Available at www.MATHEMATICSweb.org
POWERED BY SCIENCE DIRECT
ELSEVIER
Journal of Computational and Applied Mathematics 153 (2003) 243–248
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

On the Bernstein-type inequalities for ultraspherical polynomials

C. Giordano[a,*], A. Laforgia[b]

[a] Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
[b] Dipartimento di Matematica, Università di Roma Tre, Largo S. Leonardo Murialdo, 1, Rome, Italy

Received 19 March 2002

Abstract

We present a survey of the most recent results and inequalities for the gamma function and the ratio of the gamma functions and study, among other things, the relation between these results and known inequalities for ultraspherical polynomials. In particular, we discuss the inequality

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(n+3/2\lambda)}{\Gamma(n+1+1/2\lambda)}, \qquad 0 \leqslant \theta \leqslant \pi,$$

where $P_n^{(\lambda)}(\cos\theta)$ denotes the ultraspherical polynomial of degree n, established by Alzer (Arch. Math. 69 (1997) 487) and the one established by Durand (In: R.A. Askey (Ed.), Theory and Application of Special Functions, Proceedings of the Advanced Seminar on Mathematical Research Center, University of Wisconsin, Madison, Vol. 35, Academic Press, New York, 1975, p. 353)

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| \leqslant \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}, \qquad 0 \leqslant \theta \leqslant \pi.$$

MSC: 33B15; 33C45

Keywords: Gamma function; Inequality; Ultraspherical polynomials

1. Introduction

The importance of the function

$$y_n(\theta) = (\sin\theta)^{\lambda}P_n^{(\lambda)}(\cos\theta),$$

* Corresponding author.
E-mail address: giordano@dm.unito.it (C. Giordano).

PII: S0377-0427(02)00591-5

is well known where $P_n^{(\lambda)}(\cos\theta)$ are the ultraspherical polynomials of degree $n=0,1,\dots$ and parameter λ, in different fields of analysis and physics and to find related inequalities. Especially in the case of the Legendre polynomials the role of this function and the need to improve the known bounds to get physically interesting results were pointed out, for example, in [15] in the study of scattering problems.

This function is solution of the differential equation

$$\frac{d^2 y(x)}{d\theta^2} + \left[(n+\lambda)^2 + \frac{\lambda(1-\lambda)}{\sin^2\theta}\right] y = 0$$

and classical inequalities were obtained by using Sonin–Polya Theorem.

Let us consider the case $0 < \lambda < 1$. Then we have [16, Theorem 7.33.2], if n is even

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| \leqslant \frac{\Gamma(n/2+\lambda)}{\Gamma(n/2+1)\Gamma(\lambda)} \tag{1.1}$$

or if n is odd

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < [\lambda(1-\lambda)+(n+\lambda)^2]^{-1/2}(n+1)\frac{\Gamma((n+1)/2+\lambda)}{\Gamma((n+1)/2+1)\Gamma(\lambda)}. \tag{1.2}$$

The sign of equality holds only for even n and $\theta = \pi/2$ because in this case we have

$$|P_n^{(\lambda)}(0)| = \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}$$

and as a consequence of [16, Theorem 7.33.2], only for even n,

$$\max_{0\leqslant\theta\leqslant\pi} (\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| = \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}.$$

From inequalities (1.1) and (1.2) the remarkable inequality follows [16], $0 < \lambda < 1$,

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}[\Gamma(\lambda)]^{-1} n^{\lambda-1}, \quad 0 \leqslant \theta \leqslant \pi \tag{1.3}$$

for both the cases n even and n odd. The special case $\lambda = \frac{1}{2}$ leads to the well-known inequality for the Legendre polynomials P_n

$$(\sin\theta)^{1/2} |P_n(\cos\theta)| < (2/\pi)^{1/2} n^{-1/2}. \tag{1.4}$$

The proof of (1.4) is due to Bernstein [4] and it was the first leading to the constant $(2/\pi)^{1/2}$. The constant $2^{1-\lambda}\Gamma(\lambda)^{-1}$ in (1.3) and the constant $(2/\pi)^{1/2}$ in (1.4) cannot be replaced by a smaller one, taking into account that

$$\lim_{n\to\infty}\left[\max_{0\leqslant\theta\leqslant\pi} (\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)|\right] = 2^{1-\lambda}[\Gamma(\lambda)]^{-1}, \quad 0 \leqslant \theta \leqslant \pi.$$

In this paper, we present our investigation about the improvements of the classical Bernstein inequality recently obtained as consequences of the results for the ratio of gamma functions [9,10] and we give a comparison of the refinements due to Durand, Lorch, Laforgia, Alzer and other authors.

