_j)|^s}{|x-x_j|^s} |f(x_j)| \]

\[ \times \sum_{k \neq j} \frac{\lambda_k^{s/2} |p_{n-1}(x_k)|^{s/2}}{|x-x_k|^{s+1}} \]

\[ \leq C \|w\varphi\| \frac{|x-x_j|^s}{n^{-s/2} (1-x_j^2)^{s/2}} \varphi(x) \sum_{k \neq j} \frac{(1-x_k^2)^{s/2}}{n^{s/2} |x-x_k|^{s+1}} \]

\[ \leq Cn \|w\varphi\| \left( \frac{\sqrt{1-x^2} + 1/n}{1-x^2} \right)^s \sum_{k \neq j} \frac{1 (1-x_k^2)^{s/2}}{n^2 |x-x_k|^s} \]

\[ \leq Cn \|w\varphi\|. \]

Similarly by Lemma 3.1 we can prove that

\[ \|w\varphi S_i\| \leq Cn \|w\varphi\|, \quad i = 1, 2, 3, 4, 6. \]

Hence the assertion follows.

**Lemma 3.3.** Let \( s \geq 2\alpha + 2 \). Then

\[ \|w\varphi N'_n(f)\| \leq C \|w\varphi f'\|. \]

**Proof.** We assume \( x \neq x_k, k = 0, \ldots, n \). It results that

\[ N_n(f; x) = \sum_{k=1}^{n} A_k(x) f(x_k) = f(x) + \sum_{k=1}^{n} A_k(x) [f(x_k) - f(x)], \quad (44) \]

with

\[ A_k(x) = \frac{|\ell_{n,k}(x)|^{s/\lambda_k^{s/2}}}{\sum_{k=1}^{n} |\ell_{n,k}(x)|^{s/\lambda_k^{s/2}}}. \]

Since \( \sum_{k=1}^{n} A_k'(x) = 0 \), from (44) it follows that

\[ N_n'(f; x) = \sum_{k=1}^{n} A_k'(x) [f(x_k) - f(x)]. \]
Hence by the mean value theorem

\[ w(x) \varphi(x)|N'_x(f; x)| \leq w(x) \varphi(x) \sum_{k=1}^{n} |A'_k(x)||g(\theta_k) - g(\theta)| \]

\[ \leq C w(x) \varphi(x) \sum_{k=1}^{n} \frac{|A'_k(x)|}{w(\xi_k)} \|g'|_{[0, \pi]} |\theta - \theta_k| \]

\[ \leq C w(x) \varphi(x) \|f'w\| \sum_{k=1}^{n} \frac{|A'_k(x)||\theta - \theta_k|}{w(\xi_k)} \]

where \( g(\theta) = f(\cos \theta) \), \( \varphi(\theta) = w(\cos \theta) \), \( \|g'w\|_{[0, \pi]} \) is the usual supremum norm on \([0, \pi]\) of \( g'w \) and \( \xi_k \) lies between \( x \) and \( x_k \).

Now if \( w(\xi_k) > w(x) \), then we need to estimate \( \sum_{k=1}^{n} |A'_k(x)||\theta - \theta_k|\varphi(x) \).

From (42) and (24), if \( x_j \) denotes again the closest zero to \( x \), we have

\[
\varphi(x) \sum_{k=1}^{n} |A'_k(x)||\theta - \theta_k| \leq C \varphi(x) \left[ \sum_{k=1}^{n} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \right]^{1/2} \]

\[
= C \varphi(x) \left[ \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} |k-j| \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \right]^{1/2} \]

\[
+ \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} |k-j| \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \]

\[
+ \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} |k-j| \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \]

\[
+ \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} |k-j| \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \]

\[
+ \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} |k-j| \frac{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}}{|x-x_j|^s} \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \]

\[
:= T_1 + T_2 + T_3 + T_4 + T_5 + T_6.\]

We start in estimating \( T_5 \). Indeed by (43), (25)–(27), and (29)–(31),

\[
T_5 \leq C \frac{|x-x_j|^s}{\lambda_j^{1/2}|p_{n-1}(x_j)|^{s}} \sum_{k \neq j} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^{s}}{|x-x_k|^s} \]

\[
\leq C \frac{(1-x^2+1/n)^s}{n^{1/2}(1-x_j^2)^{1/2}} \sum_{k \neq j} \frac{(1-x_k^2)^{1/2}}{n^{1/2}|x-x_k|^s} \leq C.\]
Working similarly

\[ T_i \leq C, \quad i = 1, 2, 3, 4, 6. \]

If \( w(\xi_k) < w(x) \), then we work similarly and by following the proof of Lemma 3.1 with \( s - 1 \geq 2 \alpha + 1 \), finally we get the assertion.

We remark that from Lemmas 3.2 and 3.3 we obtain for \( h \in C_{\text{Loc}}((-1, 1)) \), \( \|wh\| < +\infty \), and \( \|w'\phi\| < +\infty \)

\[
\|w\varphi N_s(f)\| \leq \|w\varphi N_s(f - h)\| + \|w\varphi N_s(h)\|
\leq C\{\|w(f - h)\| + \|w\varphi h\|\}
\leq Cn \left\{ \|w(f - h)\| + \frac{1}{n}\|w\varphi h\| \right\}.
\]

By (11) we get

\[
\|w\varphi N_s(f)\| \leq CnK^\varphi \left( f; \frac{1}{n} \right) \leq Cn\omega^\varphi \left( f; \frac{1}{n} \right).
\]

(45)

Now we demonstrate Theorem 2.1.

**Proof.** Obviously we assume \( x \neq x_k, k = 1, \ldots, n \). Then

\[
w(x)|f(x) - N_s(f; x)| \leq w(x) \sum_{k=1}^{n} \frac{|e_{n,k}(x)|^r}{\lambda_k^{q/2}} |f(x) - f(x_k)|
= w(x) \sum_{k=1}^{n} \frac{|e_{n,k}(x)|^r}{\lambda_k^{q/2}} |g(\theta) - g(\theta_k)|
\]

(46)

with \( g(\theta) = f(\cos \theta) \).

Hence if \( \bar{w}(\theta) = w(\cos \theta) \) and \( x_j \) denotes again a closest knot to \( x \), by (46) and (24) we obtain for \( \|g'\bar{w}\|_{[0,\pi]} < +\infty \)

\[
\sum_{k=1}^{n} \frac{|e_{n,k}(x)|^r}{\lambda_k^{q/2}} \left\| g'\bar{w} \right\|_{[0,\pi]} \left\| \frac{j}{n} - \frac{k}{n} \right\|
\]

\[ w(x)|f(x) - N_s(f; x)| \leq Cw(x) \sum_{k \neq j}^{n} \frac{|e_{n,k}(x)|^r}{\lambda_k^{q/2}} \bar{w}(\xi_k)
+ C \frac{\|g'\bar{w}\|_{[0,\pi]}}{n}, \]
where \( \|w\|_{[0,s]} \) denotes the usual supremum norm on \([0, \pi]\) of the bounded function \( g' \tilde{w} \) and \( \xi_k \) is between \( \theta \) and \( \theta_k \). By (34)

\[
\frac{w(x)}{n} \sum_{\alpha(\xi_k) > w(x)} \frac{|f_{n,k}(x)|^+ |j-k|}{\lambda_k^{1/2} w(\xi_k)} \leq C \sum_{n \neq j} \frac{1}{|j-k|^{s-1}} \leq C/n.
\]