2. Comparison of some Bernstein-type inequalities

Several authors presented refinements of (1.3) and (1.4) replacing the factor $n^{\lambda-1}$ and the factor $n^{-1/2}$, respectively, with a smaller one. By using complex variable methods, Antonov and Holševnikov [3] deduced that the factor $n^{-1/2}$ in (1.4) can be replaced by $(n+1/2)^{-1/2}$.

More generally Lorch [14] refined (1.3) replacing the factor $n^{\lambda-1}$ by $(n+\lambda)^{\lambda-1}$

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}[\Gamma(\lambda)]^{-1}(n+\lambda)^{\lambda-1}, \quad 0 \leqslant \theta \leqslant \pi \tag{2.1}$$

by using the inequality, $n=0,1\ldots,$

$$\frac{1}{(n+\lambda)^{1-\lambda}} < \frac{\Gamma(n+\lambda)}{\Gamma(n+1)} < \frac{1}{(n+\frac{1}{2}\lambda)^{1-\lambda}}, \quad 0<\lambda<1. \tag{2.2}$$

Earlier, Durand derived a Nicholson-type formula [7] for Gegenbauer functions of the first and second kind, $C_\mu^{(\alpha)}(x)$ and $D_\mu^{(\alpha)}(x)$, that is, he expressed the sum of two squares $[C_\mu^{(\alpha)}(x)]^2+[D_\mu^{(\alpha)}(x)]^2$ as an integral over a Gegenbauer function of the second kind. As a consequence, he obtained the set of bounds [7, (24), p. 362]

$$|aC_\mu^{(\alpha)}(x)+bD_\mu^{(\alpha)}(x)| \leqslant (1-x^2)^{-\alpha/2}L_0\sqrt{a^2+b^2}, \quad 0<\alpha\leqslant 1,$$

where

$$L_0=\frac{\Gamma(\mu/2+\alpha)}{\Gamma(\alpha)\Gamma(\mu/2+1)},$$

which implies separate inequalities for C and D. For instance, with $a=1$ and $b=0$,

$$|C_\mu^{(\alpha)}(x)| \leqslant (1-x^2)^{-\alpha/2}L_0, \quad 0<\alpha\leqslant 1.$$

Hence for integer $\mu=n\geqslant 0$ and $\alpha=\lambda$, $x=\cos\theta$, there follows the inequality for the corresponding ultraspherical polynomials $|P_n^{(\lambda)}(\cos\theta)|$, $0<\lambda<1$,

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| \leqslant \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}, \quad 0\leqslant\theta\leqslant\pi. \tag{2.3}$$

Bound (2.3) holds for both even n and odd n and the Lorch's bound is implicit in it [14]. It suffices to apply the upper bound in (2.2), valid for $n>0$ [12], not only integer-valued, to obtain

$$\frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)} < 2^{1-\lambda}[\Gamma(\lambda)]^{-1}(n+\lambda)^{\lambda-1}, \quad 0\leqslant\theta\leqslant\pi. \tag{2.4}$$

In the case even $n=2r$, we have that the Durand upper bound (2.3) is equal to (1.1) which gives the maximum value of the function $(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)|$, for $0\leqslant\theta\leqslant\pi$ and $0<\lambda<1$.

In the case odd $n=2r+1$, Durand inequality is an upper bound of (1.2).

Now we compare Durand inequality with others that some authors have given in the recent past.