Moreover

\[
T := \frac{w(x)}{n} \sum_{\alpha(\xi_k) > w(x)} \frac{|f_{n,k}(x)|^+ |j-k|}{\lambda_k^{1/2} w(\xi_k)} \leq C \sum_{n \neq j} \frac{1}{|j-k|^{s-1}} \leq C/n.
\]

and working as in the proof of Lemma 3.1 we deduce for \( s \geq 2\alpha + 2 \)

\[
T \leq \frac{C}{n}.
\]

Hence if \( s \geq 2\alpha + 2 \)

\[
\|w[f - N_n(f)]\| \leq C \frac{\|g' \tilde{w}\|_{[0,s]}}{n} \leq C \frac{\|w f'\|}{n},
\]

with \( \varphi(x) = \sqrt{1 - x^2} \), from which by (11)

\[
\|w[f - N_n(f)]\| \leq C \omega^x \left( f; \frac{1}{n} \right)_w, \quad \forall f \in C([-1,1]),
\]

that is, (7).

Now we prove (8). By (11), (7), and (45) we obtain

\[
\omega^x \left( f; \frac{1}{n} \right)_w \leq CK^\varphi \left( f; \frac{1}{n} \right)_w \leq C \left\{ \|w[f - N_n(f)]\| + \frac{1}{n} \|w \varphi N_n^\varphi(f)\| \right\}
\]

\[
\leq C \omega^x \left( f; \frac{1}{n} \right)_w,
\]

that is, (8).
Now we prove (9). From (7) it follows that if \( \omega^\varphi(f; t) u = O(t^\beta) \), then
\[
||w[f - N_n(f)]|| = O(n^{-\beta}).
\]
Now we prove the converse implication. From the definition of \( K^\varphi(f) u \), we obtain for \( h \in C_\infty((-1, 1)) \), \( ||wh|| < + \infty \), \( ||wh\varphi|| < + \infty \)
\[
K^\varphi(f; 1/n) u \leq ||w[f - N_n(f)]|| + \frac{1}{n} ||w\varphi N_n'(f)||
\]
\[
\leq ||w[f - N_k(f)]|| + \frac{1}{n} \{ ||w\varphi N_k'(f - h)|| + ||w\varphi N_k'(h)|| \}.
\]
Now, by using Lemmas 3.2 and 3.3 we get
\[
K^\varphi(f; 1/n) u \leq ||w[f - N_n(f)]|| + C \frac{k}{n} \{ ||w[h - f]| + \frac{1}{k} ||w\varphi|| \}
\]
and consequently
\[
K^\varphi(f; 1/n) u \leq ||w[f - N_n(f)]|| + C \frac{k}{n} K^\varphi(f; 1/n) u.
\]
Now if \( ||w[f - N_n(f)]|| = O(n^{-\beta}) \), \( 0 < \beta < 1 \), then
\[
K^\varphi(f; 1/n) u \leq C k^{-\beta} + C \frac{k}{n} K^\varphi(f; 1/n) u
\]
and from a well-known lemma by Berens and Lorentz (see, e.g., [1; 8, Lemma 9.34, p. 699]), (9) follows. 

Now we give the proof of Theorem 2.2.

Proof. Estimates (13) and (14) can be deduced working similarly as in the proof of (7) and (8), respectively.

Now we prove (15). From (3) the \( N_n \) operator can be written as
\[
N_n(f; x) = \sum_{k=1}^{n} a_k(\theta) g(\theta_k) := L_n(g; \theta)
\]
with \( x = \cos \theta, x_k = \cos \theta_k, k = 1, \ldots, n \) (see 24),
\[
a_k(\theta) = \frac{[\epsilon \varphi, k(\cos \theta)]^r}{\sum_{k=1}^{\lambda_{k}/2} [\epsilon \varphi, k(\cos \theta)]^r}
\]
and \( g(\theta) = f(\cos \theta) \in C([0, \pi]) \).
Now if we prove that

\[ L_n(g; \theta) = g(\theta), \quad \text{if} \quad g = \text{constant} \quad (50) \]

\[ \sum_{|\theta - \theta_k| > d_0} |a_k(\theta)| = o \left( \frac{1}{n} \right), \quad d_0 > 0 \text{ arbitrarily fixed} \quad (51) \]

\[ a_j(\theta) > \frac{1}{2}, \quad \text{if} \quad |\theta - \theta_j| \leq \frac{\delta}{n}, \quad 0 \leq \delta < d_1 \quad (52) \]

\[ \sum_{k \neq j} |\theta - \theta_k||a_k(\theta)| \leq d_2 \frac{\delta^{1+\epsilon}}{n}, \quad \delta \text{ is as above,} \quad (53) \]

with \( x_j \) again the closest knot to \( x \) and with certain positive fixed reals \( d_1, d_2 \) and \( \epsilon \), then by [9, Theorem 2.1] (see also [14; 17, Theorem 2.1])

\[ \limsup_{n \to +\infty} n \|L_n(g) - g\|_{[0, \pi]} > KM(g), \quad (54) \]

where with \( \theta, \tau \in [0, \pi] \)

\[ M(g) := \sup_{\theta} \left\{ M(g, \theta); M(g, \theta) := \limsup_{\tau \to \theta} \left| \frac{g(\theta) - g(\tau)}{\theta - \tau} \right| \right\} \]

\( \| \cdot \|_{[0, \pi]} \) denotes the usual supremum norm on \([0, \pi]\) and \( K \) is an absolute constant.

Now we prove properties (50)–(53).

Relation (50) holds true by definition. Now we prove (51). Since \( |x - x_k| \geq D_0 \) if \( |\theta - \theta_k| \geq d_0 \), from (49) by (25) it follows

\[ \sum_{|\theta - \theta_k| > d_0} a_k(\theta) \leq C \sum_{\theta_k \neq x} \frac{\lambda_k^{1/2}|p_{n-1}(x_k)|^s}{\sum_{k=1}^{n} \lambda_k^{1/2}|p_{n-1}(x_k)|^s}. \]

Then by (26), (27), and (46)

\[ \sum_{|\theta - \theta_k| > d_0} a_k(\theta) \leq C \sum_{\theta_k \neq x} \frac{\lambda_k^{1/2}(1-x_k^2)^{1/2}}{n^{t-1}} \leq C \frac{1}{n^{t-1}} = o \left( \frac{1}{n} \right). \]
that is, (51). Now we prove (52). From (25)–(30) and (2)
\[
\sum_{k \neq j} a_k(\theta) \leq -\frac{|f_j(x)|^s}{\lambda_j^{s/2}} \left\{ \sum_{|x-x_k| \leq \delta} \frac{(1-x_k^2)^{s/2}}{n^{s/2}|x-x_k|^s} + \sum_{|x-x_k| > \delta} \frac{(1-x_k^2)^{s/2}}{n^{s/2}} \right\}
\leq C\delta^s (\phi(x) + \delta/n)^r \left\{ \sum_{|x-x_k| \leq \delta} \frac{1}{(k-j)^s} + \frac{n}{n^s} \right\}
\leq C\delta^s \left\{ 1 + n^{1-s} \right\} \leq \frac{1}{2}.
\]
if \(d_i\) is small enough. Now by \(a_k \geq 0\) and \(\sum a_k = 1\), we get (52).
Finally we verify (53). Let \(\theta < \theta_j < \theta_k\); the other cases are similar. Then
\[
|\theta - \theta_k| \geq |\theta - \theta_j| + \frac{|\theta_j - \theta_k|}{2} \geq |\theta - \theta_j| \left( 1 + C \frac{|j-k|}{\delta} \right) \geq \frac{C}{\delta} |\theta - \theta_j||k-j|.
\]
Hence working as above we get
\[
\sum_{k \neq j} |\theta - \theta_k| a_k(\theta) \leq C |\theta - \theta_j|^r \left\{ \sum_{|x-x_k| \leq \delta} \frac{1}{|\theta - \theta_k|^{s-1} + \delta^s} + \frac{n}{n^s} \right\}
\leq C \left\{ |\theta - \theta_j|^{s-1} + \frac{\delta^s}{n^{s-1}} \right\}
\leq C \frac{\delta^s}{n} + C \frac{\delta^s}{n^{s-1}} \leq C \frac{\delta^s}{n},
\]
i.e. relation (53) has been proved.
Moreover since [10]
\[
\omega^s(f; t) \sim \omega(g; t)
\]  
(55)
with \(g(\theta) = f(\cos \theta)\) and \(\omega(g)\) the usual modulus of continuity of \(g\), then (15) in Theorem 2.2 gives
\[
\|f - N_n(f)\| = \|g - L_n(g)\|_{[0, \pi]} \leq C\omega \left( g; \frac{1}{n} \right)
\]  
(56)
From (54) and (56) we obtain (cf. [17, p. 315])