Laforgia [13] in 1992 supplied a very simple proof of the Bernstein inequality (1.1),which leads to replace the term $n^{\lambda-1}$ with the ratio of two gamma functions $\Gamma(n+\lambda)/\Gamma(n+1)$,

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(n+\lambda)}{\Gamma(n+1)}, \quad 0\leqslant\theta\leqslant\pi,$$

but although this inequality provides another refinement of the Bernstein inequality, does not improve Lorch's result because from (2.2) we have

$$(n+\lambda)^{\lambda-1} < \Gamma(n+\lambda)/\Gamma(n+1).$$

Recently, Alzer [2] established a new upper bound for $|P_n^{(\lambda)}(\cos\theta)|$, for $0<\lambda<1$,

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(n+\frac{3}{2}\lambda)}{\Gamma(n+1+\frac{1}{2}\lambda)}, \quad 0\leqslant\theta\leqslant\pi, \tag{2.5}$$

which refines the inequalities given by Bernstein, Lorch and Laforgia. Now we compare Durand and Alzer upper bounds. In the case even $n=2r$, it is easily seen that Durand inequality is sharper than Alzer's because it assumes exactly the value $|P_n^{(\lambda)}(0)|$. In the case odd n, we have to show that

$$\frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)} < \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(n+\frac{3}{2}\lambda)}{\Gamma(n+1+\frac{1}{2}\lambda)}. \tag{2.6}$$

For $n=2r+1$ and the duplication formula of the gamma function, we have

$$\frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(2r+1+\frac{3}{2}\lambda)}{\Gamma(2r+2+\frac{1}{2}\lambda)} = \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma[2(r+\frac{1}{2}+\frac{3}{4}\lambda)]}{\Gamma[2(r+1+\frac{1}{4}\lambda)]} = \frac{1}{\Gamma(\lambda)}\frac{\Gamma(r+\frac{3}{4}\lambda+\frac{1}{2})}{\Gamma(r+\frac{1}{4}\lambda+1)}\frac{\Gamma(r+\frac{3}{4}\lambda+1)}{\Gamma(r+\frac{1}{4}\lambda+\frac{3}{2})}$$

and it suffices to show that for $0<\lambda<1$,

$$\frac{\Gamma(r+\frac{1}{2}+\lambda)}{\Gamma(r+\frac{3}{2})} < \frac{\Gamma(r+\frac{3}{4}\lambda+\frac{1}{2})\Gamma(r+\frac{3}{4}\lambda+1)}{\Gamma(r+\frac{1}{4}\lambda+1)\Gamma(r+\frac{1}{4}\lambda+\frac{3}{2})}.$$

that is, the function

$$f(r,\lambda) = \frac{\Gamma(r+\frac{3}{4}\lambda+\frac{1}{2})\Gamma(r+\frac{3}{4}\lambda+1)\Gamma(r+\frac{3}{2})}{\Gamma(r+\frac{1}{4}\lambda+1)\Gamma(r+\frac{1}{4}\lambda+\frac{3}{2})\Gamma(r+\lambda+\frac{1}{2})} > 1.$$

Recalling the limit relation [8]

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{\Gamma(x+b)}x^{b-a} = 1, \tag{2.7}$$

we can easily deduce that

$$\lim_{r\to\infty} f(r,\lambda) = 1, \quad 0<\lambda<1.$$

We use a generalization of some results of [5,11] given in [1] to prove that the function $f(r,\lambda)$ is greater than its limit 1.

Theorem. *Let a_i and b_i $(i=1,2\dots,n)$ be real numbers such that $0\leqslant a_1\leqslant\cdots\leqslant a_n$, $0\leqslant b_1\leqslant\cdots\leqslant b_n$, and $\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i$, $k=1,\dots,n$. Then,*

$$x \to \prod_{i=1}^{n}\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is completely monotonic on $(0,\infty)$.

Remark. Since (2.7), from the theorem it follows that the inequality

$$\prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)} \geqslant 1, \quad x>0$$

holds for all real numbers $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, $0 \leqslant b_1 \leqslant \cdots \leqslant b_n$, and $\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i$, $k=1,\ldots,n-1$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

The function $f(r,\lambda)$ which satisfies the conditions of the Remark, $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 3+\frac{3}{2}\lambda$, is greater than 1. This proves that the Durand inequality is sharper than the Alzer one. In the case of the Legendre polynomials, $\lambda = 1/2$, Durand bound gives for $n=2r$,

$$(\sin\theta)^{1/2}|P_{2r}(\cos\theta)| \leqslant \frac{(2r-1)(2r-3)\ldots 1}{2^r r!}$$

and for $n=2r+1$,

$$(\sin\theta)^{1/2}|P_{2r+1}(\cos\theta)| \leqslant \frac{2^{2r+1}(r!)^2}{\pi(2r+1)!}.$$

We have graphically and numerically compared (2.3) and (2.5) for some fixed values of n and $0<\lambda<1$. The difference increases on $0<\lambda<1/2$ until about $\lambda<1/2$ and then decreases.

In 1994, Chow et al. [6] proved a Bernstein-type inequality for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for $-1/2 \leqslant \alpha,\beta \leqslant 1/2$ and $0 \leqslant \theta \leqslant \pi$,

$$\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\beta+1/2} |P_n^{(\alpha,\beta)}(\cos\theta)| \leqslant \frac{\Gamma(q+1)}{\Gamma(\frac{1}{2})} \binom{n+q}{n} N^{-q-1/2}, \tag{2.8}$$

where $q=\max(\alpha,\beta)$ and $N=n+\frac{1}{2}(\alpha+\beta+1)$. In the case of ultraspherical polynomials, $\alpha=\beta=\lambda-\frac{1}{2}$ (2.8) gives

$$(\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| \leqslant \frac{2^{1-\lambda}}{\Gamma(\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} (n+\lambda)^{-\lambda}, \quad 0 \leqslant \theta \leqslant \pi, \tag{2.9}$$

which sharpens Lorch's result (2.1) only for $0<\lambda<1/2$. If $\frac{1}{2}<\lambda<1$, their result reduces to one which is weaker than (2.1). Alzer has proved that the inequality

$$\frac{\Gamma(n+\frac{3}{2}\lambda)}{\Gamma(n+1+\frac{1}{2}\lambda)} < \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)}(n+\lambda)^{-\lambda}$$

holds for values of $0<\lambda<1$ which are sufficiently close to 1, whereas the reverse inequality holds for all sufficiently small λ.

References

[1] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp. 66 (1997) 373–389.
[2] H. Alzer, On Bernstein's inequality for ultraspherical polynomials, Arch. Math. 69 (1997) 487–490.
[3] V.A. Antonov, K.V. Holševnikov, An estimate of the remainder in the expansion of the generating function for the Legendre polynomials (Generalization and improvement of Bernstein's inequality), Vestnik Leningrad Univ. Math. 13 (1981) 703–709.
[4] S. Bernstein, Sur les polynomes orthogonaux relatifs a un segment fini, J. Math. 10 (1931) 219–286.

[5] J. Bustoz, M.E.H. Ismail, On gamma function inequalities, Math. Comp. 47 (176) (1986) 659–667.
[6] Y. Chow, L. Gatteschi, R. Wong, A Bernstein-type inequality for the Jacobi polynomial, Proc. AMS 121 (3) (1994) 703–709.
[7] L. Durand, Nicholson-type integrals for products of Gegenbauer functions and related topics, in: R.A. Askey (Ed.), Theory and Application of Special Functions, Proceedings of the Advanced Seminar on Mathematical Research Center, University of Wisconsin, Madison, Vol. 35, Academic Press, New York, 1975, pp. 353–374.
[8] A. Erdély, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, Toronto, London, 1953.
[9] C. Giordano, A. Laforgia, Inequalities and monotonicity properties for the gamma function, J. Comput. Appl. Math. 133 (1, 2) (2001) 387–396.
[10] C. Giordano, A. Laforgia, J. Pečarić, Unified treatment of Gautschi–Kershaw type inequalities for the gamma function, J. Comput. Appl. Math. 99 (1998) 166–175.
[11] M.E.H. Ismail, M.E. Muldoon, Inequalities and monotonicity properties for gamma and q-gamma functions, ISNM Approx. Comp. 119 (1994) 309–323.
[12] D. Kershaw, Some extensions of W. Gautschi's inequalities for the gamma function, Math. Comp. 41 164 (1983) 607–611.
[13] A. Laforgia, A simple proof of the Bernstein inequality for ultraspherical polynomials, Boll. Un. Mat. Ital. A 6 (1992) 267–269.
[14] L. Lorch, Inequalities for ultraspherical polynomials and the gamma function, J. Approx. Theory 40 (1984) 115–120.
[15] A. Martin, An optimal inequality on associated Legendre functions, Ann. Numer. Math. 2 (1995) 327–344.
[16] G. Szegö, Orthogonal Polynomials, Vol. 23, 4th Edition, American Mathematical Society, Providence, RI, 1975.