\[ C_1 M(g) \leq m \| L_m(g) - g \|_{[0, \pi]} \leq C_2 \omega(\frac{1}{m}) \]

\[ \leq C_2 r_m \frac{|g(u_m + r_m) - g(u_m)|}{r_m} \leq C_2 \frac{|g(u_m + r_m) - g(u_m)|}{r_m} \]

\[ \leq C_2 \sup_{\theta \neq \eta} \frac{|g(\theta) - g(\eta)|}{|\theta - \eta|} := C_2 N(g) \]

(57)

(58)

(u_m and r_m, r_m \leq m^{-1}, (m = n_1, n_2, \ldots) are properly chosen).

Now we recall that [17, Lemma 3.1, p. 315]

\[ M(g) = N(g). \]

Therefore

\[ C_1 M(g) \leq C_2 \omega(\frac{1}{m}) \leq C_2 M(g), (m = n_1, n_2, \ldots) \]

and from (54), (57), and (58)

\[ \lim \sup_{n \to +\infty} \frac{\| L_n(g) - g \|_{[0, \pi]}}{\omega(g; 1/n)} \sim 1. \]

Finally from (56) and (55) we get (15).

Now we prove (16).

We quote an observation from [14] (see also [17, Proof of Theorem 2.2, p. 316]), namely

\[ M(g; \theta) = 0, \quad \text{for each } \theta \in [0, \pi], \text{ iff } g = \text{ constant} \]

\[ M(g) < \infty, \quad \text{iff } g \in \text{ Lip } 1. \]

(59)

Then if \( g = \text{ constant}, \) then \( L_n(g) - g = 0 = \omega(1/n); \) if \( \| L_n(g) - g \|_{[0, \pi]} = o(1/n), \) then by (54) \( M(g) = 0 \) and by (59) \( g = \text{ constant}. \) Further \( g = \text{ constant } \iff f = \text{ constant }, \) whence we get (16).

Finally we prove (17). If \( \omega^n(f; t) \leq Ct, \) then by (13) it follows that

\[ \| N_n(f) - f \| = O\left(\frac{1}{n}\right). \]

On the other hand, if \( \| N_n(f) - f \| = \| L_n(g) - g \|_{[0, \pi]} = o(1/n), \) then by (54) \( M(g) < +\infty. \) Then by (59) \( g \in \text{ Lip } 1 \) and by (55) we have (17).
Now we prove Theorem 2.3.

Proof. From (46) if \( s = 2 \), working as in [2, p. 83] by (55), we deduce (21) and (22).

Finally from the proofs of Lemmas 3.2 and 3.3 we get for \( s = 2 \)

\[
\| \varphi N_n^s (f) \| \leq C n \| f \|, \quad f \in C([-1, 1]),
\]

\[
\| \varphi N_n^s (f) \| \leq C \log n \| f' \|, \quad f' \in C([-1, 1]),
\]

and working as in [7, Corollary 1, p. 81] by (55) we deduce (23).

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